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On linear Lie algebras I.

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Introduction

Let $\mathfrak{gl}(R^n)$ be the set of all linear transformations X of an n-dimensional linear space R^n over the field of real numbers, equipped with the ordinary law of addition $X+X'$ and the law of composition $[X, Y]=XY-YX$. The purpose of the present paper, which is considered as the first of the series of papers pursuing the same purpose, is to deal with a method of studying some properties of r-dimensional subalgebras $\mathfrak{g}$ of this general linear Lie algebra $\mathfrak{gl}(R^n)$.

If we take a base S composed of a set of n linearly independent vectors $\boldsymbol{e}_\lambda$ $(\lambda=1,\cdots,n)$ of R^n, then a subalgebra $\mathfrak{g}$ is represented by an r-dimensional linear subspace in an n^2-dimensional linear space spanned by all matrices of degree n. This subspace will be denoted by $\mathfrak{K}(\mathfrak{g}, S)$, or by $\mathfrak{K}$ if there is no possibility of confusion. If we take another base $\tilde{S}(\tilde{\boldsymbol{e}}_\lambda)$ such that

$$\hat{S}=AS, \quad \tilde{\boldsymbol{e}}_\lambda=A^{\alpha}_{\cdot\lambda}\boldsymbol{e}_\alpha,$$

then a matrix K of $\mathfrak{K}$ representing an element of $\mathfrak{g}$ is transformed into $\tilde{K}=A^{-1}KA$. This fact will be denoted by

$$\mathfrak{K}(\mathfrak{g}, AS)=A^{-1}\mathfrak{K}(\mathfrak{g}, S)A.$$

The set of matrices $(V^{\lambda}_{\cdot\mu})$ where $V^{\lambda}_{\cdot\mu}$ satisfy

$$K^{\lambda}_{\cdot\mu}V^{\mu}_{\cdot\lambda}=0$$

for all matrices $(K^{\lambda}_{\cdot\mu})$ of $\mathfrak{K}$ spans an (n^2-r)-dimensional linear subspace in an n^2-dimensional linear space spanned by all matrices of degree n. This subspace will be denoted by $\mathfrak{V}(\mathfrak{g}, S)$ or by $\mathfrak{V}$ for short. The law of transformation is

$$\mathfrak{V}(\mathfrak{g}, AS)=A^{-1}\mathfrak{V}(\mathfrak{g}, S)A.$$

$\mathfrak{V}$ may be considered to represent a set of transformations of R^n which will be denoted by $\mathfrak{t}(\mathfrak{g})$ or by $\mathfrak{t}$ for short.

If we take a suitable base S in R^n and moreover, if we take a suitable base M in $\mathfrak{V}(\mathfrak{g}, S)$

$$M:\ \underset{A}{V} \qquad (A=1,\cdots,m;\ m=n^2-r),$$

then the matrices $(\underset{A}{V}{}^{\lambda}_{\cdot\mu})$ take special forms as shown in Theorem 1 (§ 6). In § 7, we shall introduce the notion of *d series* which will play an important role in our studies, in which we shall be able to find out all subalgebras $\mathfrak{g}$ of sufficiently large dimensions r and get some interesting properties of $\mathfrak{g}$ of more general dimensions.

We remark here the important

LEMMA 1. *$\mathfrak{t}$ is an invariant of $\mathfrak{g}$, that is, if $V\in\mathfrak{V}$ and $K\in\mathfrak{K}$, then* $[K, V]\in\mathfrak{V}$.

PROOF. For any matrices $\underset{a}{K}$ $(a=1,\cdots,r)$ forming a base of $\mathfrak{K}$ we have

$$\underset{c}{K}{}^{\lambda}_{\cdot\mu}(\underset{b}{K}{}^{\mu}_{\cdot\alpha}V^{\alpha}_{\cdot\lambda}-V^{\mu}_{\cdot\alpha}\underset{b}{K}{}^{\alpha}_{\cdot\lambda})$$
$$=V^{\mu}_{\cdot\lambda}(\underset{c}{K}{}^{\lambda}_{\cdot\alpha}\underset{b}{K}{}^{\alpha}_{\cdot\mu}-\underset{b}{K}{}^{\lambda}_{\cdot\alpha}\underset{c}{K}{}^{\alpha}_{\cdot\mu})=V^{\mu}_{\cdot\lambda}\underset{a}{K}{}^{\lambda}_{\cdot\mu}C_{cb}^{\cdot\cdot a}=0 \qquad (b, c=1,\cdots,r).$$

Unless otherwise specified indices are used as follows,

$$\alpha,\beta,\gamma,\lambda,\mu,\nu,\kappa=1,\cdots,n,$$
$$A, B=1,\cdots,m,$$
$$a, b, c=1,\cdots,r,$$
$$t, u, v, x, y, z=n-n_1+1,\cdots,n,$$
$$h, i, j, k, l, m=1,\cdots,n-n_1,$$
$$p, q, r, s=2,\cdots,n-n_1,$$
$$S, T, U=1,\cdots,P \text{ or } P+1,$$
$$h_T, i_T, j_T, k_T, l_T, m_T=n-n_{T-1}+1,\cdots,n-n_T,$$
$$t_T, u_T, v_T, x_T, y_T, z_T=n-n_T+1,\cdots,n.$$

We adopt the *summation convention* with respect to indices in small letters.

§ 1. Bases of the first order.

First consider that the base S of R^n and the base M of $\mathfrak{V}$ are taken arbitrarily. When a non-zero vector $\boldsymbol{v}_1$ $(=v_1^{\lambda}\boldsymbol{e}_{\lambda})$ of R^n is given, we get m vectors $\underset{A}{\boldsymbol{v}}$ $(A=1,\cdots,m)$ of R^n whose components are $\underset{A}{V}{}^{\lambda}_{\cdot\alpha}v^{\alpha}_1$. These span a linear subspace of R^n which will be denoted by $L(\mathfrak{g},\boldsymbol{v}_1)$. Of course this subspace does not depend upon the choice of S. We observe that $L(\mathfrak{g},\boldsymbol{v}_1)$ is spanned by n_1 vectors

$$v_x^{\lambda} \qquad (x=n-n_1+1,\cdots,n)$$

or by $1+n_1$ vectors

$$v_1^{\lambda},\ v_x^{\lambda} \qquad (x=n-n_1+1,\cdots,n).$$

In both cases the $1+n_1$ vectors $v_1^\lambda, v_{n-n_1+1}^\lambda, \cdots, v_n^\lambda$ are taken linearly independent. We get the first case if $\boldsymbol{v}_1 \notin L(\mathfrak{g}, \boldsymbol{v}_1)$ and the second case if $\boldsymbol{v}_1 \in L(\mathfrak{g}, \boldsymbol{v}_1)$. As the number n_1 depends upon the choice of $\boldsymbol{v}_1$, this is denoted, if necessary, by $n_1(\boldsymbol{v}_1)$.

Taking $n-n_1-1$ vectors $v_2^\lambda, \cdots, v_{n-n_1}^\lambda$, we can complete a set of n linearly independent vectors

$$v_\kappa^\lambda \qquad (\kappa=1, \cdots, n),$$

whose reciprocal set is denoted by w_μ^κ so that

$$w_\kappa^\lambda v_\mu^\kappa = \delta_\mu^\lambda .$$

Then

$$\underset{A}{\tilde{V}}{}^\lambda_{\cdot\mu} = w_\alpha^\lambda \underset{A}{V}{}^\alpha_{\cdot\beta} v_\mu^\beta$$

is the transformation of the matrices $\underset{A}{V}$ of $\mathfrak{V}$ induced by the transformation

$$\tilde{\boldsymbol{e}}_\lambda = v_\lambda^\alpha \boldsymbol{e}_\alpha$$

of the base S, and we find

$$\underset{A}{\tilde{V}}{}^2_{\cdot 1} = 0, \cdots, \underset{A}{\tilde{V}}{}^{n-n_1}_{\cdot 1} = 0 .$$

Moreover, the m vectors

$$\underset{A}{\tilde{V}}{}^y_{\cdot 1},$$

where $y=n-n_1+1, \cdots, n$ and each vector has n_1 components, span an n_1-dimensional linear space.

Thus we find that

When a vector $\boldsymbol{v}_1$ is given, we can find $n-1$ vectors $\boldsymbol{e}_2, \cdots, \boldsymbol{e}_n$ such that the space $\mathfrak{V}(\mathfrak{g}, S)$ where $S=S(\boldsymbol{v}_1, \boldsymbol{e}_2, \cdots, \boldsymbol{e}_n)$ is spanned by the matrices $\underset{A}{V}$ with the property

case 1:

$$\left\{ \begin{array}{l} \underset{A}{V}{}^\lambda_{\cdot 1} = \varepsilon_A \delta_A^\lambda \\ \varepsilon_1 = \cdots = \varepsilon_{n-n_1} = 0, \quad \varepsilon_{n-n_1+1} = \cdots = \varepsilon_n = 1 , \end{array} \right. \tag{1}$$

or

case 2:

$$\left\{ \begin{array}{l} \underset{A}{V}{}^\lambda_{\cdot 1} = \varepsilon_A \delta_A^\lambda , \\ \varepsilon_2 = \cdots = \varepsilon_{n-n_1} = 0, \ \varepsilon_1 = \varepsilon_{n-n_1+1} = \cdots = \varepsilon_n = 1 . \end{array} \right. \tag{2}$$

Such a set (S, M) of vectors and matrices will be called a *base of the first order* of $(R^n, \mathfrak{t})$.

The linear space spanned by all matrices V with the property $V^\lambda_{\cdot 1} = 0$ of

$\mathfrak{B}$ is denoted by $\mathfrak{B}'$. A base of the first order is not determined uniquely when S is given, for the matrices of M are only determined to within additive matrices of $\mathfrak{B}'$.

In short a base of the first order is a base (S, M) of $(R^n, \mathfrak{t})$ such that the given vector $\boldsymbol{v}_1$ has components $(1, 0, \cdots, 0)$ and the matrices of M satisfy (1) or (2).

If (S, M) and $(\tilde{S}, \tilde{M})$ where

$$\tilde{S}=AS, \qquad \tilde{\boldsymbol{e}}_\lambda=A^{\alpha}_{\cdot\lambda}\boldsymbol{e}_\alpha$$

are bases of the first order with the same vector $\boldsymbol{v}_1$, then we find

$$A^{\lambda}_{\cdot 1}=\delta^{\lambda}_{1}, \qquad A^{i}_{\cdot x}=0 \tag{3}$$

$$(i=1,\cdots, n-n_1;\ x=n-n_1+1,\cdots, n)$$

in case 1 and

$$A^{\lambda}_{\cdot 1}=\delta^{\lambda}_{1}, \qquad A^{q}_{\cdot x}=0 \tag{4}$$

$$(q=2,\cdots, n-n_1;\ x=n-n_1+1,\cdots, n)$$

in case 2. Conversely this is a sufficient condition that A induce a transformation of the bases of the first order with the vector $\boldsymbol{v}_1(1, 0,\cdots, 0)$. In case 1 $\tilde{M}$ is obtained from M when $\underset{x}{V}$ are replaced with $A^{y}_{\cdot x}A^{-1}\underset{y}{V}A$ and the matrices $\underset{B'}{V}$ in $\mathfrak{B}'$ with $A^{-1}\underset{B'}{V}A$. But in case 2 $\tilde{M}$ is obtained when $\underset{1}{V}, \underset{x}{V}$ are replaced with $A^{-1}\underset{1}{V}A$, $A^{y}_{\cdot x}A^{-1}\underset{y}{V}A-A^{y}_{\cdot x}(A^{-1})^{1}_{\cdot y}A^{-1}\underset{1}{V}A$ respectively and $\underset{B'}{V}$ with $A^{-1}\underset{B'}{V}A$.

Later we shall assume the vector $\boldsymbol{v}_1$ to be such that the number n_1 takes the maximum possible value, but for the present $\boldsymbol{v}_1$ is assumed to be given arbitrarily.

§ 2. Bases of the second order.

Among the bases of the first order we can choose a special one in the following manner.

1) Case 1.

In this case the matrix

$$\overset{1}{\underset{1}{K}}{}^{\lambda}_{\cdot\mu}\overset{d}{=}\delta^{\lambda}_{1}\delta^{1}_{\mu}\ ^{1)}$$

1) In such a formula a superscript indicates rows and a subscript columns, so that for example $K^{\lambda}_{\cdot\mu}$ is the element of the λ-th row and of the μ-th column of the matrix $(K^{\lambda}_{\cdot\mu})$. If there is no possibility of confusion we drop the parentheses so that such an expression often denotes not an individual element but a matrix.

is an element of $\mathfrak{K}$. Hence according to Lemma 1 $[\overset{1}{\underset{1}{K}}, V]\in\mathfrak{V}$ for every $V\in\mathfrak{V}$. We also get $[\overset{1}{\underset{1}{K}},[\overset{1}{\underset{1}{K}}, V]]\in\mathfrak{V}$, and, as we have

$$[\overset{1}{\underset{1}{K}}, V]^{\lambda}_{\cdot\mu}=\delta^{\lambda}_{1}V^{1}_{\cdot\mu}-V^{\lambda}_{\cdot 1}\delta^{1}_{\mu}$$

and

$$[\overset{1}{\underset{1}{K}},[\overset{1}{\underset{1}{K}}, V]]^{\lambda}_{\cdot\mu}=\delta^{\lambda}_{1}V^{1}_{\cdot\mu}+V^{\lambda}_{\cdot 1}\delta^{1}_{\mu}-2V^{1}_{\cdot 1}\delta^{\lambda}_{1}\delta^{1}_{\mu},$$

we find that the matrix $V^{\lambda}_{\cdot 1}\delta^{1}_{\mu}-V^{1}_{\cdot 1}\delta^{\lambda}_{1}\delta^{1}_{\mu}$ [1] is an element of $\mathfrak{V}$. Moreover, as we have $V^{1}_{\cdot 1}=0$, we get

$$V^{\lambda}_{\cdot 1}\delta^{1}_{\mu}\in\mathfrak{V}.$$

Especially, if we take a matrix $\underset{x}{V}$, we find $\delta^{\lambda}_{x}\delta^{1}_{\mu}\in\mathfrak{V}$, as we have $\underset{x}{V}{}^{\lambda}_{\cdot 1}=\delta^{\lambda}_{x}$. Hence the matrix

$$\overset{1}{\underset{x}{V}}{}^{\lambda}_{\cdot\mu}=\delta^{\lambda}_{x}\delta^{1}_{\mu}$$

can play the part of $\underset{x}{V}$ in M.

Again, as we have

$$\overset{i}{\underset{1}{K}}{}^{\lambda}_{\cdot\mu}\overset{d}{=}\delta^{\lambda}_{1}\delta^{i}_{\mu}\in\mathfrak{K},$$

we get

$$[\overset{i}{\underset{1}{K}}, \overset{1}{\underset{x}{V}}]^{\lambda}_{\cdot\mu}=-\delta^{\lambda}_{x}\delta^{i}_{\mu}\in\mathfrak{V}.$$

Hence the $n_1(n-n_1)$ linearly independent matrices

$$\overset{i}{\underset{x}{V}}{}^{\lambda}_{\cdot\mu}=\delta^{\lambda}_{x}\delta^{i}_{\mu} \tag{5}$$

can form a part of a base M of $\mathfrak{V}$.

The subspace of $\mathfrak{V}$ spanned by (5) is denoted by $\mathfrak{V}_1$, and the subspace of $\mathfrak{V}$ spanned by matrices with the property $V^{y}_{\cdot i}=0$ is denoted by $\mathfrak{V}_{\hat{1}}$. Of course the elements of the first column vanish in all matrices of $\mathfrak{V}_{\hat{1}}$. $\mathfrak{V}_1$ and $\mathfrak{V}_{\hat{1}}$ span $\mathfrak{V}$.

If a base M is composed of the matrices (5) and an arbitrary base $M_{\hat{1}}$ of $\mathfrak{V}_{\hat{1}}$, then the (S, M) is called a *base of the second order.*

2) Case 2.

In this case we have

$$\overset{p}{\underset{1}{K}}{}^{\lambda}_{\cdot\mu}\overset{d}{=}\delta^{\lambda}_{1}\delta^{p}_{\mu}\in\mathfrak{K} \qquad (p=2,\cdots,n-n_1)$$

and we get

$$[\overset{p}{\underset{1}{K}}, \underset{x}{V}]^{\lambda}_{\cdot\mu}=\delta^{\lambda}_{1}\delta^{p}_{\alpha}\underset{x}{V}{}^{\alpha}_{\cdot\mu}-\underset{x}{V}{}^{\lambda}_{\cdot\alpha}\delta^{\alpha}_{1}\delta^{p}_{\mu},$$

hence the matrices

(6) $$\overset{p}{\underset{x}{V}}{}^{\lambda}_{\bullet\mu} = -\delta^{\lambda}_{1}\overset{1}{\underset{x}{V}}{}^{p}_{\bullet\mu} + \delta^{\lambda}_{x}\delta^{p}_{\mu}$$

lie in $\mathfrak{V}$ if for example

$$\overset{1}{\underset{x}{V}} = \underset{x}{V}.$$

It is also evident that (6) lie in $\mathfrak{V}'$. But, as we have

$$\overset{p}{\underset{x}{V}}{}^{q}_{\bullet\mu} = 0 \qquad (q=2,\cdots,n-n_1),$$

we obtain the same matrices $\overset{p}{\underset{x}{V}}$ if we put

$$\overset{1}{\underset{x}{V}} = \underset{x}{V} + \lambda^{y}_{xq}\overset{q}{\underset{y}{V}} \qquad (y=n-n_1+1,\cdots,n)$$

in (6). Especially, if we put $\lambda^{y}_{xq} = -\underset{x}{V}{}^{y}_{\bullet q}$, we get

$$\overset{1}{\underset{x}{V}}{}^{y}_{\bullet p} = 0,$$

hence we consider hereafter that $\overset{1}{\underset{x}{V}}$ satisfy

(7) $$\overset{1}{\underset{x}{V}}{}^{\lambda}_{\bullet 1} = \delta^{\lambda}_{x}, \qquad \overset{1}{\underset{x}{V}}{}^{y}_{\bullet p} = 0.$$

We also obtain a matrix $\overset{1}{\underset{1}{V}}$ such that

(8) $$\overset{1}{\underset{1}{V}}{}^{\lambda}_{\bullet 1} = \delta^{\lambda}_{1}, \qquad \overset{1}{\underset{1}{V}}{}^{y}_{\bullet p} = 0$$

by adding to $\underset{1}{V}$ some linear combination of (6).

The subspace of $\mathfrak{V}$ spanned by the $1+n_1(n-n_1)$ linearly independent matrices $\overset{1}{\underset{1}{V}}$, $\overset{i}{\underset{x}{V}}$ obtained above will be called $\mathfrak{V}_1$. The subspace of $\mathfrak{V}$ spanned by matrices V of $\mathfrak{V}'$ such that $V^{y}_{\bullet p}=0$ (hence $V^{y}_{\bullet i}=0$) will be called $\mathfrak{V}_{\hat{1}}$. $\mathfrak{V}_1$ and $\mathfrak{V}_{\hat{1}}$ span $\mathfrak{V}$.

A base (S, M) of the first order is called a *base of the second order* if M is composed of the base of $\mathfrak{V}_1$ mentioned above and an arbitrary base of $\mathfrak{V}_{\hat{1}}$.

In the following only bases of the second order are used.

§ 3. Choice of the vector v_1, (I).

Hitherto the vector v_1 was not specified. Now we consider that v_1 is such that the number $n_1(v_1)$ takes the maximum of its possible values. Then the matrices of the base of $\mathfrak{V}$ can assume simpler forms if $n_1 \leqq n-2$. Since we have two cases,

case 1: $\quad v_1 \notin L(\mathfrak{g}, v_1),$

case 2: $\quad v_1 \in L(\mathfrak{g}, v_1),$

case 1 is studied at first.

Although we have $\boldsymbol{v}_1 \notin L(\mathfrak{g}, \boldsymbol{v}_1)$, we do not know whether we have $\boldsymbol{v} \notin L(\mathfrak{g}, \boldsymbol{v})$ or not for an arbitrary vector $\boldsymbol{v}$. But, as we know that $n_1(\boldsymbol{v}_1)$, or n_1 for short, is not less than $n_1(\boldsymbol{v})$ for any vector $\boldsymbol{v}$, the number of linearly independent vectors in the set of vectors

$$(9) \qquad v^\lambda, \qquad \underset{A}{V}{}^{\lambda}_{\cdot\alpha} v^\alpha$$

is no more than n_1+1.

On the other hand we have

$$\overset{i}{\underset{x}{V}}{}^{\lambda}_{\cdot\alpha} v^\alpha = \delta^\lambda_x v^i ,$$

hence the number of linearly independent vectors in

$$v^\lambda, \quad v^i \delta^\lambda_x, \quad V^{\lambda}_{\cdot\alpha} v^\alpha$$

is no more than n_1+1 for any matrix V in $\mathfrak{V}_{\hat{1}}$. Consequently $V^{\lambda}_{\cdot\alpha} v^\alpha$ is a linear combination of v^λ and δ^λ_x, and in particular we obtain

$$(10) \qquad V^{i}_{\cdot\alpha} v^\alpha \propto v^i ,$$

where $\propto$ means "proportional to".

If $n-n_1=1$, we get no relation.

If $n-n_1 \geqq 2$, we get $V^{i}_{\cdot x}=0$ and $V^{i}_{\cdot j} \propto \delta^i_j$. But since $V^{i}_{\cdot 1}=0$ for any matrix V of $\mathfrak{V}_{\hat{1}}$, we find that $V^{i}_{\cdot j}=0$. Thus we obtain the result,

If $n_1 \leqq n-2$, every matrices V in $\mathfrak{V}_{\hat{1}}$ have the form

$$V^{\lambda}_{\cdot\mu} = \delta^\lambda_y V^{y}_{\cdot x} \delta^x_\mu .$$

If $n_1=n-1$, such deduction fails. But we can divide $\mathfrak{V}_{\hat{1}}$ into two subspaces $\mathfrak{V}_{\hat{1}}'$ and $\mathfrak{V}_{\hat{1}}''$, such that the matrices of $\mathfrak{V}_{\hat{1}}'$ have the form

$$V^{\lambda}_{\cdot\mu} = \delta^\lambda_y V^{y}_{\cdot x} \delta^x_\mu ,$$

while the matrices of $\mathfrak{V}_{\hat{1}}''$ have the form

$$V^{\lambda}_{\cdot\mu} = \delta^\lambda_1 V^{1}_{\cdot x} \delta^x_\mu .$$

These matrices are obtained from matrices V of $\mathfrak{V}_{\hat{1}}$ by considering $V-[\overset{1}{\underset{1}{K}}, V]$ and $[\overset{1}{\underset{1}{K}}, V]$.

§ 4. Choice of the vector v_1, (II).

We now consider the case of $\boldsymbol{v}_1 \in L(\mathfrak{g}, \boldsymbol{v}_1)$ and assume $n-n_1 \geqq 2$.

1) As we have $n_1=n_1(\boldsymbol{v}_1) \geqq n_1(\boldsymbol{v})$, the number of linearly independent vectors among (9) is no more than n_1+1. Since the components of the vectors $\overset{p}{\underset{x}{V}}{}^{\lambda}_{\cdot\alpha}v^{\alpha}$ are for $\lambda=1, q, y$

$$-\overset{1}{\underset{x}{V}}{}^{p}_{\cdot\alpha}v^{\alpha}, \qquad 0, \qquad \delta^{y}_{x}v^{p}$$

by virture of (6), we find that the rank of the matrix of degree n_1+2

$$\begin{array}{lll} & & \overbrace{\qquad\qquad}^{n_1 \text{ columns } (x=n-n_1+1,\cdots,n)} \\ v^{1} & \overset{1}{\underset{z}{V}}{}^{1}_{\cdot\alpha}v^{\alpha} & -\overset{1}{\underset{x}{V}}{}^{p}_{\cdot\alpha}v^{\alpha} \\ v^{q} & \overset{1}{\underset{z}{V}}{}^{q}_{\cdot\alpha}v^{\alpha} & 0 \\ v^{y} & \overset{1}{\underset{z}{V}}{}^{y}_{\cdot\alpha}v^{\alpha} & \delta^{y}_{x}v^{p} \left.\vphantom{V}\right\} \; n_1 \text{ rows } (y=n-n_1+1,\cdots,n), \end{array}$$

where p, q, z are arbitrary within $p, q=2,\cdots, n-n_1$ and $z=n-n_1+1,\cdots, n$, is at most n_1+1. We thus obtain

$$v^{1}v^{p}\overset{1}{\underset{z}{V}}{}^{q}_{\cdot\alpha}v^{\alpha}-v^{p}v^{q}\overset{1}{\underset{z}{V}}{}^{1}_{\cdot\alpha}v^{\alpha}-v^{q}\overset{1}{\underset{t}{V}}{}^{p}_{\cdot\alpha}v^{\alpha}\overset{1}{\underset{z}{V}}{}^{t}_{\cdot\beta}v^{\beta}+\overset{1}{\underset{z}{V}}{}^{q}_{\cdot\alpha}v^{\alpha}\overset{1}{\underset{t}{V}}{}^{p}_{\cdot\beta}v^{\beta}v^{t}=0. \tag{11}$$

As this must be fulfilled for an arbitrary vector v^{λ}, we get by picking up the terms involving $v^{p}v^{q}v^{r}$

$$\overset{1}{\underset{z}{V}}{}^{1}_{\cdot r}=0, \tag{12}$$

for we have (7).

Then (11) becomes

$$\begin{aligned} & v^{1}v^{p}\overset{1}{\underset{z}{V}}{}^{q}_{\cdot r}v^{r}+v^{1}v^{p}\overset{1}{\underset{z}{V}}{}^{q}_{\cdot t}v^{t}-v^{p}v^{q}\overset{1}{\underset{z}{V}}{}^{1}_{\cdot t}v^{t} \\ & -v^{q}(\overset{1}{\underset{t}{V}}{}^{p}_{\cdot r}v^{r}+\overset{1}{\underset{t}{V}}{}^{p}_{\cdot u}v^{u})(\delta^{t}_{z}v^{1}+\overset{1}{\underset{z}{V}}{}^{t}_{\cdot v}v^{v}) \\ & +(\overset{1}{\underset{z}{V}}{}^{q}_{\cdot r}v^{r}+\overset{1}{\underset{z}{V}}{}^{q}_{\cdot u}v^{u})(\overset{1}{\underset{t}{V}}{}^{p}_{\cdot s}v^{s}+\overset{1}{\underset{t}{V}}{}^{p}_{\cdot v}v^{v})v^{t}=0, \end{aligned} \tag{13}$$

and we get from the terms involving v^{1}

$$v^{p}\overset{1}{\underset{z}{V}}{}^{q}_{\cdot r}v^{r}-v^{q}\overset{1}{\underset{z}{V}}{}^{p}_{\cdot r}v^{r}=0,$$

$$v^{p}\overset{1}{\underset{z}{V}}{}^{q}_{\cdot t}v^{t}-v^{q}\overset{1}{\underset{z}{V}}{}^{p}_{\cdot t}v^{t}=0.$$

If $n-n_1=2$, these give no relation. If $n-n_1\leqq 3$, we get

$$\overset{1}{\underset{z}{V}}{}^{q}_{\cdot p}\propto\delta^{q}_{p}, \qquad \overset{1}{\underset{z}{V}}{}^{p}_{\cdot t}=0 \qquad\qquad (n-n_1\geqq 3). \tag{14}$$

In (13)

$$\overset{1}{\underset{z}{V}}{}^{q}_{\cdot u}v^{u}\overset{1}{\underset{t}{V}}{}^{p}_{\cdot v}v^{v}v^{t}$$

is the only one term involving $v^{t}v^{u}v^{v}$. Since this must vanish, we get

$$(\overset{1}{\underset{z}{V}}{}^{q}_{\cdot u}v^u v^z)(\overset{1}{\underset{t}{V}}{}^{p}_{\cdot v}v^v v^t)=0\,,$$

hence

$$\overset{1}{\underset{z}{V}}{}^{q}_{\cdot u}v^u v^z=0\,.$$

Thus we obtain

$$\overset{1}{\underset{x}{V}}{}^{2}_{\cdot y}+\overset{1}{\underset{y}{V}}{}^{2}_{\cdot x}=0 \qquad (n_1=n-2)\,. \tag{15}$$

Now, collecting the terms involving $v^r v^s v^t$ and $v^r v^t v^u$ respectively, we get from (13)

$$-(\delta^p_r\delta^q_s+\delta^p_s\delta^q_r)\overset{1}{\underset{z}{V}}{}^{1}_{\cdot t}-\delta^q_s \overset{1}{\underset{u}{V}}{}^{p}_{\cdot r}\overset{1}{\underset{z}{V}}{}^{u}_{\cdot t}-\delta^q_r\overset{1}{\underset{u}{V}}{}^{p}_{\cdot s}\overset{1}{\underset{z}{V}}{}^{u}_{\cdot t} + \overset{1}{\underset{z}{V}}{}^{q}_{\cdot r}\overset{1}{\underset{t}{V}}{}^{p}_{\cdot s}+\overset{1}{\underset{z}{V}}{}^{q}_{\cdot s}\overset{1}{\underset{t}{V}}{}^{p}_{\cdot r}=0\,, \tag{16}$$

$$-\delta^q_r \overset{1}{\underset{v}{V}}{}^{p}_{\cdot u}\overset{1}{\underset{z}{V}}{}^{v}_{\cdot t}-\delta^q_r\overset{1}{\underset{v}{V}}{}^{p}_{\cdot t}\overset{1}{\underset{z}{V}}{}^{v}_{\cdot u} + \overset{1}{\underset{t}{V}}{}^{p}_{\cdot r}\overset{1}{\underset{z}{V}}{}^{q}_{\cdot u}+\overset{1}{\underset{u}{V}}{}^{p}_{\cdot r}\overset{1}{\underset{z}{V}}{}^{q}_{\cdot t}=0\,. \tag{17}$$

As we have (14) for $n_1\leqq n-3$, we can put

$$\overset{1}{\underset{z}{V}}{}^{q}_{\cdot p}=\underset{z}{v}\delta^q_p$$

for $n_1\leqq n-2$, and (16) becomes

$$\overset{1}{\underset{z}{V}}{}^{1}_{\cdot t}+\underset{u}{v}\overset{1}{\underset{z}{V}}{}^{u}_{\cdot t}-\underset{t}{v}\underset{z}{v}=0\,. \tag{18}$$

On the other hand (17) becomes

$$\overset{1}{\underset{v}{V}}{}^{2}_{\cdot u}\overset{1}{\underset{z}{V}}{}^{v}_{\cdot t}+\overset{1}{\underset{v}{V}}{}^{2}_{\cdot t}\overset{1}{\underset{z}{V}}{}^{v}_{\cdot u}-\underset{t}{v}\overset{1}{\underset{z}{V}}{}^{2}_{\cdot u}-\underset{u}{v}\overset{1}{\underset{z}{V}}{}^{2}_{\cdot t}=0 \tag{19}$$

for $n_1=n-2$, but for $n_1\leqq n-3$ (17) is fulfilled already by virtue of (14).

The results obtained, (7), (12), (14), (15), (18), (19), are arranged as follows

$$\begin{cases}
\overset{1}{\underset{z}{V}}{}^{\lambda}_{\cdot 1}=\delta^\lambda_z\,, \quad \overset{1}{\underset{z}{V}}{}^{\lambda}_{\cdot p}=\underset{z}{v}\delta^\lambda_p & (n_1\leqq n-2)\,,\\
\overset{1}{\underset{z}{V}}{}^{1}_{\cdot x}=\underset{x}{v}\underset{z}{v}-\underset{t}{v}\overset{1}{\underset{z}{V}}{}^{t}_{\cdot x} & (n_1\leqq n-2)\,,\\
\overset{1}{\underset{t}{V}}{}^{2}_{\cdot x}\overset{1}{\underset{z}{V}}{}^{t}_{\cdot y}+\overset{1}{\underset{t}{V}}{}^{2}_{\cdot y}\overset{1}{\underset{z}{V}}{}^{t}_{\cdot x}=\underset{x}{v}\overset{1}{\underset{z}{V}}{}^{2}_{\cdot y}+\underset{y}{v}\overset{1}{\underset{z}{V}}{}^{2}_{\cdot x} & (n_1=n-2)\,,\\
\overset{1}{\underset{x}{V}}{}^{2}_{\cdot y}+\overset{1}{\underset{y}{V}}{}^{2}_{\cdot x}=0 & (n_1=n-2)\,,\\
\overset{1}{\underset{x}{V}}{}^{p}_{\cdot \mu}=\underset{x}{v}\delta^p_\mu & (n_1\leqq n-3)\,.
\end{cases} \tag{20}$$

2) Let us study the matrix $\overset{1}{\underset{1}{V}}$. We obtain the matrix of degree n_1+2

$$\begin{array}{ccc} & & \overbrace{\qquad\qquad}^{n_1\text{ columns}} \\ v^1 & \overset{1}{\underset{1}{V}}{}^{1}_{\cdot\alpha}v^\alpha & -\overset{1}{\underset{x}{V}}{}^{p}_{\cdot\alpha}v^\alpha \\ v^q & \overset{1}{\underset{1}{V}}{}^{q}_{\cdot\alpha}v^\alpha & 0 \\ v^y & \overset{1}{\underset{1}{V}}{}^{y}_{\cdot\alpha}v^\alpha & \delta^y_x v^p \end{array} \Big\}\ n_1 \text{ rows}$$

and, as the determinant must vanish, we get

$$v^1v^p\overset{1}{\underset{1}{V}}{}^{q}_{\cdot\alpha}v^\alpha-v^pv^q\overset{1}{\underset{1}{V}}{}^{1}_{\cdot\alpha}v^\alpha-v^q\overset{1}{\underset{t}{V}}{}^{p}_{\cdot\alpha}v^\alpha\overset{1}{\underset{1}{V}}{}^{t}_{\cdot\beta}v^\beta+\overset{1}{\underset{1}{V}}{}^{q}_{\cdot\alpha}v^\alpha\overset{1}{\underset{t}{V}}{}^{p}_{\cdot\beta}v^\beta v^t=0\,,$$

which becomes by virtue of (8) and (14) or (15)

$$\begin{aligned} & v^1v^p\overset{1}{\underset{1}{V}}{}^{q}_{\cdot r}v^r+v^1v^p\overset{1}{\underset{1}{V}}{}^{q}_{\cdot t}v^t \\ & -v^pv^qv^1-v^pv^q\overset{1}{\underset{1}{V}}{}^{1}_{\cdot r}v^r-v^pv^q\overset{1}{\underset{1}{V}}{}^{1}_{\cdot t}v^t \\ & -v^q(\overset{1}{\underset{t}{V}}{}^{p}_{\cdot r}v^r+\overset{1}{\underset{t}{V}}{}^{p}_{\cdot u}v^u)\overset{1}{\underset{1}{V}}{}^{t}_{\cdot v}v^v \\ & +(\overset{1}{\underset{1}{V}}{}^{q}_{\cdot r}v^r+\overset{1}{\underset{1}{V}}{}^{q}_{\cdot u}v^u)\overset{1}{\underset{t}{V}}{}^{p}_{\cdot s}v^sv^t=0\,. \end{aligned}$$

From the terms involving v^1 we get

$$\overset{1}{\underset{1}{V}}{}^{q}_{\cdot r}=\delta^q_r\,,\qquad \overset{1}{\underset{1}{V}}{}^{q}_{\cdot t}=0\,,$$

and from the only one term involving $v^pv^qv^r$ we get

$$\overset{1}{\underset{1}{V}}{}^{1}_{\cdot r}=0\,.$$

Then making use of (20) and the result just obtained, we get by collecting the terms involving $v^pv^qv^t$

$$-\overset{1}{\underset{1}{V}}{}^{1}_{\cdot t}-v_y\overset{1}{\underset{1}{V}}{}^{y}_{\cdot t}+v_t=0\,.$$

The remaining terms are

$$-v^q\overset{1}{\underset{t}{V}}{}^{p}_{\cdot u}v^u\overset{1}{\underset{1}{V}}{}^{t}_{\cdot v}v^v$$

from which we get

$$\overset{1}{\underset{t}{V}}{}^{p}_{\cdot x}\overset{1}{\underset{1}{V}}{}^{t}_{\cdot y}+\overset{1}{\underset{t}{V}}{}^{p}_{\cdot y}\overset{1}{\underset{1}{V}}{}^{t}_{\cdot x}=0\,.$$

This is already fulfilled if $n_1\leqq n-3$ as we have (14).

The results obtained are arranged as follows.

$$
(21)\qquad
\begin{cases}
\overset{1}{\underset{1}{V}}{}^{\lambda}{}_{\cdot 1}=\delta^{\lambda}_{1}, \quad \overset{1}{\underset{1}{V}}{}^{\lambda}{}_{\cdot p}=\delta^{\lambda}_{p} & (n_1\leqq n-2),\\
\overset{1}{\underset{1}{V}}{}^{q}{}_{\cdot \mu}=\delta^{q}_{\mu} & (n_1\leqq n-2),\\
\overset{1}{\underset{1}{V}}{}^{1}{}_{\cdot x}=\underset{x}{v}-\underset{t}{v}\,\overset{1}{\underset{1}{V}}{}^{t}{}_{\cdot x} & (n_1\leqq n-2),\\
\overset{1}{\underset{t}{V}}{}^{2}{}_{\cdot x}\overset{1}{\underset{1}{V}}{}^{t}{}_{\cdot y}+\overset{1}{\underset{t}{V}}{}^{2}{}_{\cdot y}\overset{1}{\underset{1}{V}}{}^{t}{}_{\cdot x}=0 & (n_1=n-2).
\end{cases}
$$

3) If we effect a transformation of the base of R^n,

$$\tilde{S}=AS, \qquad \tilde{\boldsymbol{e}}_{\lambda}=A^{\alpha}_{\cdot\lambda}\boldsymbol{e}_{\alpha}$$

where

$$A^{\lambda}_{\cdot\mu}=\delta^{\lambda}_{\mu}-\delta^{\lambda}_{1}\underset{x}{v}\,\delta^{x}_{\mu},$$

$$(A^{-1})^{\lambda}_{\cdot\mu}=\delta^{\lambda}_{\mu}+\delta^{\lambda}_{1}\underset{x}{v}\,\delta^{x}_{\mu},$$

then the components of $\boldsymbol{v}_1$ remain unchanged, for $\tilde{\boldsymbol{e}}_1=\boldsymbol{e}_1$. Each matrix V of $\mathfrak{V}$ is transformed into

$$\tilde{V}=A^{-1}VA$$

and we write

$$\tilde{\mathfrak{V}}=A^{-1}\mathfrak{V}A.$$

Then we can understand easily that the matrices

$$\overset{1}{\underset{1}{V}}{}'=A^{-1}\overset{1}{\underset{1}{V}}A, \quad \overset{1}{\underset{x}{V}}{}'=A^{-1}\overset{1}{\underset{x}{V}}A-\underset{x}{v}A^{-1}\overset{1}{\underset{1}{V}}A$$

of $\tilde{\mathfrak{V}}$ are such that

$$(7)' \qquad \overset{1}{\underset{1}{V}}{}'^{\lambda}{}_{\cdot 1}=\delta^{\lambda}_{1}, \qquad \overset{1}{\underset{1}{V}}{}'^{y}{}_{\cdot p}=0,$$

$$(8)' \qquad \overset{1}{\underset{x}{V}}{}'^{\lambda}{}_{\cdot 1}=\delta^{\lambda}_{x}, \qquad \overset{1}{\underset{x}{V}}{}'^{y}{}_{\cdot p}=0.$$

Consequently we get relations quite similar to (20), (21), but, as we have

$$\overset{1}{\underset{z}{V}}{}'^{q}{}_{\cdot p}=0$$

now, the quantities $\underset{z}{v}'$ which correspond to $\underset{z}{v}$ of (20), (21) must vanish.

This shows that, if we take a suitable base S in R^n, then we get (20), (21) with vanishing $\underset{z}{v}$. Thus we obtain

If $n_1\leqq n-3$,

$$
(22)\qquad
\begin{cases}
\overset{1}{\underset{z}{V}}{}^{\lambda}{}_{\cdot 1}=\delta^{\lambda}_{z}, \quad \overset{1}{\underset{z}{V}}{}^{\lambda}{}_{\cdot p}=0, \quad \overset{1}{\underset{z}{V}}{}^{i}{}_{\cdot \mu}=0,\\
\overset{1}{\underset{1}{V}}{}^{\lambda}{}_{\cdot j}=\delta^{\lambda}_{j}, \quad \overset{1}{\underset{1}{V}}{}^{i}{}_{\cdot \mu}=\delta^{i}_{\mu}.
\end{cases}
$$

If $n_1=n-2$,

$$\left\{\begin{aligned} &\overset{1}{\underset{z}{V}}{}^{\lambda}_{\cdot 1}=\delta^{\lambda}_{z}, \qquad \overset{1}{\underset{z}{V}}{}^{\lambda}_{\cdot 2}=0, \qquad \overset{1}{\underset{z}{V}}{}^{1}_{\cdot \mu}=0, \\ &\overset{1}{\underset{1}{V}}{}^{\lambda}_{\cdot j}=\delta^{\lambda}_{j}, \qquad \overset{1}{\underset{1}{V}}{}^{i}_{\cdot \mu}=\delta^{i}_{\mu}, \end{aligned}\right. \tag{23}$$

$$\left\{\begin{aligned} &\overset{1}{\underset{t}{V}}{}^{2}_{\cdot x}\overset{1}{\underset{z}{V}}{}^{t}_{\cdot y}+\overset{1}{\underset{t}{V}}{}^{2}_{\cdot y}\overset{1}{\underset{z}{V}}{}^{t}_{\cdot x}=0, \\ &\overset{1}{\underset{t}{V}}{}^{2}_{\cdot x}\overset{1}{\underset{1}{V}}{}^{t}_{\cdot y}+\overset{1}{\underset{t}{V}}{}^{2}_{\cdot y}\overset{1}{\underset{1}{V}}{}^{t}_{\cdot x}=0, \\ &\quad \overset{1}{\underset{x}{V}}{}^{2}_{\cdot y}+\overset{1}{\underset{y}{V}}{}^{2}_{\cdot x}=0. \end{aligned}\right. \tag{24}$$

In the following we use only such bases S.

4) Let us consider the matrices V of $\mathfrak{V}_{\hat{1}}$.

Then we get

$$v^{1}v^{p}V^{q}_{\cdot\alpha}v^{\alpha}-v^{p}v^{q}V^{1}_{\cdot\alpha}v^{\alpha}-v^{q}\overset{1}{\underset{t}{V}}{}^{p}_{\cdot\alpha}v^{\alpha}V^{t}_{\cdot\beta}v^{\beta}+V^{q}_{\cdot\alpha}v^{\alpha}\overset{1}{\underset{t}{V}}{}^{p}_{\cdot\beta}v^{\beta}v^{t}=0,$$

and, as we have $V^{\lambda}_{\cdot 1}=0$, $v^{1}v^{p}V^{q}_{\cdot\alpha}v^{\alpha}$ are the only terms involving v^{1}. Hence we get $V^{q}_{\cdot\mu}=0$, and from the remaining terms we obtain

$$-v^{p}v^{q}V^{1}_{\cdot r}v^{r}-v^{p}v^{q}V^{1}_{\cdot t}v^{t}-v^{q}\overset{1}{\underset{t}{V}}{}^{p}_{\cdot u}v^{u}(V^{t}_{\cdot r}v^{r}+V^{t}_{\cdot v}v^{v})=0$$

by virtue of (22) or (23). Since we have $V^{t}_{\cdot r}=0$ as stated in § 2, we get, if $n_1=n-2$,

$$V^{1}_{\cdot r}=0, \quad \overset{1}{\underset{t}{V}}{}^{2}_{\cdot x}V^{t}_{\cdot y}+\overset{1}{\underset{t}{V}}{}^{2}_{\cdot y}V^{t}_{\cdot x}=0, \quad V^{1}_{\cdot x}=0$$

and, if $n_1\leqq n-3$,

$$V^{1}_{\cdot r}=0, \qquad V^{1}_{\cdot x}=0.$$

We can summarize the result as

$$V^{\lambda}_{\cdot\mu}=\delta^{\lambda}_{y}V^{y}_{\cdot x}\delta^{x}_{\mu} \qquad (n_1\leqq n-2), \tag{25}$$

$$\overset{1}{\underset{t}{V}}{}^{2}_{\cdot x}V^{t}_{\cdot y}+\overset{1}{\underset{t}{V}}{}^{2}_{\cdot y}V^{t}_{\cdot x}=0 \qquad (n_1=n-2). \tag{26}$$

5) Thus we find that for all matrices V in $\mathfrak{V}$ we have $V^{1}_{\cdot 2}=0$ as long as $n_1\leqq n-2$. Hence the matrix

$$K^{\lambda}_{\cdot\mu}=\delta^{\lambda}_{2}\delta^{1}_{\mu}$$

lies in $\mathfrak{K}$, and, as we get

$$[K, \overset{2}{\underset{x}{V}}]^{\lambda}_{\cdot\mu}=\delta^{\lambda}_{2}\overset{2}{\underset{x}{V}}{}^{1}_{\cdot\mu}-\overset{2}{\underset{x}{V}}{}^{\lambda}_{\cdot 2}\delta^{1}_{\mu}=-\delta^{\lambda}_{2}\overset{1}{\underset{x}{V}}{}^{2}_{\cdot\mu}+(\delta^{\lambda}_{1}\overset{1}{\underset{x}{V}}{}^{2}_{\cdot 2}-\delta^{\lambda}_{x})\delta^{1}_{\mu}$$

and

$$\overset{1}{\underset{x}{V}}{}^{2}_{\cdot 2}=0,$$

the matrix

$$\delta^{\lambda}_{x}\delta^{1}_{\mu}+\delta^{\lambda}_{2}\overset{1}{\underset{x}{V}}{}^{2}_{\cdot\mu}$$

lies in $\mathfrak{B}$. We can choose this for $\overset{1}{\underset{x}{V}}$,

$$\overset{1}{\underset{x}{V}}{}^{\lambda}_{\cdot\mu}=\delta^{\lambda}_{x}\delta^{1}_{\mu}+\delta^{\lambda}_{2}\overset{1}{\underset{x}{V}}{}^{2}_{\cdot\mu}.$$

If $n_1\leqq n-3$, we get

$$\overset{1}{\underset{x}{V}}{}^{\lambda}_{\cdot\mu}=\delta^{\lambda}_{x}\delta^{1}_{\mu}.$$

6) If $n_1=n-1$, such deduction fails.

§ 5. Bases of the third order, resumé of the first step.

In foregoing paragraphs we obtained a base $S(\boldsymbol{v}_1, \boldsymbol{e}_2,\cdots, \boldsymbol{e}_n)$ of R^n such that $n_1(n-n_1)$ or $n_1(n-n_1)+1$ matrices with special property compose a base M_1 of $\mathfrak{B}_1$ and $m-n_1(n-n_1)$ or $m-n_1(n-n_1)-1$ matrices with the property

$$V^{\lambda}_{\cdot\mu}=\delta^{\lambda}_{y}V^{y}_{\cdot x}\delta^{x}_{\mu}$$

compose a base $M_{\hat{1}}$ of $\mathfrak{B}_{\hat{1}}$, by assuming that $\boldsymbol{v}_1$ satisfies the inequality $n_1(\boldsymbol{v}_1)\geqq n_1(\boldsymbol{v})$ for all vectors $\boldsymbol{v}$ of R^n. The only one exceptional case is the case of $n_1=n-1$.

The results obtained are resumed as follows.

We can find a base (S, M) such that M is composed of the base M_1 of $\mathfrak{B}_1$ and the base $M_{\hat{1}}$ of $\mathfrak{B}_{\hat{1}}$ mentioned below.

(i) case 1, $n_1=n-1$

$$M_1\ :\ \overset{1}{\underset{x}{V}}{}^{\lambda}_{\cdot\mu}=\delta^{\lambda}_{x}\delta^{1}_{\mu},$$

$M_{\hat{1}}$:

- $M_{\hat{1}}'$: some matrices V with the property $$V^{\lambda}_{\cdot\mu}=\delta^{\lambda}_{y}V^{y}_{\cdot x}\delta^{x}_{\mu},$$
- $M_{\hat{1}}''$: some matrices V with the property $$V^{\lambda}_{\cdot\mu}=\delta^{\lambda}_{1}V^{1}_{\cdot x}\delta^{x}_{\mu},$$

(ii) case 1, $n_1\leqq n-2$

$$M_1:\ \overset{i}{\underset{x}{V}}{}^{\lambda}_{\cdot\mu}=\delta^{\lambda}_{x}\delta^{i}_{\mu},$$

$M_{\hat{1}}$: some matrices V with the property

$$V^{\lambda}_{\cdot\mu}=\delta^{\lambda}_{y}V^{y}_{\cdot x}\delta^{x}_{\mu},$$

(iii) case 2, $n_1=n-1$

M_1: n matrices $\overset{1}{\underset{\kappa}{V}}$ with the property

$$\overset{1}{\underset{\kappa}{V}}{}^{\lambda}_{\cdot 1}=\delta^{\lambda}_{\kappa},$$

$M_{\hat{1}}$: some matrices V with the property

$$V^{\lambda}_{\cdot 1}=0,$$

(iv) case 2, $n_1=n-2$

M_1: a matrix $\overset{1}{\underset{1}{V}}$ with the property

$$\overset{1}{\underset{1}{V}}{}^{\lambda}_{\cdot j}=\delta^{\lambda}_{j}, \qquad \overset{1}{\underset{1}{V}}{}^{i}_{\cdot \mu}=\delta^{i}_{\mu},$$

n_1 matrices $\overset{1}{\underset{x}{V}}$ with the property

$$\overset{1}{\underset{x}{V}}{}^{\lambda}_{\cdot j}=\delta^{\lambda}_{x}\delta^{1}_{j}, \qquad \overset{1}{\underset{x}{V}}{}^{\lambda}_{\cdot y}=\delta^{\lambda}_{2}\overset{1}{\underset{x}{V}}{}^{2}_{\cdot y},$$

n_1 matrices $\overset{2}{\underset{x}{V}}$ with the property

$$\overset{2}{\underset{x}{V}}{}^{\lambda}_{\cdot j}=\delta^{\lambda}_{x}\delta^{2}_{j}, \qquad \overset{2}{\underset{x}{V}}{}^{\lambda}_{\cdot y}=-\delta^{\lambda}_{1}\overset{1}{\underset{x}{V}}{}^{2}_{\cdot y},$$

$M_{\hat{1}}$: some matrices V with the property

$$V^{\lambda}_{\cdot \mu}=\delta^{\lambda}_{y}V^{y}_{\cdot x}\delta^{x}_{\mu},$$

where

$$\overset{1}{\underset{x}{V}}{}^{2}_{\cdot y}+\overset{1}{\underset{y}{V}}{}^{2}_{\cdot x}=0,$$

$$\overset{1}{\underset{t}{V}}{}^{2}_{\cdot x}\overset{1}{\underset{1}{V}}{}^{t}_{\cdot y}+\overset{1}{\underset{t}{V}}{}^{2}_{\cdot y}\overset{1}{\underset{1}{V}}{}^{t}_{\cdot x}=0,$$

$$\overset{1}{\underset{t}{V}}{}^{2}_{\cdot x}V^{t}_{\cdot y}+\overset{1}{\underset{t}{V}}{}^{2}_{\cdot y}V^{t}_{\cdot x}=0,$$

(v) case 2, $n_1\leqq n-3$

M_1: a matrix $\overset{1}{\underset{1}{V}}$ with the property

$$\overset{1}{\underset{1}{V}}{}^{\lambda}_{\cdot j}=\delta^{\lambda}_{j}, \qquad \overset{1}{\underset{1}{V}}{}^{i}_{\cdot \mu}=\delta^{i}_{\mu},$$

matrices $\overset{i}{\underset{x}{V}}{}^{\lambda}_{\cdot \mu}=\delta^{\lambda}_{x}\delta^{i}_{\mu}$,

$M_{\hat{1}}$: some matrices V with the property

$$V^{\lambda}_{\cdot \mu}=\delta^{\lambda}_{y}V^{y}_{\cdot x}\delta^{x}_{\mu}.$$

Now let us assume that we have bases M_1 and $M_{\hat{1}}$ with the property just mentioned without assuming $n_1(\boldsymbol{v}_1)\geqq n_1(\boldsymbol{v})$. The matrices obtained by arranging the vectors v^{λ} and $\underset{A}{V}{}^{\lambda}_{\cdot\alpha}v^{\alpha}$ in columns are as follows where n_1 denotes $n_1(\boldsymbol{v}_1)$.

(i)′

$$
\begin{array}{cccc l}
 & \overbrace{}^{n-1\ \text{columns}} & & & \\
v^1 & 0 & 0 & \underset{B''}{V}{}^{1}_{\cdot t}v^t & \\
v^y & \delta^y_x v^1 & \underset{B'}{V}{}^{y}_{\cdot t}v^t & 0 & \Big\}\ n-1\ \text{rows}
\end{array}
$$

(ii)′

$$
\begin{array}{ccc l}
 & \overbrace{}^{n_1(n-n_1)\ \text{columns}} & & \\
v^k & 0 & 0 & \Big\}\ n-n_1\ \text{rows} \\
v^y & \delta^y_x v^i & \underset{B'}{V}{}^{y}_{\cdot t}v^t & \Big\}\ n_1\ \text{rows}
\end{array}
$$

(iv)′

$$
\begin{array}{cccccl}
 & & \overbrace{}^{n-2\ \text{columns}} & \overbrace{}^{n-2\ \text{columns}} & & \\
v^1 & v^1 & 0 & -\overset{1}{\underset{v}{V}}{}^{2}_{\cdot t}v^t & 0 & \\
v^2 & v^2 & \overset{1}{\underset{u}{V}}{}^{2}_{\cdot t}v^t & 0 & 0 & \\
v^y & \overset{1}{\underset{1}{V}}{}^{y}_{\cdot t}v^t & \delta^y_u v^1 & \delta^y_v v^2 & \underset{B'}{V}{}^{y}_{\cdot t}v^t & \Big\}\ n-2\ \text{rows}
\end{array}
$$

(v)′

$$
\begin{array}{ccccl}
 & & \overbrace{}^{n_1(n-n_1)\ \text{columns}} & & \\
v^1 & v^1 & 0 & 0 & \\
v^q & v^q & 0 & 0 & \Big\}\ n-n_1-1\ \text{rows} \\
v^y & \overset{1}{\underset{1}{V}}{}^{y}_{\cdot t}v^t & \delta^y_x v^i & \underset{B'}{V}{}^{y}_{\cdot t}v^t & \Big\}\ n_1\ \text{rows}
\end{array}
$$

The matrix (iii)′ for case 2, $n_1=n-1$ is not described as it is needless.

Then we find that the rank R is at most n_1+1 for any of them. For (i)′ and (iii)′ this is due to $n_1+1=n$. For (ii)′ and (v)′ this is found easily by inspection. For (iv)′ this is proved as follows.

If we put

$$a_1=v^2, \qquad a_2=-v^1, \qquad a_x=\overset{1}{\underset{x}{V}}{}^{2}_{\cdot t}v^t,$$

then we can verify that the column vectors v^λ, $\underset{A}{V}{}^{\lambda}_{\cdot\alpha}v^\alpha$ of (iv)′ satisfy

$$a_\beta v^\beta=0, \qquad a_\beta \underset{A}{V}{}^{\beta}_{\cdot\alpha}v^\alpha=0$$

by making use of (24) and (26). If the vector a_μ is not zero, this proves that $R\leqq n-1$, hence $R\leqq n_1+1$. If $a_\mu=0$, the matrix (iv)′ becomes

$$
\begin{array}{ccccc}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
v^y & \overset{1}{\underset{1}{V}}{}^{y}_{\cdot t}v^t & 0 & 0 & \underset{B'}{V}{}^{y}_{\cdot t}v^t ,
\end{array}
$$

hence we get $R<n_1+1$.

Since $n_1(\boldsymbol{v})=R-1$, we get the following result.

If a base (S, M) *of the first order is such that* M *is composed of* M_1 *and* $M_{\hat{1}}$ *formed by matrices stated in* (i), (ii), (iii), (iv) *or* (v), *then we get* $n_1(\boldsymbol{v}_1)\geqq n_1(\boldsymbol{v})$.

Such a base (S, M) will be called a *base of the third order.* Decomposition of $\mathfrak{V}$ into $\mathfrak{V}_1$ and $\mathfrak{V}_{\hat{1}}$ now becomes significant.

If $n_1=n-1$, it may happen that we get, for some $\boldsymbol{v}$ satisfying $n_1(\boldsymbol{v})=n-1$, $\boldsymbol{v}\in L(\mathfrak{g}, \boldsymbol{v})$ when $\boldsymbol{v}_1\notin L(\mathfrak{g}, \boldsymbol{v}_1)$. This is easily found from (i)$'$, and this means that for $n_1=n-1$ it is not an intrinsic property of $\mathfrak{g}$ whether we get case 1 or case 2. On the contrary this is an intrinsic property of $\mathfrak{g}$ if $n_1\leqq n-2$, for we find from (ii)$'$ that we get $\boldsymbol{v}\notin L(\mathfrak{g}, \boldsymbol{v})$ as long as we have $n_1(\boldsymbol{v})=n_1$.

Thus we obtain the

LEMMA 2. *When a subalgebra* $\mathfrak{g}$ *of a general linear Lie algebra* $\mathfrak{gl}(R^n)$ *is given, we can always find a base* (S, M) *of the third order which is characterized by* (i), (ii), (iii), (iv) *or* (v). *Especially, if we get* (ii), (iv), *or* (v), *this is an intrinsic property of* $\mathfrak{g}$.

§ 6. Bases of order 4, complete decomposition of $\mathfrak{V}$.

1) Let us begin the second step.

We have found that, if (S, M) is a base of the third order, then the elements in the first $n-n_1$ columns of the matrices in $\mathfrak{V}_{\hat{1}}$ are all zero, while any non-zero linear combination of matrices of M_1 has at least one non-zero element in the first $n-n_1$ columns. As we only use bases of the third order hereafter, we can consider that this property characterizes $\mathfrak{V}_{\hat{1}}$.

Now, if we take a transformation of S

$$(27)\qquad \begin{cases} \tilde{\boldsymbol{e}}_\lambda=A^{\alpha}_{\cdot\lambda}\boldsymbol{e}_\alpha\,, \\ A^{\lambda}_{\cdot j}=\delta^{\lambda}_{j}\,, \qquad A^{i}_{\cdot\mu}=\delta^{i}_{\mu}\,, \end{cases}$$

the elements in the first $n-n_1$ columns of the matrices in $A^{-1}\mathfrak{V}_{\hat{1}}A$ are all zero. On the other hand, the matrices

$$(28)\qquad \tilde{M}_1(A^{-1}\overset{1}{\underset{1}{V}}A,\ A^{z}_{\cdot x}A^{-1}\overset{i}{\underset{z}{V}}A)$$

in $A^{-1}\mathfrak{V}_1A$ have the same form as the matrices $M_1(\overset{1}{\underset{1}{V}}, \overset{i}{\underset{x}{V}})$ in $\mathfrak{V}_1$, hence (28) plays the same role in $A^{-1}\mathfrak{V}_1A$ as $M_1(\overset{1}{\underset{1}{V}}, \overset{i}{\underset{x}{V}})$ does in $\mathfrak{V}_1$. Consequently a base of the third order $(\tilde{S}, \tilde{M})$ is obtained for $\tilde{S}=AS$ when $\tilde{M}$ is constructed of (28) and some base $\tilde{M}_{\hat{1}}$ of $A^{-1}\mathfrak{V}_{\hat{1}}A$.

If we take a suitable transformation (27) and choose $\tilde{M}_{\hat{1}}$ suitably, the matrices[2] of degree n_1 constructed of the elements $V^{y}_{\cdot x}$ of the matrices $(V^{\lambda}_{\cdot\mu})$ of $\tilde{M}_{\hat{1}}$ take special forms. This implies that, if we take a suitable base (S, M) of the third order, the n_1-submatrices of $M_{\hat{1}}$ take special forms. This is obtained in a quite similar manner as in the first step with the use of the following lemma.

Lemma 3. *If a matrix* $(K^{y}_{\cdot x})$ *of degree* n_1 *satisfies*

$$K^{y}_{\cdot x} V^{x}_{\cdot y}=0$$

for all matrices V *of* $\mathfrak{V}_{\hat{1}}$, *then there is a matrix* $(K^{\lambda}_{\cdot\mu})$ *of* $\mathfrak{K}$ *such that its* n_1-*submatrix is the matrix* $(K^{y}_{\cdot x})$ *given above and moreover such that*

$$K^{y}_{\cdot j}=0 .$$

Proof. We only need to find a matrix

$$K^{\lambda}_{\cdot\mu}=\delta^{\lambda}_{y} K^{y}_{\cdot x} \delta^{x}_{\mu}+\delta^{\lambda}_{i} K^{i}_{\cdot\mu}$$

such that $K^{\lambda}_{\cdot\mu} V^{\mu}_{\cdot\lambda}=0$ for all matrices V of $\mathfrak{V}_{\hat{1}}$ and for

$$V=\overset{1}{\underset{1}{V}},\ \overset{i}{\underset{x}{V}} .$$

If V lines in $\mathfrak{V}_{\hat{1}}$, we get

$$K^{\lambda}_{\cdot\mu} V^{\mu}_{\cdot\lambda}=K^{y}_{\cdot\mu} V^{\mu}_{\cdot y}=K^{y}_{\cdot x} V^{x}_{\cdot y}=0 .$$

If $V=\overset{k}{\underset{z}{V}}$, we get

$$\begin{aligned} K^{\lambda}_{\cdot\mu} \overset{k}{\underset{z}{V}}{}^{\mu}_{\cdot\lambda} &=\delta^{\lambda}_{y} K^{y}_{\cdot x} \delta^{x}_{\mu} \overset{k}{\underset{z}{V}}{}^{\mu}_{\cdot\lambda}+\delta^{\lambda}_{i} K^{i}_{\cdot\mu} \overset{k}{\underset{z}{V}}{}^{\mu}_{\cdot\lambda} \\ &=K^{y}_{\cdot x} \overset{k}{\underset{z}{V}}{}^{x}_{\cdot y}+K^{i}_{\cdot\mu} \overset{k}{\underset{z}{V}}{}^{\mu}_{\cdot i} \\ &=K^{y}_{\cdot x} \overset{k}{\underset{z}{V}}{}^{x}_{\cdot y}+K^{k}_{\cdot z} , \end{aligned}$$

while, if $V=\overset{1}{\underset{1}{V}}$, we get

$$\begin{aligned} K^{\lambda}_{\cdot\mu} \overset{1}{\underset{1}{V}}{}^{\mu}_{\cdot\lambda} &=K^{y}_{\cdot x} \overset{1}{\underset{1}{V}}{}^{x}_{\cdot y}+K^{i}_{\cdot\mu} \overset{1}{\underset{1}{V}}{}^{\mu}_{\cdot i} \\ &=K^{y}_{\cdot x} \overset{1}{\underset{1}{V}}{}^{x}_{\cdot y}+K^{i}_{\cdot i} . \end{aligned}$$

Hence $K \in \mathfrak{K}$ is satisfied if we put

in case 1

$$K^{k}_{\cdot z}=-K^{y}_{\cdot x} \overset{k}{\underset{z}{V}}{}^{x}_{\cdot y}=0 , \tag{29}$$

---

2) In the following such matrices are called n_1-submatrices for short.

in case 2

$$(30)\qquad \begin{cases} K^{k}_{\cdot z} = -K^{y}_{\cdot x} \overset{k}{\underset{z}{V}}{}^{x}_{\cdot y}\,, \\ K^{i}_{\cdot i} = -K^{y}_{\cdot x} \overset{1}{\underset{1}{V}}{}^{x}_{\cdot y}\,. \end{cases}$$

This proves the lemma.

Conversely, if a matrix K of $\mathfrak{K}$ satisfies $K^{y}_{\cdot j}=0$, then it satisfies $K^{y}_{\cdot x} V^{x}_{\cdot y} =0$ for all V in $\mathfrak{V}_{\hat{1}}$.

Consequently the subspace $\mathfrak{K}_1$ of $\mathfrak{K}$ spanned by such matrices $(K^{\lambda}_{\cdot\mu})$ can be considered to be an isomorphic representation of a subalgebra $\mathfrak{g}_1$ of the Lie algebra $\mathfrak{g}$. The n_1-submatrices $(K^{y}_{\cdot x})$ of the matrices $(K^{\lambda}_{\cdot\mu})$ in $\mathfrak{K}_1$ span a linear subspace $\mathfrak{K}_{(1)}$ in an n_1^2-dimensional linear space spanned by all matrices of degree n_1. $\mathfrak{K}_{(1)}$ can be considered to be an isomorphic representation of a Lie algebra $\mathfrak{g}_{(1)}$ which is isomorphic or homomorphic to $\mathfrak{g}_1$. As $\mathfrak{K}$ determined $\mathfrak{V}$, so $\mathfrak{K}_{(1)}$ determines $\mathfrak{V}_{(1)}$. But any matrix of $\mathfrak{V}_{(1)}$ is the n_1-submatrix $(V^{y}_{\cdot x})$ of a matrix $(V^{\lambda}_{\cdot\mu})$ in $\mathfrak{V}_{\hat{1}}$, and the n_1-submatrix of any matrix V in $\mathfrak{V}_{\hat{1}}$ is a matrix of $\mathfrak{V}_{(1)}$. This is easily understood as $\mathfrak{K}_{(1)}$ is spanned by the matrices $(K^{y}_{\cdot x})$ satisfying $K^{y}_{\cdot x} V^{x}_{\cdot y}=0$ for all matrices V in $\mathfrak{V}_{\hat{1}}$, and the n_1-submatrices of matrices in $\mathfrak{V}_{\hat{1}}$ span $\mathfrak{V}_{(1)}$. As $\mathfrak{k}(\mathfrak{g})$ is an invariant of $\mathfrak{g}$, so $\mathfrak{k}(\mathfrak{g}_{(1)})$ is an invariant of $\mathfrak{g}_{(1)}$.

Now, an element of $\mathfrak{g}_{(1)}$ can be considered as a transformation of an n_1-dimensional space $R_{(1)}$ which is spanned by the vectors $\boldsymbol{e}_x (x=n-n_1+1,\cdots, n)$ in a base (S, M) of the third order. Let the base of $R_{(1)}$ formed by these vectors be denoted by $S_{(1)}$. If S is transformed into $\tilde{S}=AS$ by means of (27), then $S_{(1)}$ is transformed into $\tilde{S}_{(1)}=A_{(1)}S_{(1)}$ which is composed of the vectors $\tilde{\boldsymbol{e}}_x = A^{y}_{\cdot x}\boldsymbol{e}_y$. The matrix $A_{(1)}$ is the n_1-submatrix $(A^{y}_{\cdot x})$ of A. If a base $\tilde{M}$ of $A^{-1}\mathfrak{V}A$ is chosen suitably, then the base $(\tilde{S}, \tilde{M})$ becomes a base of the third order, and, as we have seen, the portion (28) of $\tilde{M}$ has the same form as M_1. On the other hand, as for $R_{(1)}, \mathfrak{g}_{(1)}$, we can construct a base of the third order $(\tilde{S}_{(1)}, \tilde{M}_{(1)})$ by choosing $A_{(1)}$ and the base $\tilde{M}_{(1)}$ of $A_{(1)}^{-1}\mathfrak{V}_{(1)}A_{(1)}$ suitably.

Thus we find that we can choose a base (S, M) of the third order for the given $(R^n, \mathfrak{k}(\mathfrak{g}))$ in such a way that the corresponding base $(S_{(1)}, M_{(1)})$ of $(R_{(1)}, \mathfrak{k}(\mathfrak{g}_{(1)}))$ is also a base of the third order. From this fact we see that

If we take a particular base of the third order, then the base $M_{\hat{1}}$ of $\mathfrak{V}_{\hat{1}}$ decomposes into two portions M_2 and $M_{\hat{2}}$ such that the n_1-submatrices of the matrices of $M_2, M_{\hat{2}}$ are quite similar in form to the matrices of $M_1, M_{\hat{1}}$ stated in § 5 *respectively, with the understanding that the numbers n_1, n_2 play the parts of n, n_1.*

M_2 contains just $n_2(n_1-n_2)$ or $n_2(n_1-n_2)+1$ linearly independent matrices.

The two cases are distinguished by writing $2 \in C1$ or $2 \in C2$ as we are now in the second step. The linear spaces spanned by $M_2, M_{\hat{2}}$ are denoted by $\mathfrak{V}_2, \mathfrak{V}_{\hat{2}}$ respectively.

2) We can proceed in this way step by step. Let us assume that we have completed the P-th step. If we have $n_{P+1} \neq 0$, then the $P+1$-th step is carried out as follows.

By assumption $\mathfrak{V}$ is decomposed in the form

$$\mathfrak{V} = \mathfrak{V}_1 + \cdots + \mathfrak{V}_{T-1} + \mathfrak{V}_{(\widehat{T-1})} \qquad (T \leqq P).$$

We also have

$$\mathfrak{V}_{(\widehat{T-1})} = \mathfrak{V}_T + \mathfrak{V}_{\hat{T}},$$

where the matrices V of $\mathfrak{V}_{\hat{T}}$ satisfy $V^{\lambda}_{\cdot\mu} = 0$ for

$$\lambda = 1, \cdots, n; \quad \mu = 1, \cdots, n - n_1,$$

$$\lambda = n - n_1 + 1, \cdots, n; \quad \mu = n - n_1 + 1, \cdots, n - n_2,$$

$$\cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdots\cdots$$

$$\lambda = n - n_{T-1} + 1, \cdots, n; \quad \mu = n - n_{T-1} + 1, \cdots, n - n_T,$$

and a base M_T of $\mathfrak{V}_T$ is such that the n_{T-1}-submatrices

$$V^{\lambda}_{\cdot\mu} \ (\lambda, \mu = n - n_{T-1} + 1, \cdots, n)$$

of the matrices of M_T have the same forms as the matrices of M_1 written in § 5, with the understanding that the numbers n_{T-1}, n_T play the parts of n, n_1. Moreover, the n_{T-1}-submatrices of the matrices in $\mathfrak{V}_{\hat{T}}$ also have the same form as the matrices in $\mathfrak{V}_{\hat{1}}$. This fact becomes significant when $n_{T-1} - n_T \geqq 2$.

If we effect a transformation of S where

$$A^{\lambda}_{\cdot j} = \delta^{\lambda}_{j}, \qquad A^{i}_{\cdot\mu} = \delta^{i}_{\mu} \tag{31}$$

$$(\lambda, \mu = 1, \cdots, n; \, i, j = 1, \cdots, n - n_P),$$

then the spaces $A^{-1}\mathfrak{V}_T A, A^{-1}\mathfrak{V}_{\hat{T}} A$ keep the properties mentioned above.

Now let us consider $\mathfrak{K}$. If $(K^{y}_{\cdot x})$ where $x, y = n - n_P + 1, \cdots, n$ is a matrix of degree n_P satisfying

$$K^{y}_{\cdot x} V^{x}_{\cdot y} = 0 \tag{32}$$

for all matrices V in $\mathfrak{V}_{\hat{P}}$, then we can find a matrix $(K^{\lambda}_{\cdot\mu})$ of $\mathfrak{K}$ containing this $(K^{y}_{\cdot x})$ as the n_P-submatrix and such that $K^{\lambda}_{\cdot\mu} = 0$ for

$$\lambda = n - n_1 + 1, \cdots, n; \quad \mu = 1, \cdots, n - n_1,$$

$$\lambda = n - n_2 + 1, \cdots, n; \quad \mu = n - n_1 + 1, \cdots, n - n_2,$$

$$\cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdots\cdots$$

$$\lambda = n - n_P + 1, \cdots, n; \quad \mu = n - n_{P-1} + 1, \cdots, n - n_P.$$

This is an extension of Lemma 3 and is proved as follows.

We put

$$K^{\lambda}_{\cdot\mu}=\delta^{\lambda}_{y}K^{y}_{\cdot x}\delta^{x}_{\mu}+\delta^{\lambda}_{i_1}K^{i_1}_{\cdot\mu}+\cdots+\delta^{\lambda}_{i_P}K^{i_P}_{\cdot\mu} \tag{33}$$

$$(i_1=1,\cdots,n-n_1;\cdots;i_P=n-n_{P-1}+1,\cdots,n-n_P),$$

where

$$K^{x_T}_{\cdot j_T}=0 \tag{34}$$

$$(T=1,\cdots,P;\ x_T=n-n_T+1,\cdots,n),$$

and determine $K^{i_1}_{\cdot\mu},\cdots,K^{i_P}_{\cdot\mu}$ so as to get $(K^{\lambda}_{\cdot\mu})\in\mathfrak{K}$. If $V\in\mathfrak{V}_{\hat{T}}$, we get from

$$\begin{cases} V^{\mu}_{\cdot\lambda}=\delta^{\mu}_{i_1}V^{i_1}_{\cdot\lambda}+\cdots+\delta^{\mu}_{i_T}V^{i_T}_{\cdot\lambda}+\delta^{\mu}_{x_T}V^{x_T}_{\cdot\lambda}, \\ V^{i_S}_{\cdot j_1}=0,\cdots,V^{i_S}_{\cdot j_S}=0 \qquad (S=1,\cdots,T), \\ V^{x_T}_{\cdot j_1}=0,\cdots,V^{x_T}_{\cdot j_T}=0 \end{cases} \tag{35}$$

and (34)

$$K^{\lambda}_{\cdot\mu}V^{\mu}_{\cdot\lambda}=K^{y_T}_{\cdot x_T}V^{x_T}_{\cdot y_T},$$

hence, if $V\in\mathfrak{V}_{T+1}(T<P)$, the equation $K^{\lambda}_{\cdot\mu}V^{\mu}_{\cdot\lambda}=0$ becomes simply

$$K^{y_T}_{\cdot x_T}V^{x_T}_{\cdot y_T}=0. \tag{36}$$

But $\mathfrak{V}_{T+1}$ is spanned by the base M_{T+1} composed of

$$\overset{k_{T+1}}{\underset{z_{T+1}}{V}} \tag*{$(37)_{T+1}$}$$

or

$$\underset{T+1}{V},\qquad \overset{k_{T+1}}{\underset{z_{T+1}}{V}}, \tag*{$(38)_{T+1}$}$$

where the $n_{T+1}(n_T-n_{T+1})$ or $n_{T+1}(n_T-n_{T+1})+1$ matrices are such that their n_T-submatrices have the same form as the matrices of M_1 stated in § 5, with the numbers n_T, n_{T+1} playing the parts of n, n_1. Hence, for $T+1=P$, that is, in regard to the condition that $K^{\lambda}_{\cdot\mu}V^{\mu}_{\cdot\lambda}=0$ for all matrices V in $\mathfrak{V}_P$, we get, just as we obtained (29) or (30) in the second step,

$$K^{k_P}_{\cdot z_P}=-K^{y_P}_{\cdot x_P}\overset{k_P}{\underset{z_P}{V}}{}^{x_P}_{\cdot y_P}=0 \qquad (P\in C1) \tag*{$(39)_P$}$$

or

$$\left.\begin{aligned} K^{k_P}_{\cdot z_P}&=-K^{y_P}_{\cdot x_P}\overset{k_P}{\underset{z_P}{V}}{}^{x_P}_{\cdot y_P}, \\ K^{i_P}_{\cdot i_P}&=-K^{y_P}_{\cdot x_P}\underset{P}{V}{}^{x_P}_{\cdot y_P} \end{aligned}\right\} \qquad (P\in C2). \tag*{$(40)_P$}$$

$P\in C1$ or $C2$ means that the dimension of $\mathfrak{V}_P$ is $n_P(n_{P-1}-n_P)$ or $n_P(n_{P-1}-n_P)+1$ respectively. Similarly we get for an arbitrary $T(1\leqq T\leqq P)$ $(39)_T$ or $(40)_T$ according as $T\in C1$ or $T\in C2$, and all these together are a necessary and sufficient condition that the matrix $(K^{\lambda}_{\cdot\mu})$ given above lie in $\mathfrak{K}$.

We determine $K^{i_P}_{\cdot\mu}$ at first by means of $(39)_P$ or $(40)_P$, and then $K^{i_{P-1}}_{\cdot\mu}$ by means of $(39)_{P-1}$ or $(40)_{P-1}$ and so on. It will be seen that, if $T\in C1$, we can choose the matrix $(K^{i_T}_{\cdot j_T})$ of degree $n_{T-1}-n_T$ arbitrarily, while, if $T\in C2$, only the trace of $(K^{i_T}_{\cdot j_T})$ is given by the second equation of $(40)_T$. This fact will become important later.

By virtue of the fact that we can find such matrices K of $\mathfrak{K}$ and that the spaces $A^{-1}\mathfrak{V}_T A$, $A^{-1}\mathfrak{V}_{\hat{T}} A$ obtained from $\mathfrak{V}_T, \mathfrak{V}_{\hat{T}}$ by a transformation of S satisfying (31) have the same property as $\mathfrak{V}_T, \mathfrak{V}_{\hat{T}}$, it will be understood that the $P+1$-th step is completed in the same manner as the former ones.

Such process can be continued until we get $n_{P+1}=0$. Then the P-th step is the last step and we can not decompose $\mathfrak{V}_{\hat{P}}$ further in a similar manner. The n_P-submatrices of the matrices in $\mathfrak{V}_{\hat{P}}$ are scalar matrices, and, if these are all zero, we write $P+1\in C1$. If some are not zero, we write $P+1\in C2$, and in this case we can find just one linearly independent matrix $\underset{P+1}{V}$ in $\mathfrak{V}_{\hat{P}}$ such that the n_P-submatrix is a unit matrix. If $\mathfrak{V}'_{P+1}$ denotes a subspace of $\mathfrak{V}_{\hat{P}}$ spanned by the matrices whose n_P-submatrices are zero, then we have

$$\mathfrak{V}_{\hat{P}}=\mathfrak{V}'_{P+1} \qquad (P+1\in C1),$$
$$\mathfrak{V}_{\hat{P}}=\underset{P+1}{V}+\mathfrak{V}'_{P+1} \qquad (P+1\in C2).$$

It may happen that $\mathfrak{V}_{\hat{P}}$ is an empty set.

We thus obtain the

THEOREM 1. *Let $\mathfrak{g}$ be an r-dimensional subalgebra of a general linear Lie algebra $\mathfrak{gl}(R^n)$ operating on an n-dimensional linear space R^n over the field of real numbers. An isomorphic representation of $\mathfrak{g}$ is taken in an n^2-dimensional linear space spanned by all matrices of degree n and is denoted by $\mathfrak{K}(\mathfrak{g}, S)$ where S is a base of R^n. The matrices $(V^{\lambda}_{\cdot\mu})$ which satisfy*

$$K^{\lambda}_{\cdot\mu}V^{\mu}_{\cdot\lambda}=0$$

for all matrices K of $\mathfrak{K}$ span a subspace $\mathfrak{V}(\mathfrak{g}, S)$ of dimension $m=n^2-r$ in an n^2-dimensional linear space spanned by all matrices of degree n. Then we can choose S such that $\mathfrak{V}$ is decomposed as follows

$$(41)_T \qquad \mathfrak{V}=\mathfrak{V}_1+\cdots+\mathfrak{V}_T+\mathfrak{V}_{\hat{T}} \qquad (1\leqq T\leqq P),$$

$$(42) \qquad \begin{cases} \mathfrak{V}_{\hat{P}}=\mathfrak{V}'_{P+1} & (P+1\in C1), \\ \mathfrak{V}_{\hat{P}}=\underset{P+1}{V}+\mathfrak{V}'_{P+1} & (P+1\in C2), \end{cases}$$

where the matrices V of $\mathfrak{V}_{\hat{P}}$ satisfy (35) *and*

$$\overset{i_T}{V}_{\bullet\mu}=0 \quad \textit{if} \quad n_{T-1}-n_T\geqq 2\,,$$

while $\mathfrak{V}_T$ $(1\leqq T\leqq P)$ *is spanned by* $(37)_T$ *or* $(38)_T$ *according as* $T\in$C1 *or* $T\in$C2. *Especially the matrices of* $\mathfrak{V}'_{P+1}$ *are such that their* n_P*-submatrices are zero, and the* n_P*-submatrix af* $\underset{P+1}{V}$ *is a unit matrix.*

The base M_T of $\mathfrak{V}_T$ as given by $(37)_T$ or $(38)_T$ for each $T(1\leqq T\leqq P)$, $\underset{P+1}{V}$ if $P+1\in$C2, and an arbitrary base M'_{P+1} of $\mathfrak{V}'_{P+1}$ together compose a base M of $\mathfrak{V}$. Such a base (S, M) will be called a *base of the fourth order.*

§ 7. The d series and the dimension of $\mathfrak{g}$.

Let us put $n_0=n$ and

$$d_T=n_{T-1}-n_T\,,\ d_{P+1}=n_P \qquad (1\leqq T\leqq P)\,. \tag{43}$$

Then we get

$$n=d_1+\cdots+d_{P+1} \tag{44}$$

and the space $\mathfrak{V}_T$ is spanned by $d_T(n-d_1-\cdots-d_T)$ or $d_T(n-d_1-\cdots-d_T)+1$ linearly independent matrices according as $T\in$C1 or $T\in$C2. Thus we obtain the

Theorem 2. *Let* $\mathfrak{V}$ *decompose in the form stated in Theorem* 1. *Then its dimension* $m=n^2-r$ *satisfies the inequality*

$$m\geqq d_1(n-d_1)+\cdots+d_P(n-d_1-\cdots-d_P)+c\,, \tag{45}$$

hence

$$m\geqq \sum_{1\leqq S<T\leqq P+1} d_S d_T+c\,, \tag{46}$$

where c is the number of times T satisfies $T\in$C2 when it takes the values from 1 *to* $P+1$,

$$c=\sum_{T\in \mathrm{C2}} 1\,.$$

The ordered set of numbers $d_1,\cdots,d_{P+1}$ will be called the *d series* of $\mathfrak{g}$.

The exact value of m is obtained by the addition of the dimension of $\mathfrak{V}'_{P+1}$ to the right hand member of (46). It is evident that

$$\dim \mathfrak{V}'_{P+1}\leqq \sum_{1\leqq S<T\leqq P+1} d_S d_T\,.$$

But according to § 5 the matrices V of $\mathfrak{V}'_{P+1}$ satisfy, besides

$$\overset{i_T}{V}_{\bullet j_S}=0 \qquad (1\leqq S\leqq T\leqq P+1)\,, \tag{47}$$

the equations

(48) $$V_{\cdot j_S}^{i_T}=0 \qquad (1\leqq S,\ T\leqq P+1;\ d_T\geqq 2)$$

if $d_T\geqq 2$. If T^* denotes T such that $d_T=1$, then we get

$$\dim \mathfrak{B}'_{P+1}\leqq \sum_{T^*}\Big(\sum_{T>T^*} d_T\Big),$$

hence the

THEOREM 3. *If r is the dimension of $\mathfrak{g}$, the number $m=n^2-r$ satisfies*

(49) $$\sum_{T^*}\Big(\sum_{T>T^*} d_T\Big)+\sum_{1\leqq S<T\leqq P+1} d_S d_T+c\geqq m\geqq \sum_{1\leqq S<T\leqq P+1} d_S d_T+c\,.$$

EXAMPLE. According to Theorem 3 we can obtain in principle all possible d series for any given value of r. This is especially easy for sufficiently large r. For example, if $r\geqq n^2-3n+9$, that is, if $m\leqq 3n-9$, the possible d series and the corresponding values of m are as follows. Of course some of them must be omitted.

P	d series	m
0	n	0, 1
1	$1,\ n-1$	$n-1,\cdots,2n$
	$2,\ n-2$	$2(n-2),\ 2(n-2)+1,\ 2(n-2)+2$
	$3,\ n-3$	$3(n-3),\ 3(n-3)+1,\ 3(n-3)+2$
	$n-3,\ 3$	$3(n-3),\ 3(n-3)+1,\ 3(n-3)+2$
	$n-2,\ 2$	$2(n-2),\ 2(n-2)+1,\ 2(n-2)+2$
	$n-1,\ 1$	$n-1,\ n,\ n+1$
2	$1,\ 1,\ n-2$	$2n-3,\cdots,4n-3$
	$1,\ n-2,\ 1$	$2n-3,\cdots,3n-1$
	$n-2,\ 1,\ 1$	$2n-3,\ 2n-2,\ 2n-1,\ 2n,\ 2n+1$.

Some of the values of m must be omitted not because of $m>3n-9$ but because of the property of $\mathfrak{g}$. Its study will be continued.

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