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On the fundamental conjecture of *GLC* V.

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In this paper we shall introduce the notion of regular proof-figures in *GLC* (§ 1), and prove that the end-sequence of such a proof-figure is provable without cut (Chap. I). This generalizes our result of [6].

From this result and the restriction theory in [2] follows immediately that no end-sequence of a regular proof-figure in G^1LC can contain an inconsistency of the theory of natural numbers, i.e. the logical system consisting of regular proof-figures of G^1LC and the theory of natural numbers—we shall denote this system with R^1NN for a while—is consistent. The system obtained in replacing G^1LC in the above definition of R^1NN by *GLC* will be denoted by *RNN*. The consistency of *RNN* could be also proved as an application of the result of Chapter I, but this would involve some complications. In Chapter II we prove the consistency of *RNN* by an anologous method as in Chapter I.

In this paper, we make use of the theory of ordinal diagrams, as developed in [7]. We shall show in [8] that the theory of ordinal diagrams can be formalized in *RNN*.

Chapter I. The regular proof-figure and the fundamental conjecture.

§ 1. Several concepts concerning a proof-figure of *GLC* and lemmas on ordinal diagrams.

We refer to [6], Chapter I as to the notations and the notions on *GLC* such as *t*-variables, *f*-variables, words, positive and negative, proper and inproper, degenerate and non-degenerate. We remind further that we have introduced in [5] 1.1, the notions of a formula *in a proof-figure* $\mathfrak{P}$, and of a logical symbol or a variable *in a formula A*. As these notions are of frequent use in the sequel, we shall illustrate them by an example. The same logical symbol ∀ may appear in a formula A as the outermost symbol and again several times. (E. g. $A=\forall\varphi\forall\psi\neg\forall\xi\neg\forall x(\xi[x]\vdash\!\dashv\varphi[x]\wedge\psi[x])$.) To distinguish these ∀'s, we shall designate the outermost one by $\curlyvee$, the second one by $\sharp$, the third one by $\natural$ etc. (so that $A=\curlyvee\varphi\sharp\psi\neg\natural\xi\neg\forall x(\xi[x]\vdash\!\dashv\varphi[x]\wedge\psi[x])$ in the above example). These $\curlyvee,\sharp,\natural,\cdots$, *symbols considered together with the places they occupy in the formula A* are examples of symbols *in the formula*

A (in this example they are $\forall$'s in A).

1.1. We say ' γ ties an f-variable α, or a logical symbol or a variable $\#$ in some formula' in the following case: γ is the outermost $\forall$ on an f-variable in a word of the form $\forall\varphi C(\varphi)$ and α or $\#$ appears in $C(\varphi)$.

1.2. We say ' γ affects $\#$' in the following case: γ is the outermost $\forall$ on an f-variable in a word of the form $\forall\varphi C(\varphi)$, $\#$ is $\forall$ on an f-variable tied by γ, and $\#$ ties φ.

1.3. Let A be a formula and γ be a proper $\forall$ on an f-variable in A. We say ' γ is *isolated*', if and only if the following conditions are fulfilled:

1.3.1. No free variable is tied by γ.

1.3.2. No $\forall$ on an f-variable affects γ.

1.3.3. γ affects no proper $\forall$ on an f-variable in A.

1.4. Let A be a formula in a proof-figure $\mathfrak{P}$ and γ be a proper $\forall$ on an f-variable in A. γ is called an $\forall$ left in $\mathfrak{P}$, if and only if one of the following conditions is satisfied:

1.4.1. A is placed in the left side of a sequence and γ is positive to A.

1.4.2. A is placed in the right side of a sequence and γ is negative to A.

Otherwise γ is called an $\forall$ right in $\mathfrak{P}$.

1.5. Lemmas on ordinal diagrams.

Let α, β and γ be c. o. d. 's (See [7].) and i be an integer satisfying $1 < i \leqq n$. By $R_i(\gamma, \alpha, \beta)$, we shall mean the following conditions:

1.5.1. γ is an i-section of α.

1.5.2. If α' is a k-section of α and is neither γ nor a k-section of γ, and k is an integer satisfying $1 < k \leqq n$, there exists a k-section β' of β such that $\alpha' \leqq_k \beta'$.

1.5.3. $\alpha <_1 \beta$.

Let α and β be c. o. d.'s. By $R(\alpha, \beta)$, we shall mean the following conditions:

1.5.4. If α' is a k-section of α, and k is an integer satisfying $1 < k \leqq n$, there exists a k-section β' of β such that $\alpha' \leqq_k \beta'$.

1.5.5. $\alpha <_1 \beta$.

The following lemmas are easily verified.

LEMMA 1. *$R_i(\gamma, \alpha, \beta)$ implies* $\alpha <_k \beta$ $(1 \leqq k < i)$.

LEMMA 2. *$R(\alpha, \beta)$ implies* $\alpha <_k \beta$ $(1 \leqq k \leqq n)$.

LEMMA 3. *Let j be an integer satisfying* $1 \leqq j < i$ *and a be a positive integer. $R_i(\gamma, \alpha, \beta)$ implies $R_i(\gamma, (j; a, \alpha\#\delta), (j; a, \beta\#\delta))$ where δ is a c. o. d., or δ is void in which case $\alpha\#\delta$, $\beta\#\delta$ mean α, β respectively.*

LEMMA 4. *Let j be an integer satisfying $i \leqq j \leqq n$ and a be a positive integer. $R(\gamma, \beta)$ and $R_i(\gamma, \alpha, \beta)$ imply $R((j; a, \alpha), (j; a, \beta))$ and $\alpha <_k \beta$ $(1 \leqq k \leqq n)$.*

LEMMA 5. *$R(\alpha, \beta)$ implies $R((j; a, \alpha \# \delta), (j; a, \beta \# \delta))$ $(1 \leqq j \leqq n)$ where δ is as in Lemma 3.*

§ 2. Regular proof-figures.

In this section, we define first the concept of *regular* proof-figures and next the concept of proof-figures of *order* n and correspondence of the ordinal diagram to a proof-figure of order n.

2.1. A formula A is regular, if the following condition is fulfilled: Let $\gamma, \natural$ be any pair of proper $\forall$'s on f-variables in A. If γ ties $\natural$ and $\natural$ is not isolated, then γ is positive to $\natural$.

2.2. A proof-figure $\mathfrak{P}$ is regular, if and only if the following condition is fulfilled: If $\mathfrak{P}$ contains an implicit $\forall$ left on an f-variable of the form

$$\frac{F(V), \Gamma \rightarrow \Delta}{\forall \varphi F(\varphi), \Gamma \rightarrow \Delta},$$

then $\forall \varphi F(\varphi)$ is regular.

With this terminology, we now fomulate our principal theorem:

THEOREM 1. *The end-sequence of a regular proof-figure is provable without cut.*

2.3. An isolated degree of a regular formula.

Let A be a regular formula and γ be an isolated $\forall$ on an f-variable in A. The *isolated degree* of γ in A is defined recursively as follows:

2.3.1. If γ ties no isolated $\forall$ on an f-variable, then the isolated degree of γ is one.

2.3.2. If γ ties an isolated $\forall$ on an f-variable, then the isolated degree of γ is $n+1$ where n is the maximal number of the isolated degrees of the isolated $\forall$'s on f-variables tied by γ.

Let A be a regular formula. The isolated degree of A is the maximal number of the isolated degrees of the isolated $\forall$'s on f-variables in A, if such exist; otherwise the isolated degree of A is defined to be zero.

2.4. We introduce the following inference called 'substitution' in *GLC*.

Inference-schema on substitution:

$$\frac{A_1, \cdots, A_n \rightarrow B_1, \cdots, B_m}{A_1\binom{V}{\alpha}, \cdots, A_n\binom{V}{\alpha} \rightarrow B_1\binom{V}{\alpha}, \cdots, B_m\binom{V}{\alpha}},$$

where α is a free f-variable and V is a variety of the same type as α. (See [2], §5 for $\binom{V}{\alpha}$.) α is called the eigenvariable of this substitution. The formula of $A_j\binom{V}{\alpha}$ or $B_k\binom{V}{\alpha}$ in the lower sequence of this substitution is called the *successor* of the formula A_j or B_k in the upper sequence respectively.

As we have shown in [2] 6.9,

$$A_1\binom{V}{\alpha},\cdots, A_n\binom{V}{\alpha}\rightarrow B_1\binom{V}{\alpha},\cdots, B_m\binom{V}{\alpha}$$

is provable, if

$$A_1,\cdots, A_n\rightarrow B_1,\cdots, B_m$$

is provable, so that the inference schema on substitution is in principle reducdant in *GLC*, but the introduction of this inference schema facilitates us the reduction of regular proof-figures, as we shall show in the following.

2.5. Proof-figures of order n.

Let $\mathfrak{P}$ be a regular proof-figure. We attach an integer i greater than 1 to every substitution in $\mathfrak{P}$ and call i the index of the substitution. We call $\mathfrak{P}$ (considered together with i's and a positive integer n) a proof-figure of order n, if $\mathfrak{P}$, i's and n satisfy the following conditions.

2.5.1. Every substitution is in the end-place.

2.5.2. Every $i\leqq n$.

2.5.3. Let A be an arbitrary implicit regular formula in $\mathfrak{P}$. Then the isolated degree of A is less than n.

2.5.4. Let $\mathfrak{J}$ be an arbitrary substitution with the index i in $\mathfrak{P}$ and A be an arbitrary implicit formula in the upper sequence of $\mathfrak{J}$. If A is regular and the isolated degree of A is j, then $i+j-1\leqq n$. If there exsists a proper non-isolated $\forall$ on an f-variable in A, it is an $\forall$ right in $\mathfrak{P}$ and i must be 2.

Since every regular proof-figure may be, in introducing adequately i's and n, considered as a proof-figure of order n for sufficiently great n, we have only to prove that the end-sequence of a proof-figure of order n is provable without cut.

2.6. 'i-loader' of a sequence.

Let $\mathfrak{P}$ be a proof-figure of order n and $\mathfrak{S}$ be a sequence in $\mathfrak{P}$. The *i-loader* of $\mathfrak{S}$ is the upper sequence of the uppermost substitution under $\mathfrak{S}$, whose index is not less than i, if such exists; otherwise the i-loader of $\mathfrak{S}$ is the end-sequence.

2.7. Correspondence of an ordinal diagram of order n to a proof-figure of order n.

Now we assign an ordinal diagram of order n to every sequence of a proof-figure of order n recursively as follows:

2.7.1. The ordinal diagram of a beginning sequence is 1.

2.7.2. If $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are the upper sequence and the lower sequence of an inference $\mathfrak{J}$ on structure, then the ordinal diagram of $\mathfrak{S}_2$ is equal to that of $\mathfrak{S}_1$.

2.7.3. If $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are the upper sequence and the lower sequence of an inference $\neg$, $\wedge$ left, $\forall$ on a t-variable, $\forall$ right on an f-variable or explicit $\forall$ left on an f-variable respectively, then the ordinal diagram of $\mathfrak{S}_2$ is $(1; 1, \sigma)$, where σ is the ordinal diagram of $\mathfrak{S}_1$.

2.7.4. If $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are the upper sequences and $\mathfrak{S}$ is the lower sequence of an inference $\wedge$ right, then the ordinal diagram of $\mathfrak{S}$ is $(1; 1, \sigma_1 \# \sigma_2)$, where σ_1 and σ_2 are the ordinal diagrams of $\mathfrak{S}_1$ and $\mathfrak{S}_2$ respectively.

2.7.5. If $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are the upper sequence and the lower sequence of an implicit inference $\forall$ left $\mathfrak{J}$ on an f-variable respectively, then the ordinal diagram of $\mathfrak{S}_2$ is $(1; a+2, \sigma)$, where σ is the ordinal diagram of $\mathfrak{S}_1$, and a is the number of the proper logical symbols in the subformula of $\mathfrak{J}$.

2.7.6. If $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are the upper sequences and $\mathfrak{S}$ is the lower sequence of a cut $\mathfrak{J}$, then the ordinal diagram of $\mathfrak{S}$ is $(1; a+1, \sigma_1 \# \sigma_2)$, where σ_1 and σ_2 are the ordinal diagrams of $\mathfrak{S}_1$ and $\mathfrak{S}_2$ respectively and a is the number of the proper logical symbols in the cut-formula of $\mathfrak{J}$.

2.7.7. If $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are the upper sequence and the lower sequence of a substitution $\mathfrak{J}$ with the index i respectively, then the ordinal diagram of $\mathfrak{S}_2$ is $(i; 1, \sigma)$ where σ is the ordinal diagram of $\mathfrak{S}_1$.

We call the *ordinal diagram of a proof-figure of order n* the ordinal diagram assigned to its end-sequence.

§ 3. Preparation to the essential reduction.

3.1. Let $\mathfrak{S}_1, \cdots, \mathfrak{S}_m$ and $\mathfrak{S}$ be sequences. '$\mathfrak{S}$ is reducible to $\mathfrak{S}_1, \cdots, \mathfrak{S}_m$' will mean 'if $\mathfrak{S}_1, \cdots, \mathfrak{S}_m$ are provable without cut, then $\mathfrak{S}$ is provable without cut'.

Let $\mathfrak{P}_1, \cdots, \mathfrak{P}_m$ and $\mathfrak{P}$ be proof-figures of order n. We say $\mathfrak{P}$ is *reduced* to $\mathfrak{P}_1, \cdots, \mathfrak{P}_m$ if and only if the following conditions are satisfied:

3.1.1. For each i $(1 \leqq i \leqq m)$, the ordinal diagram of $\mathfrak{P}_i$ is less than that of $\mathfrak{P}$.

3.1.2. The end-sequence of $\mathfrak{P}$ is reducible to the end-sequences of $\mathfrak{P}_1, \cdots, \mathfrak{P}_m$.

As we have proved in [7] that the ordinal diagrams of order n form a well-ordered system, we have only to show for the proof of our theorem that we can find proof-figures $\mathfrak{P}_1, \cdots, \mathfrak{P}_m$ of order n for a given proof-figure $\mathfrak{P}$ of order n, such that '$\mathfrak{P}$ is reduced to $\mathfrak{P}_1, \cdots, \mathfrak{P}_m$' in the sense just defined.

3.2. Reduction for the case that the end-place contains an explicit logical inference.

Let $\mathfrak{P}$ be a proof-figure of order n and $\mathfrak{J}$ be the lowermost explicit logical inferece contained in the end-place of $\mathfrak{P}$. The inference $\mathfrak{J}$ may have various forms, but since all cases are similarly treated, we may assume that $\mathfrak{P}$ is of the following form:

3.2.1.

$$\begin{array}{c} \vdots \\ \dfrac{\Gamma \to \Delta, F(\alpha)}{\Gamma \to \Delta, \forall\varphi F(\varphi)}\ \mathfrak{J} \\ \vdots \\ \Gamma_0 \to \Delta_0 \end{array}$$

Without loss of generality, we may assume moreover that α is not an eigenvariable of any substitution in $\mathfrak{P}$ and that $\Gamma_0 \to \Delta_0$ contains no α.

3.2.2. Now we consider the following proof-figure $\mathfrak{P}'$:

$$\begin{array}{c} \vdots \\ \Gamma \to \Delta, F(\alpha) \\ \hline \text{Some exchanges} \\ \hline \Gamma \to F(\alpha), \Delta \\ \hline \Gamma \to F(\alpha), \Delta, \forall\varphi F(\varphi) \\ \vdots \\ \Gamma_0 \to \tilde{F}(\alpha), \Delta_0 \end{array}$$

where every substitution in $\mathfrak{P}'$ has the same index as the corresponding one in $\mathfrak{P}$ and $\tilde{F}(\alpha)$ denotes the descendant of $F(\alpha)$.

We now show that $\mathfrak{P}'$ is a proof-figure of order n and $\mathfrak{P}$ is reduced to $\mathfrak{P}'$. For every substitution in $\mathfrak{P}'$, there exsists corresponding one in $\mathfrak{P}$ with the same index, and $\mathfrak{P}$ is a proof-figure of order n. So $\mathfrak{P}'$ is of order n. Let τ be the ordinal diagram of the sequence $\Gamma \to \Delta, F(\alpha)$ in $\mathfrak{P}$. Then the ordinal diagrams of $\Gamma \to F(\alpha), \Delta, \forall\varphi F(\varphi)$ and $\Gamma \to \Delta, \forall\varphi F(\varphi)$ are τ and $(1;1,\tau)$, respectively. If τ' is a k-section of τ and $k>1, \tau'$ is also a k-section of $(1;1,\tau)$. Then clearly $R(\tau,(1;1,\tau))$. From this we see that the ordinal diagram of $\mathfrak{P}'$ is less than that of $\mathfrak{P}$, by the help of Lemmas 5 and 2 and induction on the number of sequences under $\Gamma \to F(\alpha), \Delta, \forall\varphi F(\varphi)$. Since $\forall\varphi F(\varphi)$ is an explicit formula in $\mathfrak{P}$ and $\mathfrak{P}$ has no logical inference under $\mathfrak{J}, \Delta_0$ contains a formula of the form $\forall\varphi \tilde{F}(\varphi)$. Thus $\mathfrak{P}$ is reduced to $\mathfrak{P}'$.

3.3. Reuction for the case that the end-place contains an implicit beginning sequence.

Hereafter we consider only proof-figures of order n whose end-places contain no logical inferences. Here we consider the case that the end-place of the proof-figure $\mathfrak{P}$ of order n contains an implicit beginning sequence.

3.3.1. Let $\mathfrak{P}$ be of the following form and $D \to D$ be one of the beginning sequences in the end-place of $\mathfrak{P}$:

$$
\begin{array}{c}
\begin{array}{cc}
 & D \to D \\
\Downarrow & \Downarrow \\
\Gamma \to \Delta, \tilde{D} & \tilde{D}, \Pi \to \Lambda_1, \tilde{D}, \Lambda_2
\end{array} \\
\hline
\Gamma, \Pi \to \Delta, \Lambda_1, \tilde{D}, \Lambda_2 \\
\Downarrow \\
\Gamma_0 \to \Delta_0
\end{array}
$$

where two $\tilde{D}$'s in the right side of the cut denote the descendants of the D's occuring in the beginning sequence.

3.3.2. Now we consider the proof-figure $\mathfrak{P}'$ of the following form:

$$
\begin{array}{c}
\Downarrow \\
\Gamma \to \Delta, \tilde{D} \\
\hline
\text{Some weakenings and exchanges} \\
\hline
\Gamma, \Pi \to \Delta, \Lambda_1, \tilde{D}, \Lambda_2 \\
\Downarrow \\
\Gamma_0 \to \Delta_0
\end{array}
$$

where every substitution in $\mathfrak{P}'$ has the same index as the corresponding one in $\mathfrak{P}$. We see in the same way as in 3.2.2, that $\mathfrak{P}'$ is a proof-figure of order n.

Let λ and μ be the ordinal diagrams of $\Gamma \to \Delta, \tilde{D}$ and $\tilde{D}, \Pi \to \Lambda, \tilde{D}, \Lambda_2$ in $\mathfrak{P}$ respectively. Then the ordinal diagrams of $\Gamma, \Pi \to \Delta, \Lambda, \tilde{D}, \Lambda_2$ in $\mathfrak{P}$ and in $\mathfrak{P}'$ are $(1; a+1, \lambda \# \mu)$ and λ where a is the number of the proper logical symbols in $\tilde{D}$, rsepectively. We see easily $R(\lambda, (1; a+1, \lambda \# \mu))$. Then, in the same way as in 3.2.2, we see the ordinal diagram of $\mathfrak{P}'$ is less than that of $\mathfrak{P}$.

§ 4. **Essential reduction.**

4.1. According to 3.2 and 3.3, we may assume that the end-place of a proof-figure of order n contains no logical inference and no implicit beginning sequence. Then in the same way as in [3], § 6, we may assume that the end-place contains a 'suitable cut' as defined in [3]. Moreover, without loss of generality, we may assume that every free variable used as an eigenvariable in a proof-figure is different from each other and is not contained in the sequences under the inference in which it is used as an eigenvariable.

Let $\mathfrak{P}$ be a proof-figure of order n and $\mathfrak{I}$ be a suitable cut in $\mathfrak{P}$. To define the essential reduction, we must treat separately several cases according to the form of the outermost logical symbol of the cut-formulas of $\mathfrak{P}$.

4.2. First we treat the case that the outermost logical symbol of $\mathfrak{I}$ is $\forall$ on an f-variable. Then $\mathfrak{P}$ is of the following form:

4.2.1.

$$
\begin{array}{c}
\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Gamma_2 \overset{\lambda_2}{\rightarrow} \Delta_2, \forall\varphi\tilde{F}(\varphi)\end{array}\quad \begin{array}{c}\vdots\\ \forall\varphi\tilde{F}(\varphi), \Pi_2 \overset{\mu_2}{\rightarrow} \Lambda_2\end{array}}{\Gamma_2, \Pi_2 \rightarrow \Delta_2, \Lambda_2}\,\mathfrak{I}}{}
\end{array}
$$

$$
\begin{array}{ccc}
\begin{array}{c}\vdots\\ \dfrac{\Gamma_1 \overset{\lambda_1}{\rightarrow} \Delta_1, F(\alpha)}{\Gamma_1 \rightarrow \Delta_1, \forall\varphi F(\varphi)}\\ \vdots\\ \Gamma_2 \overset{\lambda_2}{\rightarrow} \Delta_2, \forall\varphi\tilde{F}(\varphi)\end{array} & &
\begin{array}{c}\vdots\\ \dfrac{F_1(V), \Pi_1 \overset{\mu_1}{\rightarrow} \Lambda_1}{\forall\varphi F_1(\varphi), \Pi_1 \rightarrow \Lambda_1}\\ \vdots\\ \forall\varphi\tilde{F}(\varphi), \Pi_2 \overset{\mu_2}{\rightarrow} \Lambda_2\end{array}\\
\hline
\multicolumn{3}{c}{\Gamma_2, \Pi_2 \rightarrow \Delta_2, \Lambda_2 \quad \mathfrak{I}}\\
\multicolumn{3}{c}{\vdots}\\
\multicolumn{3}{c}{\Gamma_3 \overset{\nu_3}{\rightarrow} \Delta_3}\\
\multicolumn{3}{c}{\vdots}\\
\multicolumn{3}{c}{\Gamma_0 \overset{\sigma}{\rightarrow} \Delta_0}
\end{array}
$$

Here and the following, the small Greek letters $\lambda_1, \lambda_2, \mu_1, \mu_2, \cdots$ in the figure denote respectively the ordinal diagrams of the sequences, on the arrows of which they are written. Let j be the isolated degree of $\tilde{F}(\alpha)$. Let i mean 2 or $n-j+1$ according as $\forall\varphi\tilde{F}(\varphi)$ has a proper non-isolated $\forall$ on an f-variable or not. Let $\Gamma_3 \rightarrow \Delta_3$ be the i-loader of $\Gamma_2, \Pi_2 \rightarrow \Delta_2, \Lambda_2$. Generally $\forall\varphi F_1(\varphi)$ is different from $\forall\varphi\tilde{F}(\varphi)$, as some substitutions may appear between $\forall\varphi F_1(\varphi), \Pi_1 \rightarrow \Lambda_1$ and $\forall\varphi\tilde{F}(\varphi), \Pi_2 \rightarrow \Lambda_2$. But now this is not the case, because every substitution in $\mathfrak{P}$ satisfies 2.5.4 and $\forall\varphi F_1(\varphi)$ is in the left side of the sequence. Thus $\forall\varphi F_1(\varphi)$ is $\forall\varphi\tilde{F}(\varphi)$.

4.2.2.

$$\vdots$$
$$\frac{\Gamma_1 \xrightarrow{\lambda_1} \Delta_1, F(\alpha)}{\Gamma_1 \to F(\alpha), \Delta_1, \forall\varphi F(\varphi)}\ \text{Some exchanges and a weakening}$$
$$\vdots$$
$$\frac{\dfrac{\Gamma_3 \xrightarrow{\lambda_3} \Delta_3, \tilde{F}(\alpha)}{\Gamma_3 \to \Delta_3, \tilde{F}(V)}\,\mathfrak{J}_1 \qquad \begin{matrix}\vdots\\ \tilde{F}(V), \Pi_1 \xrightarrow{\mu_1} \Lambda_1\end{matrix}}{\dfrac{\Gamma_3, \Pi_1 \to \Delta_3, \Lambda_1}{\forall\varphi\tilde{F}(\varphi), \Pi_1, \Gamma_3 \to \Delta_3, \Lambda_1}\ \text{Some exchanges and a weakening}}$$
$$\vdots$$
$$\frac{\begin{matrix}\vdots\\ \Gamma_2 \xrightarrow{\lambda_2} \Delta_2, \forall\varphi\tilde{F}(\varphi)\end{matrix} \qquad \forall\varphi\tilde{F}(\varphi), \Pi_2, \Gamma_3 \xrightarrow{\mu_2} \Delta_3, \Lambda_2}{\dfrac{\Gamma_2, \Pi_2, \Gamma_3 \to \Delta_2, \Delta_3, \Lambda_2}{\Gamma_2, \Pi_2, \Gamma_3 \to \Delta_3, \Delta_2, \Lambda_2}\ \text{Some exchanges}}$$
$$\vdots$$
$$\frac{\Gamma_3, \Gamma_3 \to \Delta_3, \Delta_3}{\Gamma_3 \xrightarrow{\nu_2} \Delta_3}\ \text{Some exchanges and contractions}$$
$$\vdots$$
$$\Gamma_0 \xrightarrow{\sigma'} \Delta_0$$

where $\mathfrak{J}_1$ is a substitution whose eigenvariable is α and whose index is defined to be i. Every substitution in this proof-figure, except $\mathfrak{J}_1$, has the same index as the corresponding one in $\mathfrak{P}$.

We shall prove in 4.2.3 that $\mathfrak{P}$ is reduced to a proof-figure $\mathfrak{P}'$ of the form 4.2.2. Here we should remark that the formula in the upper sequence of $\mathfrak{J}_1$, which is the descendant of $F(\alpha)$, is $\tilde{F}(\alpha)$. In fact, i is 2, or $\tilde{F}(\alpha)$ contains no other free f-variables than α, according as $\forall\varphi\tilde{F}(\varphi)$ contains a proper non-isolated $\forall$ on an f-variable or not. In the former case, no substitution is used between $\Gamma_2, \Pi_2 \to \Delta_2, \Lambda_2$ and $\Gamma_3 \to \Delta_3$, and in the latter case, any substitution does not influence $\tilde{F}(\alpha)$. So in both cases the upper sequence of $\mathfrak{J}_1$ may be denoted as $\Gamma_3 \to \Delta_3, \tilde{F}(\alpha)$.

4.2.3. Now we prove that $\mathfrak{P}'$ is a proof-figure of order n, i. e. $\mathfrak{P}'$ satisfies the conditions described in 2.5. The conditions 2.5.1 and 2.5.2 for $\mathfrak{P}'$ follow from those for $\mathfrak{P}$, as the new substitution $\mathfrak{J}_1$ is in the end-place and its index is defined to satisfy 2.5.2.

4.2.3.1. To prove 2.5.3 for $\mathfrak{P}'$, it is sufficient to show that the isolated degree of $\tilde{F}(\alpha)$ is less than n. This is clear because $\forall\varphi\tilde{F}(\varphi)$ is implicit in $\mathfrak{P}$ and has consequently an isolated degree $<n$, and the isolated degree of

$\tilde{F}(\alpha) \leqq$ the isolated degree of $\forall\varphi\tilde{F}(\varphi)$, as every proper $\forall$ on an f-variable isolated in $\tilde{F}(\alpha)$ is also isolated in $\forall\varphi\tilde{F}(\varphi)$.

4.2.3.2. Now we prove 2.5.4 for $\mathfrak{P}'$. $\Gamma_3 \to \Delta_3$ is either an upper sequence of a substitution in $\mathfrak{P}$, whose index k is not less than i, or the end-sequence of $\mathfrak{P}$. In the former case, let l be the isolated degree of an arbitrary implicit formula in $\Gamma_3 \to \Delta_3$, then $k+l-1 \leqq n$. This and $i \leqq k$ imply $i+l-1 \leqq n$. In the latter case, no implicit formula is in $\Gamma_3 \to \Delta_3$. Now we show 2.6.4 on $\tilde{F}(\alpha)$. We have $i=n-j+1$ by our assumption, if no proper non-isolated $\forall$ on an f-variable is in $\forall\varphi F(\varphi)$. Moreover, if there exists a proper non-isolated $\forall$ on an f-variable in $\tilde{F}(\alpha)$, which is denoted by γ, it is also proper non-isolated in $\forall\varphi\tilde{F}(\varphi)$. Now let $\natural$ be the outermost logical symbol of $\forall\varphi\tilde{F}(\varphi)$. Then $\natural$ ties γ. This implies that $\natural$ must be positive to γ by regularity of $\forall\varphi F(\varphi)$. $\tilde{F}(\alpha)$ being in the right side of the sequence, γ is an $\forall$ right in $\mathfrak{P}'$, and i is 2 by our assumption.

4.2.4. New we prove that σ' is less than σ.

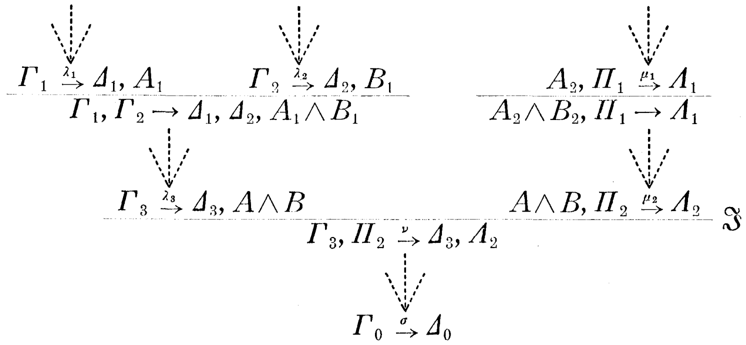
4.2.4.1. First we show that the index k of every substitution between $\Gamma_3, \Pi_1 \to \Delta_3, \Lambda_1$ and $\Gamma_3 \to \Delta_3$ is less than i. If $\forall\varphi\tilde{F}(\varphi)$ contains a proper non-isolated $\forall$ on an f-variable, which is denoted by γ, γ is positive to $\forall\varphi\tilde{F}(\varphi)$ by regularity of $\forall\varphi\tilde{F}(\varphi)$. Then, $\forall\varphi\tilde{F}(\varphi)$ being in the left side of the sequence, no substitution is used above $\forall\varphi\tilde{F}(\varphi), \Pi_2 \to \Lambda_2$ in $\mathfrak{P}$. And if $\forall\varphi\tilde{F}(\varphi)$ contains no proper non-isolated $\forall$ on an f-variable, necessarily the outermost logical symbol is isolated. Then the isolated degree of $\forall\varphi\tilde{F}(\varphi)$ must be $j+1$ where j is that of $\tilde{F}(\alpha)$, and we see $k+j \leqq n$ by 2.5.4 for $\mathfrak{P}$, that is, $k<i$. For each sequence between $\Gamma_2, \Pi_2, \Gamma_3 \to \Delta_2, \Delta_3, \Lambda_2$ and $\Gamma_3 \to \Delta_3$, our assertion follows from the fact that $\Gamma_3 \to \Delta_3$ is the i-loader of $\Gamma_2, \Pi_2 \to \Delta_2, \Lambda_2$.

4.2.4.2. Let τ' and τ be the ordinal diagrams of $\forall\varphi\tilde{F}(\varphi), \Pi_1, \Gamma_3 \to \Delta_3, \Lambda_1$ and $\forall\varphi\tilde{F}(\varphi), \Pi_1 \to \Lambda_1$ respectively. We have to prove $R_i(\lambda_3, \tau', \tau)$. τ' and τ are $(1; a+1, (i; 1, \lambda_3)\# \mu_1)$ and $(1; a+2, \mu_1)$ respectively. Since $(i; 1, \lambda_3)$ contains no 1-section, we see easily $\tau' <_1 \tau$. Other conditions 1.5.1, 1.5.2 are clearly obtained. Then we can obtain $R_i(\lambda_3, \nu_2, \nu_1)$ by the help of 4.2.4.1, Lemma 3 and induction on the number of sequences under $\forall\varphi\tilde{F}(\varphi), \Pi_1, \Gamma_3 \to \Delta_3, \Lambda_1$. On the other hand we have $R(\lambda_3, \nu_1)$ in the same way as in 3.2. Then, by Lemma 4, we obtain $R((k; 1, \nu_2), (k; 1, \nu_1))$ where k is the index of the substitution whose upper sequence is $\Gamma_3 \to \Delta_3$, and $\nu_2 < \nu_1$. Then $R(\sigma', \sigma)$ by the help of Lemma 5 and induction on the number of sequences under the substitution. From this follows $\sigma' < \sigma$ by Lemma 2.

4.3. Next we treat the case that the outermost logical symbol of the cut-formulas of $\mathfrak{I}$ is $\wedge$.

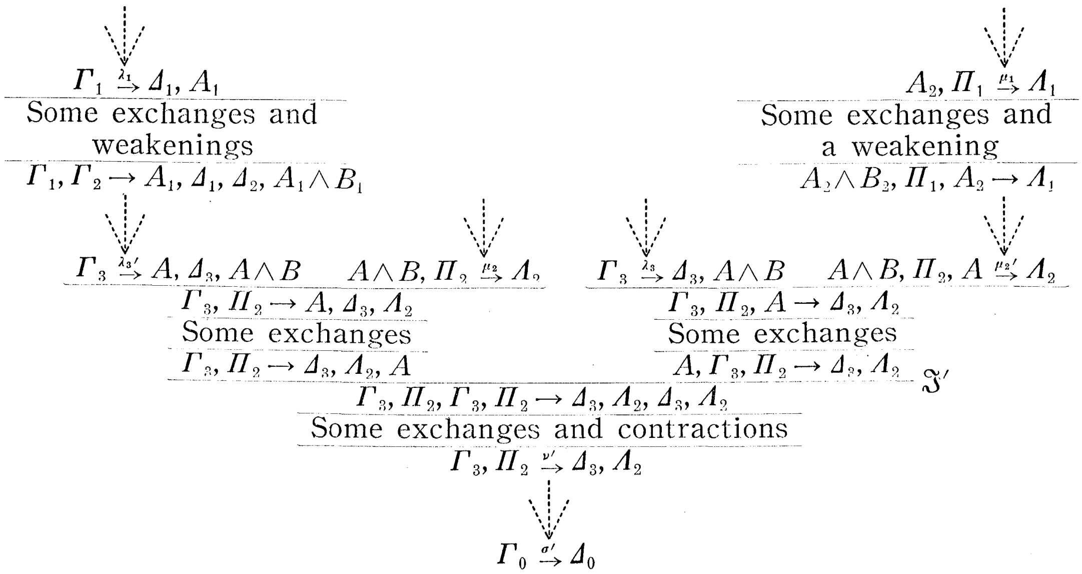
Then $\mathfrak{P}$ is of the following form:

4.3.1.



We shall prove that $\mathfrak{P}$ is reduced to $\mathfrak{P}'$ of the following form.

4.3.2.



Every substitution in $\mathfrak{P}'$ has the same index as the corresponding one in $\mathfrak{P}$.

4.3.3. We prove first that $\mathfrak{P}'$ is a proof-figure of order n. It follows from the fact that for every substitution in $\mathfrak{P}'$, there exists corresponding one in $\mathfrak{P}$ with the same index, and $\mathfrak{P}$ is a proof-figure of order n.

4.3.4. We have to prove $\sigma' < \sigma$. Let a and b be the numbers of the proper logical symbols in A and in B respectively. In the same was as in 3.2, we have $R(\lambda_3', \lambda_3)$, $R(\mu_2', \mu_2)$, $R((1; a+b+2, \lambda_3' \# \mu_2), \nu)$ and $R((1; a+b+2, \lambda_3 \# \mu_2'), \nu)$. Then we see easily $R(\nu', \nu)$. Then the proof is concluded by Lemmas 5 and 2 and induction on the number of sequences under $\mathfrak{J}'$.

4.4. Now we consider the case that the outermost logical symbol of the cut-formulas of $\mathfrak{J}$ is $\forall$ on a variable.

Then $\mathfrak{P}$ is of the following form:

4.4.1.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\Gamma_1 \overset{\lambda_1}{\rightarrow} \Delta_1, F(a)}}{\Gamma_1 \rightarrow \Delta_1, \forall x F_1(x)}}{\vdots}}{\Gamma_2 \overset{\lambda_2}{\rightarrow} \Delta_2, \forall x F(x)}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\vdots}{F_2(T), \Pi_1 \overset{\mu_1}{\rightarrow} \Lambda_1}}{\forall x F_2(x), \Pi_1 \rightarrow \Lambda_1}}{\vdots}}{\forall x F(x), \Pi_2 \overset{\mu_2}{\rightarrow} \Lambda_2}
$$

$$
\dfrac{\Gamma_2 \overset{\lambda_2}{\rightarrow} \Delta_2, \forall x F(x) \qquad \forall x F(x), \Pi_2 \overset{\mu_2}{\rightarrow} \Lambda_2}{\Gamma_2, \Pi_2 \overset{\nu}{\rightarrow} \Delta_2, \Lambda_2}
$$

$$
\begin{array}{c} \vdots \\ \Gamma_0 \overset{\sigma}{\rightarrow} \Delta_0 \end{array}
$$

We can prove in the same way as in 4.3 that $\mathfrak{P}$ is reduced to $\mathfrak{P}'$ of the following form.

4.4.2.

$$
\dfrac{\dfrac{\vdots}{\Gamma_1 \overset{\lambda_1}{\rightarrow} \Pi_1, F_1(\tilde{T})}}{\Gamma_1 \rightarrow F_1(\tilde{T}), \Delta, \forall x F_1(x)}\ \text{Some exchanges and a weakening}
\qquad
\dfrac{\dfrac{\vdots}{F_2(T), \Pi_1 \overset{\mu_1}{\rightarrow} \Lambda_1}}{\forall x F_2(x), \Pi_1, F_2(T) \rightarrow \Lambda_1}\ \text{Some exchanges and a weakening}
$$

$$
\dfrac{\dfrac{\Gamma_2 \overset{\lambda_2'}{\rightarrow} F(\tilde{T}), \Delta_2, \forall x F(x) \quad \forall x F(x), \Pi_2 \overset{\mu_2}{\rightarrow} \Lambda_2}{\Gamma_2, \Pi_2 \rightarrow F(\tilde{T}), \Delta_2, \Lambda_2}}{\Gamma_2, \Pi_2 \rightarrow \Delta_2, \Lambda_2, F(\tilde{T})}\ \text{Some exchanges}
\qquad
\dfrac{\dfrac{\Gamma_2 \overset{\lambda_2}{\rightarrow} \Delta_2, \forall x F(x) \quad \forall x F(x), \Pi_2, F(\tilde{T}) \overset{\mu_2}{\rightarrow} \Lambda_2}{\Gamma_2, \Pi_2, F(\tilde{T}) \rightarrow \Delta_2, \Lambda_2}}{F(\tilde{T}), \Gamma_2, \Pi_2 \rightarrow \Delta_2, \Lambda_2}\ \text{Some exchanges}
$$

$$
\dfrac{\dfrac{\Gamma_2, \Pi_2 \rightarrow \Delta_2, \Lambda_2, F(\tilde{T}) \qquad F(\tilde{T}), \Gamma_2, \Pi_2 \rightarrow \Delta_2, \Lambda_2}{\Gamma_2, \Pi_2, \Gamma_2, \Pi_2 \rightarrow \Delta_2, \Lambda_2, \Delta_2, \Lambda_2}}{\Gamma_2, \Pi_2 \overset{\nu'}{\rightarrow} \Delta_2, \Lambda_2}\ \text{Some exchanges and contractions}
$$

$$
\begin{array}{c} \vdots \\ \Gamma_0 \overset{\sigma'}{\rightarrow} \Delta_0 \end{array}
$$

where every substitution has the same index as the corresponding one in $\mathfrak{P}$, and the proof-figure to $\Gamma_1 \rightarrow \Delta_1, F_1(\tilde{T})$, is obtained from the proof-figure to $\Gamma_1 \rightarrow \Delta_1, F_1(a)$ by substituting everywhere $\tilde{T}$ for a. Here we should remark that the ordinal diagram of the sequence $\Gamma_1 \rightarrow \Delta_1, F_1(\tilde{T})$ is the same λ_1 as that of the sequence $\Gamma_1 \rightarrow \Delta_1, F_1(a)$, because the logical symbols in $\Gamma_1 \rightarrow \Delta_1, F_1(\tilde{T})$, which are not contained in $\Gamma_1 \rightarrow \Delta_1, F_1(a)$, are degenerate in $F_1(\tilde{T})$.

4.5. The remaining case, that the outermost logical symbol is $\neg$, is treated in the same way as in 4.3.

Chapter II. On the theory of natural numbers.

1. The system *RNN*.

We obtain the logical system *RNN* from *GLC* modifying it as follows:
1.1. Every beginning sequence of *RNN* is of the form $D \to D$ or of the form $a=b, A(a) \to A(b)$ or the "mathematische Grundsequenz" in Gentzen [1].

1.2. The following inference-schema called 'induction' is added:

$$\frac{A(a), \Gamma \to \Delta, A(a')}{A(0), \Gamma \to \Delta, A(t)}$$

where a is contained in none of $A(0), \Gamma, \Delta$, and t is an arbitrary term. $A(a)$ and $A(a')$ are called the *chief-formulas* and a is called an eigenvariable of this induction. We call every ancestor of $A(a)$ or $A(a')$ implicit.

1.3. The inference $\forall$ left on an f-variable of the form

$$\frac{F(V), \Gamma \to \Delta}{\forall \varphi F(\varphi), \Gamma \to \Delta}$$

is restricted by the condition that $\forall \varphi F(\varphi)$ is regular.

2. The purpose of the present chapter is to prove the following theorem.

Theorem 2. *RNN is consistent, that is,* $\to$ *is not provable in RNN.*

Proof. We introduce the inference 'substitution' in *RNN* too, and generalize the notion of a proof-figure of order n in *RNN*. Moreover, we assign an ordinal diagram to every sequence of a proof-figure of order n in *RNN* by the method as in 2.8, and by the following additional condition:
2.1. If $\mathfrak{J}$ is an inference 'induction' and $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are the upper and the lower sequences of $\mathfrak{J}$ respectively, then the ordinal diagram of $\mathfrak{S}_2$ is $(1; a+2, \sigma)$, where σ is the ordinal diagram of $\mathfrak{S}_1$, and a is the number of the proper logical symbols in one of the chief-formulas of $\mathfrak{J}$.

Then the consistency of *RNN* is easily proved by the proof of Theorem 1 of this paper and the "VJ-Reduktion" in Gentzen [1].

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