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On the theory of ordinal numbers, II.

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In his former paper [2], the author formalized a system of axioms for the theory of ordinal numbers, which will be denoted by Γ_0 in this paper. He proved in [2] that the consistency of the set theory follows from that of Γ_0. In this paper the proof for its converse is given, namely we shall show that Γ_0 is consistent, if the set theory is consistent.

For this purpose we shall give three formalizations Γ_1, Γ_2 and Γ_3 of set theory. Among them Γ_1 is the 'weakest' and the Γ_3 the 'strongest' one, and Gödel's axiom system A, B, C, D and E lies between Γ_1 and Γ_2.

More precisely, the system Γ_1 is the system obtained from Gödel's axiom system A, B, C, D and E by replacing any class variable by a well formed formula of Gentzen's *LK* (cf. [6]). The system Γ_2 is obtained again from Gödel's system by replacing any class variable by a variable for formulas. However, in Γ_2 we use the logic *HLC* [3], which means the predicate logic of the second order and the first level. The system Γ_3 is so constructed that it contains enough axioms for our purpose to reduce Γ_0 to Γ_3. In Γ_3 we use the logic *FLC* [2], which means the logic without variables for formulas but with bound variables for functions of any order.

We first prove the consistency of Γ_2 under the assumption of the consistency of Γ_1 (§2 of Chapter I). Then we show that Γ_3 is consistent, if Γ_2 is consistent (§3 of Chapter I). Finally, by using the restriction theory in the author's paper [4], we construct a model for Γ_0 in the set theory Γ_3 (§2 of Chapter II). Consequently the consistency of the ordinal number theory Γ_0 is proved, provided that the set theory Γ_1 is consistent (§2 of Chapter II).

Chapter I. Three formalizations of set theory.

§1. The first formalization.

We give the first formalization of set theory by the following axioms Γ_1 in *LK*.

1.1. $\forall x \forall y(\forall z(z \in x \leftrightarrow z \in y) \leftrightarrow x = y)$

1.2. $\forall x \forall y \forall z(z \in \{x, y\} \leftrightarrow x = z \vee y = z)$

1.3. $\forall x \forall y(y \in \mathrm{U}(x) \leftrightarrow \exists z(y \in z \wedge z \in x))$

1.4. $\forall x \neg (x \in 0)$

1.5. $0 \in \omega$

1.6. $\forall x (x \in \omega \vdash x \cup \{x\} \in \omega)$

where $\{a\}$ means $\{a, a\}$ and $a \cup b$ means $\mathrm{U}(\{a, b\})$.

1.7. $\forall x (\forall y (y \in x \vdash y \cup \{y\} \in x) \wedge 0 \in x \vdash \omega \subset x)$

where $a \subset b$ means $\forall x (x \in a \vdash x \in b)$.

1.8. $\forall x \forall y (y \in \mathrm{P}(x) \vdash y \subset x)$

1.9. $\forall x (\exists y (y \in x) \vdash \mathrm{h}(x) \in x)$

1.10. $\forall x \forall y (x = y \vdash \mathrm{h}(x) = \mathrm{h}(y))$

1.11. $\forall \mathfrak{A} \forall x (\forall u \forall v \forall w (\mathfrak{A}(u, v) \wedge \mathfrak{A}(u, w) \vdash v = w) \vdash \exists y \forall z (z \in y \vdash\dashv \exists u (u \in x \wedge \mathfrak{A}(u, z))))$

(For the meaning of $\forall \mathfrak{A}$, see [3], § 1.)

1.12. $\forall \mathfrak{A} (\exists x \mathfrak{A}(x) \vdash \exists x (\mathfrak{A}(x) \wedge \forall y \neg (y \in x \wedge \mathfrak{A}(y))))$

In these axioms the symbols or notations $\in$, $=$, $\{a, b\}$, $\mathrm{U}(a)$, 0, ω and $\mathrm{P}(a)$ correspond to ε, $=$, $\{ab\}$, $S(a)$ (the sum of a), 0, ω, and $\mathrm{P}(a)$ (the power class of a) in Gödel's [1], and $\mathrm{h}(a)$ means the function of choice. 1.7, 1.9 with 1.10, 1.11, and 1.12 are the axioms corresponding to Group C, Axiom 1 (the axiom of infinity); Axiom E (the axiom of choice); Group C, Axiom 4 (the axiom of replacement); and Axiom D in Gödel's [1], respectively.

By Gödel [1], M 1, we see easily that if A, B, C, D and E in Gödel [1] are consistent, or if the set theory constructed under Γ_0 in [2] is consistent, then Γ_1 is consistent. In this sense Γ_1 may be considered as the weakest formalization of set theory.

$\tilde{\Gamma}_1$ is the axiom system consisting of, besides those of Γ_1, the following axioms.

2.1. $\forall x \forall y \forall z (x = y \vdash (\mathrm{abz}(x, z) \vdash\dashv \mathrm{abz}(y, z)))$

2.2. $\forall x \forall y \forall z (y = z \vdash (\mathrm{abz}(x, y) \vdash\dashv \mathrm{abz}(x, z)))$

2.3. $\forall x \exists y (y \in \omega \wedge \mathrm{abz}(x, y))$

2.4. $\forall x \forall y \forall z (\mathrm{abz}(x, z) \wedge \mathrm{abz}(y, z) \vdash x = y)$

where we restrict the notation $\forall \mathfrak{A}$ in $\tilde{\Gamma}_1$ only to the case that $\mathfrak{A}$ has no predicate abz (cf. [5]).

§ 2. The second formalization.

In our former paper [3], § 1, we formalized the logic calculus *HLC*. The second formalization Γ_2 of set theory is given by the system of the following axioms in *HLC*.

1.1—1.9 and

1.13. $\forall \varphi \forall x \forall y (x = y \vdash (\varphi[x] \vdash\dashv \varphi[y]))$

1.14. $\forall \varphi \forall x (\forall u \forall v \forall w (\varphi[u, v] \wedge \varphi[u, w] \vdash v = w) \vdash \exists y \forall z (z \in y \vdash\dashv \exists u (u \in x \wedge \varphi[u, z])))$

1.15. $\forall \varphi (\exists x \varphi[x] \vdash \exists x (\varphi[x] \wedge \forall y \neg (y \in x \wedge \varphi[y])))$

1.14 and 1.15 are obtained from 1.11 and 1.12 respectively by replacing $\mathfrak{A}$ by φ (cf. [3]).

$\tilde{\Gamma}_2$ is the axiom system consisting of the axioms of Γ_2, and 2.1—2.4. In case of $\tilde{\Gamma}_2$, we restrict the inference-figure $\forall$ left on f-variable on *HLC* only to the case that in the variety V in the following inference

$$\frac{F(V), \Gamma \to \Delta}{\forall\varphi F(\varphi), \Gamma \to \Delta}$$

occurs neither $\forall\varphi$, nor $\exists\varphi$, nor the predicate abz.

Then by [3], § 1, we have the following proposition.

PROPOSITION 1. *If Γ_1 is consistent, then Γ_2 is consistent. Moreover, if $\tilde{\Gamma}_1$ is consistent, then $\tilde{\Gamma}_2$ is consistent.*

§ 3. The third formalization.

In our former paper [2], we defined the logical system *FLC*. We denote now by $f, g, \cdots$ free and bound functions if no confusion is to be feared. The third formalization Γ_3 of set theory is given by the system of the following axioms in *FLC*.

1.1—1.8 and

1.16. $\forall f \forall x \forall y(x=y \vdash f(x)=f(y))$

1.17. $\forall f \forall x \exists y \forall z(z \in y \vdash\dashv \exists u(u \in x \wedge z=f(u)))$

1.18. $\forall f(\exists x(f(x)=0) \vdash \exists x(f(x)=0 \wedge \forall y \neg(y \in x \wedge f(y)=0)))$

1.19. $\exists f \forall x(\exists y(y \in x) \vdash f(x) \in x)$

1.20. $\forall x(\mathrm{B}(x)=0 \vdash\dashv \exists y \exists z(x=<y, z> \wedge y \in z))$

where $<a, b>$ means $\{a, \{a, b\}\}$.

1.21. $\forall x(\mathrm{B}(x)=0 \vee \mathrm{B}(x)=1)$

where 1 means $\{0\}$.

1.22. $\forall f \forall g \forall x(\mathrm{Int}(f, g, x)=0 \vdash\dashv f(x)=0 \wedge g(x)=0)$

1.23. $\forall f \forall g \forall x(\mathrm{Int}(f, g, x)=0 \vee \mathrm{Int}(f, g, x)=1)$

1.24. $\forall f \forall x(\mathrm{G}(f, x)=0 \vdash\dashv \exists y \exists z(x=<y, z> \wedge z=f(y)))$

1.25. $\forall f \forall x(\mathrm{G}(f, x)=0 \vee \mathrm{G}(f, x)=1)$

1.26. $\forall f \forall x(\mathrm{C}(f, x)=0 \vdash\dashv \neg f(x)=0)$

1.27. $\forall f \forall x(\mathrm{C}(f, x)=0 \vee \mathrm{C}(f, x)=1)$

1.28. $\forall f \forall x(\mathrm{D}(f, x)=0 \vdash\dashv \exists y(f(<x, y>)=0))$

1.29. $\forall f \forall x(\mathrm{D}(f, x)=0 \vee \mathrm{D}(f, x)=1)$

1.30. $\forall f \forall x(\mathrm{V}(f, x)=0 \vdash\dashv \exists y \exists z(x=<y, z> \wedge f(y)=0))$

1.31. $\forall f \forall x(\mathrm{V}(f, x)=0 \vee \mathrm{V}(f, x)=1)$

1.32. $\forall f \forall x(\mathrm{Inv}(f, x)=0 \vdash\dashv \exists y \exists z(x=<y, z> \wedge f(<z, y>)=0))$

1.33. $\forall f \forall x(\mathrm{Inv}(f, x)=0 \vee \mathrm{Inv}(f, x)=1)$

1.34. $\forall f \forall x(\mathrm{Cnv}_2(f, x)=0 \vdash\dashv \exists u \exists v \exists w(x=<u, v, w> \wedge f(<v, w, u>)=0))$

1.35. $\forall f\forall x(\mathrm{Cnv}_2(f,x)=0\vee\mathrm{Cnv}_2(f,x)=1)$

1.36. $\forall f\forall x(\mathrm{Cnv}_3(f,x)=0\mapsto\exists u\exists v\exists w(x=<u,v,w>\wedge f(<u,w,v>)=0))$

1.37. $\forall f\forall x(\mathrm{Cnv}_3(f,x)=0\vee\mathrm{Cnv}_3(f,x)=1)$

1.38. $\forall f(\forall x\exists y(f(<x,y>)=0)\wedge\forall x\forall y\forall z(f(<x,y>)=0\wedge f(<x,z>)=0\vdash y=z)$
$\vdash\forall x(f(<x,\mathrm{Fon}(f,x)>)=0))$

1.39. $\forall f(\neg(\forall x\exists y(f(<x,y>)=0)\wedge\forall x\forall y\forall z(f(<x,y>)=0\wedge f(<x,z>)=0\vdash y=z))$
$\vdash\forall x(\mathrm{Fon}(f,x)=0))$

1.40. $\forall f\forall g\forall x(\mathrm{K}(f,g,x)=0\mapsto\exists y\exists z(x=<y,z>\wedge f(y)\in g(z)))$

1.41. $\forall f\forall g\forall x(\mathrm{K}(f,g,x)=0\vee\mathrm{K}(f,g,x)=1)$

1.42. $\forall f\forall x(\mathrm{L}(f,x)=0\mapsto\exists u\exists v\exists w(x=<u,v,w>\wedge u\in f(v,w)))$

1.43. $\forall f\forall x(\mathrm{L}(f,x)=0\vee\mathrm{L}(f,x)=1)$

To characterize the set theory by the help of the notion $\forall f$ instead of $\forall\varphi$, we have introduced here B, Int, C, Inv, etc., which mean the characteristic functions of E, $\cdot$, $\overline{}$, Cnv etc. in Gödel's [1].

$\tilde{\Gamma}_3$ is the axiom system consisting of the axioms of Γ_3, and 2.1—2.4.

In the rest of this section we shall prove the following proposition:

Proposition 2. *If Γ_2 is consistent, then Γ_3 is consistent. Moreover, if $\tilde{\Gamma}_2$ is consistent, then $\tilde{\Gamma}_3$ is consistent.*

First we define *-operation which transforms a formula in *FLC* to a formula in *HLC*, as in our former paper [3], § 2. *-operation is defined recursively as follows.

a^* is $\{x\}(x=a)$.

$(t_1\in t_2)^*$ is $\forall x\forall y(t_1^*(x)\wedge t_2^*(y)\vdash x\in y)$.

(Remark. Since $a=b$ is considered as $\forall x(x\in a\mapsto x\in b)$, $(t_1=t_2)^*$ is defined as $(\forall x(x\in t_1\mapsto x\in t_2))^*$.)

$(\mathrm{abz}(t_1,t_2))^*$ is $\forall x\forall y(t_1^*(x)\wedge t_2^*(y)\vdash\mathrm{abz}(x,y))$.

$(f(t_1,\cdots,t_n))^*(a)$ is $\forall x_1\cdots\forall x_n(t_1^*(x_1)\wedge\cdots\wedge t_n^*(x_n)\vdash\bar{f}[x_1,\cdots,x_n,a])$,

where $\bar{f}[*_1,\cdots,*_{n+1}]$ is considered as free variable of type $[0,\cdots,0]$.

$(\{t_1,t_2\})^*(a)$ is $\forall x\forall y(t_1^*(x)\wedge t_2^*(y)\vdash a=\{x,y\})$.

$(\mathrm{U}(t))^*(a)$ is $\forall x(t^*(x)\vdash a=\mathrm{U}(x))$.

$(\mathrm{P}(t))^*(a)$ is $\forall x(t^*(x)\vdash a=\mathrm{P}(x))$.

$(\mathrm{B}(t))^*(a)$ is $\forall x(t^*(x)\vdash(\exists y\exists z(x=<y,z>\wedge y\in z)\vdash a=0)$
$\wedge(\neg\exists y\exists z(x=<y,z>\wedge y\in z)\vdash a=1))$.

$(\mathrm{Int}(f,g,t))^*(a)$ is $\forall x(t^*(x)\vdash(\bar{f}[x,0]\wedge\bar{g}[x,0]\vdash a=0)$
$\wedge(\neg(\bar{f}[x,0]\wedge\bar{g}[x,0])\vdash a=1))$.

$(\mathrm{G}(f,t))^*(a)$ is $\forall x(t^*(x)\vdash(\exists y\exists z(x=<y,z>\wedge\bar{f}[y,z])\vdash a=0)$
$\wedge(\neg\exists y\exists z(x=<y,z>\wedge\bar{f}[y,z])\vdash a=1))$.

$(\mathrm{C}(f,t))^*(a)$ is $\forall x(t^*(x)\vdash(\neg\bar{f}[x,0]\vdash a=0)\wedge(\bar{f}[x,0]\vdash a=1))$.

$(\mathrm{D}(f,t))^*(a)$ is $\forall x(t^*(x)\vdash(\exists y\bar{f}[<x,y>,0]\vdash a=0)$
$\wedge(\neg\exists y\bar{f}[<x,y>,0]\vdash a=1))$.

$(\mathrm{V}(f, t))^*(a)$ is $\forall x(t^*(x) \vdash (\exists y \exists z(x = \langle y, z\rangle \wedge \bar{f}[y, 0]) \vdash a = 0)$
$\wedge (\neg \exists y \exists z(x = \langle y, z\rangle \wedge \bar{f}[y, 0]) \vdash a = 1))$.

$(\mathrm{Inv}(f, t))^*(a)$ is $\forall x(t^*(x) \vdash (\exists y \exists z(x = \langle y, z\rangle \wedge \bar{f}[\langle z, y\rangle, 0]) \vdash a = 0)$
$\wedge (\neg \exists y \exists z(x = \langle y, z\rangle \wedge \bar{f}[\langle z, y\rangle, 0]) \vdash a = 1))$.

$(\mathrm{Cnv}_2(f, t))^*(a)$ is
$\forall x(t^*(x) \vdash (\exists u \exists v \exists w(x = \langle u, v, w\rangle \wedge \bar{f}[\langle v, w, u\rangle, 0]) \vdash a = 0)$
$\wedge (\neg \exists u \exists v \exists w(x = \langle u, v, w\rangle \wedge \bar{f}[\langle v, w, u\rangle, 0]) \vdash a = 1))$.

$(\mathrm{Cnv}_3(f, t))^*(a)$ is
$\forall x(t^*(x) \vdash (\exists u \exists v \exists w(x = \langle u, v, w\rangle \wedge \bar{f}[\langle u, w, v\rangle, 0]) \vdash a = 0)$
$\wedge (\neg \exists u \exists v \exists w(x = \langle u, v, w\rangle \wedge \bar{f}[\langle u, w, v\rangle, 0]) \vdash a = 1))$.

$(\mathrm{Fon}(f, t))^*(a)$ is
$\forall x(t^*(x) \vdash (\forall y \exists z(\bar{f}[\langle y, z\rangle, 0])$
$\wedge \forall u \forall v \forall w(\bar{f}[\langle u, v\rangle, 0] \wedge \bar{f}[\langle u, w\rangle, 0] \vdash v = w) \vdash \bar{f}[\langle x, a\rangle, 0])$
$\wedge (\neg(\forall y \exists z(\bar{f}[\langle y, z\rangle, 0])$
$\wedge \forall u \forall v \forall w(\bar{f}[\langle u, v\rangle, 0] \wedge \bar{f}[\langle u, w\rangle, 0] \vdash v = w) \vdash a = 0)))$.

$(\mathrm{K}(f, g, t))^*(a)$ is
$\forall x(t^*(x) \vdash (\exists y \exists z \exists u \exists v(x = \langle y, z\rangle \wedge \bar{f}[y, u] \wedge \bar{g}[z, v] \wedge u \in v) \vdash a = 0)$
$\wedge (\neg \exists y \exists z \exists u \exists v(x = \langle y, z\rangle \wedge \bar{f}[y, u] \wedge \bar{g}[z, v] \wedge u \in v) \vdash a = 1))$.

$(\mathrm{L}(f, t))^*(a)$ is
$\forall x(t^*(x) \vdash (\exists y \exists z \exists u \exists v(\langle y, z, u\rangle = x \wedge \bar{f}[y, z, v] \wedge u \in v) \vdash a = 0)$
$\wedge (\neg \exists y \exists z \exists u \exists v(\langle y, z, u\rangle = x \wedge \bar{f}[y, z, v] \wedge u \in v) \vdash a = 1))$.

$(\{x\}t(x))^*$ is $\{x, y\}(t(x))^*(y)$.

$(\{x, y\}t(x, y))^*$ is $\{x, y, z\}(t(x, y))^*(z)$.

t^* is $\{x\}t^*(x)$.

$(\forall x A(x))^*$ is $\forall x(A(x))^*$,

which is sometimes denoted by $\forall x A^*(x)$.

$(A \wedge B)^*$ is $A^* \wedge B^*$.

$(\neg A)^*$ is $\neg A^*$.

$(\forall f A(f))^*$ is $\forall \bar{f}(\mathrm{un}(\bar{f}) \vdash (A(f))^*)$,

where $\mathrm{un}(\bar{f})$ is defined as

$$\forall x \exists y(\bar{f}[x, y]) \wedge \forall x \forall y \forall z(\bar{f}[x, y] \wedge \bar{f}[x, z] \vdash y = z)$$

if the type of f is (0), and as

$$\forall x \forall y \exists z(\bar{f}[x, y, z]) \wedge \forall x \forall y \forall u \forall v(\bar{f}[x, y, u] \wedge \bar{f}[x, y, v] \vdash u = v)$$

if the type of f is (0, 0) (cf. [1]), and $(A(f))^*$ is sometimes denoted by $A^*(f)$.

Now let A be a formula or a term in *FLC*. A is called *regular*, if and only if A satisfies the following conditions: Every special function or predicate contained in A is contained in $\bar{\Gamma}_3$; If A has a free function f, then f is of the type (0) or (0, 0). A functional $\{x\}t(x)$ is called *regular*, if and only if $t(a)$ is regular. Then we have the following propositions.

PROPOSITION 3. *Let t and $T(a)$ be regular terms and all the free functions* *contained in them be $f_1, f_2, \cdots$. Then the following sequences are provable.*

$$\Gamma_2,\ \mathrm{un}(\bar{f}_1),\ \mathrm{un}(\bar{f}_2),\cdots, t^*(a), t^*(b) \to a=b$$
$$\Gamma_2,\ \mathrm{un}(\bar{f}_1),\ \mathrm{un}(\bar{f}_2),\cdots \to \exists x(t^*(x))$$
$$\Gamma_2,\ \mathrm{un}(\bar{f}_1),\ \mathrm{un}(\bar{f}_2),\cdots \to \mathrm{un}((\{x\}T(x))^*)$$
$$\Gamma_2,\ \mathrm{un}(\bar{f}_1),\ \mathrm{un}(\bar{f}_2),\cdots \to \mathrm{un}((\{x,y\}T(x,y))^*).$$

Proof. We prove the proposition in the same way as in [3], § 2. In case of a newly introduced special function, we denote the result of $t^*(x)$ in the $*$-operation by $F(x,a)$. Then we see easily $c=d \to F(c,a) \mapsfrom\!\!\!\mapsto F(d,a)$ and $F(c,a), F(c,b) \to a=b$. So we can treat this also as in [3].

The following propositions 4—6 are also proved in the same way as in [3], § 2.

Proposition 4. *Let $A(t)$ and $T(t)$ be a regular formula and a regular term respectively, and all the free functions contained in them be $f_1, f_2 \cdots$. Then the following sequences are provable.*

$$\Gamma_2,\ \mathrm{un}(\bar{f}_1),\ \mathrm{un}(\bar{f}_2),\cdots \to (A(t))^* \mapsfrom\!\!\!\mapsto \forall x(t^*(x) \vdash A^*(x))$$
$$\Gamma_2,\ \mathrm{un}(\bar{f}_1),\ \mathrm{un}(\bar{f}_2),\cdots \to (T(t))^*(a) \mapsfrom\!\!\!\mapsto \forall x(t^*(x) \vdash (T(x))^*(a)).$$

Proposition 5. *Let $A(\{x\}t(x))$, $A(\{x,y\}t(x,y))$, $T(\{x\}t(x))$ and $T(\{x,y\}t(x,y))$ be regular formulas and terms, and all the free functions contained in them be $f_1, f_2, \cdots$. Then the following sequences are provable:*

$$\Gamma_2,\ \mathrm{un}(\bar{f}_1),\ \mathrm{un}(\bar{f}_2),\cdots \to (A(\{x\}t(x)))^* \mapsfrom\!\!\!\mapsto A^*(f)\begin{pmatrix}(\{x\}t(x))^* \\ \bar{f}\end{pmatrix}$$
$$\Gamma_2,\ \mathrm{un}(\bar{f}_1),\ \mathrm{un}(\bar{f}_2),\cdots \to (A(\{x,y\}t(x,y)))^* \mapsfrom\!\!\!\mapsto A^*(f)\begin{pmatrix}(\{x,y\}t(x,y))^* \\ \bar{f}\end{pmatrix}$$
$$\Gamma_2,\ \mathrm{un}(\bar{f}_1),\ \mathrm{un}(\bar{f}_2),\cdots \to (T(\{x\}t(x)))^*(a) \mapsfrom\!\!\!\mapsto (T(f))^*(a)\begin{pmatrix}(\{x\}t(x))^* \\ \bar{f}\end{pmatrix}$$
$$\Gamma_2,\ \mathrm{un}(\bar{f}_1),\ \mathrm{un}(\bar{f}_2),\cdots \to (T(\{x,y\}t(x,y)))^*(a) \mapsfrom\!\!\!\mapsto (T(f))^*(a)\begin{pmatrix}(\{x,y\}t(x,y))^* \\ \bar{f}\end{pmatrix}$$

where $B\begin{pmatrix}V \\ \alpha\end{pmatrix}$ denotes the formula obtained from the formula B by substituting the variety V whose type is same as that of α, for the free f-variable α in all the places in B where α occurs. The precise definition is given in [4], § 5.

Proposition 6. *Let $\Gamma \to \Delta$ be provable in FLC and every formula in Γ and Δ be regular, and the free functions contained in Γ and Δ be $f_1, f_2, \cdots$. Then the following sequence is provable in HLC:*

$$\Gamma_2,\ \mathrm{un}(\bar{f}_1),\ \mathrm{un}(\bar{f}_2),\cdots, \Gamma^* \to \Delta^*.$$

Proposition 7. *Let A be an arbitrary axiom of Γ_3. Then $\Gamma_2 \to A^*$ is provable. Moreover, let A be an arbitrary axiom of $\tilde{\Gamma}_3$. Then $\tilde{\Gamma}_2 \to A^*$ is provable.*

Proof. First we give some lemmas. Let a formula B and terms t, t_1 and t_2 contain no free function and consist of the predicates and the special functions contained in Γ_2 or $\tilde{\Gamma}_2$, respectively. Then we see easily the following lemmas.

LEMMA 1. *Under the above condition, the following sequences are provable:*

$$\Gamma_2 \rightarrow B^* \vdash\!\dashv B$$
$$\Gamma_2 \rightarrow t^*(a) \vdash\!\dashv a=t,$$

or

$$\tilde{\Gamma}_2 \rightarrow B^* \vdash\!\dashv B$$
$$\tilde{\Gamma}_2 \rightarrow t^*(a) \vdash\!\dashv a=t,$$

respectively.

LEMMA 2. *Under the above condition, we have the following sequences:*

$$\Gamma_2,\ \mathrm{un}(\bar{f}) \rightarrow (f(t)=a)^* \vdash\!\dashv \bar{f}[t, a]$$
$$\Gamma_2,\ \mathrm{un}(\bar{f}) \rightarrow (f(t_1, t_2)=a)^* \vdash\!\dashv \bar{f}[t_1, t_2, a],$$

or

$$\tilde{\Gamma}_2,\ \mathrm{un}(\bar{f}) \rightarrow (f(t)=a)^* \vdash\!\dashv \bar{f}[t, a]$$
$$\tilde{\Gamma}_2,\ \mathrm{un}(\bar{f}) \rightarrow (f(t_1, t_2)=a)^* \vdash\!\dashv \bar{f}[t_1, t_2, a],$$

respectively.

LEMMA 3. *Under the above condition, we have the following sequences:*

$$\Gamma_2,\ \mathrm{un}(\bar{f}) \rightarrow (f(t))^*(a) \vdash\!\dashv \bar{f}[t, a]$$
$$\Gamma_2,\ \mathrm{un}(\bar{f}) \rightarrow (f(t_1, t_2))^*(a) \vdash\!\dashv \bar{f}[t_1, t_2, a],$$

or

$$\tilde{\Gamma}_2,\ \mathrm{un}(\bar{f}) \rightarrow (f(t))^*(a) \vdash\!\dashv \bar{f}[t, a]$$
$$\tilde{\Gamma}_2,\ \mathrm{un}(\bar{f}) \rightarrow (f(t_1, t_2))^*(a) \vdash\!\dashv \bar{f}[t_1, t_2, a],$$

respectively.

The part of the proposition 7 containing the axioms which occur also in Γ_2 or $\tilde{\Gamma}_2$ is evident by means of the lemmas. To prove the remaining part, we use the following abbreviations:

$F(\mathrm{B})(a, b)$ for $(\exists x \exists y(a=\langle x, y\rangle \wedge x \in y) \vdash b=0) \wedge (\neg \exists x \exists y(a=\langle x, y\rangle \wedge x \in y) \vdash b=1)$,

$F(\mathrm{Int})(f, g, a, b)$ for $(f(a)=0 \wedge g(a)=0 \vdash b=0) \wedge (\neg(f(a)=0 \wedge g(a)=0) \vdash b=1)$,

$F(\mathrm{G})(f, a, b)$ for $(\exists x \exists y(a=\langle x, y\rangle \wedge f(x)=y) \vdash b=0) \wedge (\neg \exists x \exists y(a=\langle x, y\rangle \wedge f(x)=y) \vdash b=1)$,

$F(\mathrm{C})(f, a, b)$ for $(\neg f(a)=0 \vdash b=0) \wedge (f(a)=0 \vdash b=1)$,

$F(\mathrm{D})(f, a, b)$ for $(\exists x(f(\langle a, x\rangle)=0) \vdash b=0) \wedge (\neg \exists x(f(\langle a, x\rangle)=0) \vdash b=1)$,

$F(\mathrm{Inv})(f, a, b)$ for $(\exists x \exists y(a=\langle x, y\rangle \wedge f(\langle y, x\rangle)=0) \vdash b=0) \wedge (\neg \exists x \exists y(a=\langle x, y\rangle \wedge f(\langle y, x\rangle)=0) \vdash b=1)$,

$F(\mathrm{Cnv}_2)(f, a, b)$ for $(\exists x \exists y \exists z(a=\langle x, y, z\rangle \wedge f(\langle y, z, x\rangle)=0) \vdash b=0) \wedge (\neg \exists x \exists y \exists z(a=\langle x, y, z\rangle \wedge f(\langle y, z, x\rangle)=0) \vdash b=1)$,

$F(\mathrm{Cnv}_3)(f, a, b)$ for $(\exists x \exists y \exists z(a=\langle x, y, z\rangle \wedge f(\langle x, z, y\rangle)=0) \vdash b=0) \wedge (\neg \exists x \exists y \exists z(a=\langle x, y, z\rangle \wedge f(\langle x, z, y\rangle)=0) \vdash b=1)$,

$F(\mathrm{Fon})(f, a, b)$ for

$$(\forall x \exists y(f(\langle x, y\rangle)=0) \wedge \forall x \forall y \forall z(f(\langle x, y\rangle)=0 \wedge f(\langle x, z\rangle)=0 \vdash y=z) \vdash f(\langle a, b\rangle)=0)$$
$$\wedge (\neg(\forall x \exists y(f(\langle x, y\rangle)=0) \wedge \forall x \forall y \forall z(f(\langle x, y\rangle)=0 \wedge f(\langle x, z\rangle)=0 \vdash y=z)) \vdash b=0),$$

$F(\mathrm{K})(f, g, a, b)$ for $(\exists x \exists y \exists u \exists v(a=<x, y> \wedge f(x)=u \wedge g(y)=v \wedge u \in v) \vdash b=0)$
$\wedge(\neg \exists x \exists y \exists u \exists v(a=<x, y> \wedge f(x)=u \wedge g(y)=v \wedge u \in v) \vdash b=1)$,

and

$F(\mathrm{L})(f, a, b)$ for $(\exists x \exists y \exists z \exists u(a=<x, y, z> \wedge f(x, y)=u \wedge z \in u) \vdash b=0)$
$\wedge(\neg \exists x \exists y \exists z \exists u(a=<x, y, z> \wedge f(x, y)=u \wedge z \in u) \vdash b=1)$.

We shall show now that $(F(\mathrm{B})(a, \mathrm{B}(a)))^*$, $(F(\mathrm{Int})(f, g, a, \mathrm{Int}(f, g, a)))^*$, $(F(\mathrm{G})(f, a, \mathrm{G}(f, a)))^*$, $(F(\mathrm{C})(f, a, \mathrm{C}(f, a)))^*$, $(F(\mathrm{D})(f, a, \mathrm{D}(f, a)))^*$, $(F(\mathrm{V})(f, a, \mathrm{V}(f, a)))^*$, $(F(\mathrm{Inv})(f, a, \mathrm{Inv}(f, a)))^*$, $(F(\mathrm{Cnv}_2)(f, a, \mathrm{Cnv}_2(f, a)))^*$, $(F(\mathrm{Cnv}_3)(f, a, \mathrm{Cnv}_3(f, a)))^*$, $(F(\mathrm{Fon})(f, a, \mathrm{Fon}(f, a)))^*$, $(F(\mathrm{K})(f, g, a, \mathrm{K}(f, g, a)))^*$, and $(F(\mathrm{L})(f, a, \mathrm{L}(f, a)))^*$ are provable under Γ_2, provided that $\mathrm{un}(\bar{f})$ and $\mathrm{un}(\bar{g})$ hold. Then the proposition 7 will follow immediately. Since every case can be treated similarly, we show here only the case of $(F(\mathrm{K})(f, g, a, \mathrm{K}(f, g, a)))^*$. We have the following formula under Γ_2, provided that $\mathrm{un}(\bar{f})$ and $\mathrm{un}(\bar{g})$ hold:

$$(F(\mathrm{K})(f, g, a, \mathrm{K}(f, g, a)))^*$$
$$\dashv\vdash((\exists x \exists y \exists u \exists v(a=<x, y> \wedge f(x)=u \wedge g(y)=v \wedge u \in v) \vdash \mathrm{K}(f, g, a)=0)$$
$$\wedge(\neg \exists x \exists y \exists u \exists v(a=<x, y> \wedge f(x)=u \wedge g(y)=v \wedge u \in v) \vdash \mathrm{K}(f, g, a)=1))^*$$
$$\dashv\vdash \forall z((\mathrm{K}(f, g, a))^*(z) \vdash((\exists x \exists y \exists u \exists v(a=<x, y> \wedge f(x)=u \wedge g(y)=v \wedge u \in v \vdash z=0)$$
$$\wedge(\neg \exists x \exists y \exists u \exists v(a=<x, y> \wedge f(x)=u \wedge g(y)=v \wedge u \in v \vdash z=1))^*)$$

(by Proposition 4).

We have only to show

$$\Gamma_2,\ \mathrm{un}(\bar{f}),\ \mathrm{un}(\bar{g}),\ (K(f, g, a))^*(b)$$
$$\rightarrow((\exists x \exists y \exists u \exists v(a=<x, y> \wedge f(x)=u \wedge g(y)=v \wedge u \in v) \vdash b=0)$$
$$\wedge(\neg \exists x \exists y \exists u \exists v(a=<x, y> \wedge f(x)=u \wedge g(y)=v \wedge u \in v) \vdash b=1))^*.$$

By using the lemmas, we see the right side of the sequence is equivalent to

$$(\exists x \exists y \exists u \exists v(a=<x, y> \wedge \bar{f}[x, u] \wedge \bar{g}[y, v] \wedge u \in v) \vdash b=0)$$
$$\wedge(\neg \exists x \exists y \exists u \exists v(a=<x, y> \wedge \bar{f}[x, u] \wedge \bar{g}[y, v] \wedge u \in v) \vdash b=1),$$

which is $(\mathrm{K}(f, g, a))^*(b)$. q. e. d.

Proposition 2 follows now easily from Propositions 6 and 7.

Chapter II. The consistency of the theory of ordinal numbers derived from that of Γ_3.

§1. Metatheorems on Γ_3.

A formula or a term in *FLC* will be called *strictly regular,* if and only if every special function and predicate contained in it are contained in Γ_3, and every free and bound function contained in it is of type (0).

PROPOSITION 8. *Let* $A(f_1, \cdots, f_n, a_1, \cdots, a_m)$ *be a strictly regular formula and have no free functions and variables other than* $f_1, \cdots, f_n, a_1, \cdots, a_m$ *explicitly indicated in* $A(f_1, \cdots, f_n, a_1, \cdots, a_m)$. *Then there exists a regular formula* $B(f_1, \cdots, f_n, a_1, \cdots, a_m)$ *containing neither special function, nor predicate other than* $\in$, *nor free* *functions and variables other than* $f_1,\cdots,f_n, a_1,\cdots, a_m$ *explicitly indicated in* $B(f_1,\cdots, f_n, a_1,\cdots, a_m)$, *such that the following sequence is provable:*

$$\Gamma_3 \rightarrow A(f_1,\cdots,f_n, a_1,\cdots, a_m) \mapsto\!\dashv B(f_1,\cdots,f_n, a_1,\cdots, a_m).$$

Proof. Since $t_1=t_2$ may be substituted by $\forall x(x\in t_1 \mapsto\!\dashv x\in t_2)$, we may assume that $A(f_1,\cdots,f_n, a_1,\cdots, a_m)$ has no predicate other than $\in$. Clearly we have only to prove the proposition for the form $t_1\in t_2$ in $A(f_1,\cdots f_n, a_1,\cdots, a_m)$. (Here we treat the bound functions and variables contained in t_1 and t_2 as free ones.) Then the proposition is proved by the usual method by the successive application of the following substitutions:

$t_1=t_2\vee t_1=t_3$ for $t_1\in\{t_2, t_3\}$,

$\exists x(\forall y(y\in x\mapsto\!\dashv y=t_2\vee y=t_3)\wedge x\in t_1)$ for $\{t_2, t_3\}\in t_1$,

$\exists x(x=t_2\wedge t_1\in f(x))$ for $t_1\in f(t_2)$

where t_2 contains special functions,

$\exists x(x=t_1\wedge f(x)\in t_2)$ for $f(t_1)\in t_2$

where t_1 contains special functions,

$\exists x(t_1\in x\wedge x\in t_2)$ for $t_1\in \mathrm{U}(t_2)$,

$\exists x(\forall y(y\in x\mapsto\!\dashv \exists z(y\in z\wedge z\in t_1))\wedge x\in t_2)$ for $\mathrm{U}(t_1)\in t_2$,

$t_1\subset t_2$ for $t_1\in \mathrm{P}(t_2)$,

$\exists x(\forall y(y\in x\mapsto\!\dashv y\subset t_1)\wedge x\in t_2)$ for $\mathrm{P}(t_1)\in t_2$,

$(\neg\exists x\exists y(t_2=<x, y>\wedge x\in y))\wedge t_1=0$ for $t_1\in \mathrm{B}(t_2)$,

$(\exists x\exists y(t_1=<x, y>\wedge x\in y)\vdash 0\in t_2)\wedge(\neg\exists x\exists y(t_1=<x, y>\wedge x\in y)\vdash 1\in t_2)$
for $\mathrm{B}(t_1)\in t_2$,

$t_1=0\wedge\neg(T_1(t)=0\wedge T_2(t)=0)$ for $t_1\in \mathrm{Int}(\{x\}T_1(x), \{x\}T_2(x), t)$,

$(T_1(t)=0\wedge T_2(t)=0\wedge 0\in t_1)\vee(\neg(T_1(t)=0\wedge T_2(t)=0)\wedge 1\in t_1)$
for $\mathrm{Int}(\{x\}T_1(x), \{x\}T_2(x), t)\in t_1$,

$\neg\exists x\exists y(t=<x, y>\wedge y=T(x))\wedge t_1=0$ for $t_1\in \mathrm{G}(\{x\}T(x), t)$,

$(\exists x\exists y(t=<x, y>\wedge y=T(x))\wedge 0\in t_1)\vee(\neg\exists x\exists y(t=<x, y>\wedge y=T(x))\wedge 1\in t_1)$
for $\mathrm{G}(\{x\}T(x), t)\in t_1$,

$t_1=0\wedge T(t)=0$ for $t_1\in \mathrm{C}(\{x\}T(x), t)$,

$(\neg(T(t)=0)\wedge 0\in t_1)\vee(T(t)=0\wedge 1\in t_1)$ for $\mathrm{C}(\{x\}T(x), t)\in t_1$,

$\neg\exists x(T(<t, x>)=0)\wedge t_1=0$ for $t_1\in \mathrm{D}(\{x\}T(x), t)$,

$(\exists x(T(<t, x>)=0)\wedge 0\in t_1)\vee(\neg\exists x(T(<t, x>)=0)\wedge 1\in t_1)$
for $\mathrm{D}(\{x\}T(x), t)\in t_1$,

$\neg\exists x\exists y(t=<x, y>\wedge T(x)=0)\wedge t_1=0$ for $t_1\in \mathrm{V}(\{x\}T(x), t)$,

$(\exists x\exists y(t=<x, y>\wedge T(x)=0)\wedge 0\in t_1)\vee(\neg\exists x\exists y(t=<x, y>\wedge T(x)=0)\wedge 1\in t_1)$
for $\mathrm{V}(\{x\}T(x), t)\in t_1$,

$\neg\exists x\exists y(t=<x, y>\wedge T(<y, x>)=0)\wedge t_1=0$ for $t_1\in \mathrm{Inv}(\{x\}T(x), t)$,

$(\exists x\exists y(t=<x, y>\wedge T(<y, x>)=0)\wedge 0\in t_1)\vee$
$(\neg\exists x\exists y(t=<x, y>\wedge T(<y, x>)=0)\wedge 1\in t_1)$ for $\mathrm{Inv}(\{x\}T(x), t)\in t_1$,

$\neg\exists x\exists y\exists z(t=<x, y, z>\wedge T(<y, z, x>)=0)\wedge t_1=0$

for $t_1 \in \mathrm{Cnv}_2(\{x\}T(x), t)$,

$(\exists x \exists y \exists z(t = <x, y, z> \wedge T(<y, z, x>) = 0) \wedge 0 \in t_1) \vee$
$(\neg \exists x \exists y \exists z(t = <x, y, z> \wedge T(<y, z, x>) = 0) \wedge 1 \in t_1)$

for $\mathrm{Cnv}_2(\{x\}T(x), t) \in t_1$,

$\neg \exists x \exists y \exists z(t = <x, y, z> \wedge T(<x, z, y>) = 0) \wedge t_1 = 0$

for $t_1 \in \mathrm{Cnv}_3(\{x\}T(x), t)$,

$(\exists x \exists y \exists z(t = <x, y, z> \wedge T(<x, z, y>) = 0) \wedge 0 \in t_1) \vee$
$(\neg \exists x \exists y \exists z(t = <x, y, z> \wedge T(<x, z, y>) = 0) \wedge 1 \in t_1)$

for $\mathrm{Cnv}_3(\{x\}T(x), t) \in t_1$,

$\forall x \exists y(T(<x, y>) = 0) \wedge \forall x \forall y \forall z(T(<x, y>) = 0 \wedge T(<x, z>) = 0 \vdash y = z) \wedge$
$\exists x(T(<t, x>) = 0 \wedge t_1 \in x)$ for $t_1 \in \mathrm{Fon}(\{x\}T(x), t)$,

$(\neg(\forall x \exists y(T(<x, y>) = 0) \wedge \forall x \forall y \forall z(T(<x, y>) = 0 \wedge T(<x, z>) = 0 \vdash y = z)) \wedge$
$0 \in t_1) \vee (\forall x \exists y(T(<x, y>) = 0) \wedge \forall x \forall y \forall z(T(<x, y>) = 0 \wedge T(<x, z>) = 0$
$\vdash y = z) \wedge (\exists x(T(<t, x>) = 0 \wedge x \in t_1))$ for $\mathrm{Fon}(\{x\}T(x), t) \in t_1$,

$\neg \exists x \exists y(t = <x, y> \wedge T_1(x) \in T_2(y)) \wedge t_1 = 0$ for $t_1 \in \mathrm{K}(\{x\}T_1(x), \{x\}T_2(x), t)$,

$(\exists x \exists y(t = <x, y> \wedge T_1(x) \in T_2(y)) \wedge 0 \in t_1) \vee$
$(\neg \exists x \exists y(t = <x, y> \wedge T_1(x) \in T_2(y)) \wedge 1 \in t_1)$ for $\mathrm{K}(\{x\}T_1(x), \{x\}T_2(x), t) \in t_1$,

$\neg \exists x \exists y \exists z(t = <x, y, z> \wedge z \in T(x, y)) \wedge t_1 = 0$ for $t_1 \in \mathrm{L}(\{x, y\}T(x, y), t)$, and

$(\exists x \exists y \exists z(t = <x, y, z> \wedge z \in T(x, y)) \wedge 0 \in t_1) \vee$
$(\neg \exists x \exists y \exists z(t = <x, y, z> \wedge z \in T(x, y)) \wedge 1 \in t_1)$ for $\mathrm{L}(\{x, y\}T(x, y), t) \in t_1$.

A formula is called *primitive*, if and only if it is strictly regular, and it has neither $\forall f$ nor $\exists f$.

PROPOSITION 9. *Let $A(f_1, \cdots, f_n, a_1, \cdots, a_m)$ be primitive and all the free functions and variables in it be indicated in $A(f_1, \cdots, f_n, a_1, \cdots, a_m)$. Then there exists functional $\{x\}T(x, f_1, \cdots, f_n)$ containing no $a_1, \cdots, a_m$ such that the following sequence is provable.*

$$\Gamma_3 \rightarrow A(f_1, \cdots, f_n, a_1, \cdots, a_m) \leftrightarrow <a_1, \cdots, a_m> \in \{x\}T(x, f_1, \cdots, f_n)$$

where $a \in \{x\}t(x)$ means $t(a) = 0$.

PROOF. By Proposition 8, we may assume that $A(f_1, \cdots, f_n, a_1, \cdots, a_m)$ has neither special function nor predicate other than $\in$. Since $f_{i_1}(\cdots(f_{i_n}(a))\cdots) \in f_{j_1}(\cdots(f_{j_m}(b))\cdots)$ is equivalent to $<a, b> \in \{x\}\mathrm{K}(\{y\}f_{i_1}(\cdots(f_{i_n}(y))\cdots), \{y\}f_{j_1}(\cdots(f_{j_m}(y))\cdots), x)$ under Γ_3, this proposition is easily proved in the same way as in $M1$ of Gödel [1].

PROPOSITION 10. *Let $A(f_1, \cdots, f_n, a_1, \cdots, a_m, a, b)$ be primitive and all the free functions and variables in it be indicated in $A(f_1, \cdots, f_n, a_1, \cdots, a_m, a, b)$, and the following sequences be provable:*

$$\Gamma_3, \Gamma \rightarrow \forall x \exists y A(f_1, \cdots, f_n, a_1, \cdots, a_m, x, y)$$

$$\Gamma_3, \Gamma, A(f_1, \cdots, f_n, a_1, \cdots, a_m, a, b), A(f_1, \cdots, f_n, a_1, \cdots, a_m, a, c) \rightarrow b = c.$$

Then there exists a functional $\{x\}T(x, f_1, \cdots, f_n, a_1, \cdots, a_m)$ containing neither a nor *b such that the following sequence is provable:*

$$\Gamma_3, \Gamma \rightarrow A(f_1,\cdots,f_n, a_1,\cdots, a_m, a, T(a, f_1,\cdots,f_n, a_1,\cdots, a_m)).$$

PROOF. By Proposition 9 there exists a functional $\{x\}T_0(x, f_1,\cdots,f_n, a_1,\cdots, a_m)$ containing neither a nor b such that the following sequence is provable:

$$\Gamma_3 \rightarrow A(f_1,\cdots,f_n, a_1,\cdots, a_m, a, b) \mapsto\!\dashv {<}a, b{>} \in \{x\}T_0(x, f_1,\cdots,f_n, a_1,\cdots, a_m).$$

If we define $T(a, f_1,\cdots,f_n, a_1,\cdots, a_m)$ as

$$\mathrm{Fon}(\{x\}T_0(x, f_1,\cdots,f_n, a_1,\cdots, a_m), a),$$

then we have easily the proposition.

In case of Proposition 10 we define a term $T(a, f_1,\cdots,f_n, a_1,\cdots, a_m)$ by $\iota x A(f_1,\cdots,f_n, a_1,\cdots, a_m, a, x)$.

§ 2. Construction of a model of the theory of ordinal numbers under Γ_3.

Now we construct a model of the theory of ordinal numbers under Γ_3. In this section, we use the abbreviated notation $\Gamma \rightarrow \Delta$ for $\Gamma_3, \Gamma \rightarrow \Delta$.

$\mathrm{ord}(a)$ is defined as

$$\forall x \forall y (x \in a \wedge y \in a \vdash x \in y \vee x = y \vee y \in x) \wedge \forall x (x \in a \vdash x \subset a).$$

We have easily

$$\rightarrow \mathrm{ord}(0)$$

$$\mathrm{ord}(a), \mathrm{ord}(b) \rightarrow a \in b \mapsto\!\dashv a \subsetneqq b$$

where $a \subsetneqq b$ means $a \subset b \wedge \neg a = b$ as usual,

$$\mathrm{ord}(a), \mathrm{ord}(b) \rightarrow a \in b,\ a = b,\ b \in a$$

$$\mathrm{ord}(a) \rightarrow 0 \in a,\ a = 0$$

$$\mathrm{ord}(a),\ b \in a \rightarrow \mathrm{ord}(b)$$

$$\mathrm{ord}(a) \rightarrow \mathrm{ord}(a \cup \{a\})$$

$$\rightarrow \mathrm{ord}(\omega)$$

$$\mathrm{ord}(a), \mathrm{ord}(b),\ a \cup \{a\} = b \cup \{b\} \rightarrow a = b$$

$$\mathrm{ord}(a),\ b \in a \rightarrow b \cup \{b\} \subset a$$

$$\exists x (\mathrm{ord}(x) \wedge x \in f) \rightarrow \exists x (\mathrm{ord}(x) \wedge x \in f \wedge \forall y (\mathrm{ord}(y) \wedge y \in f \vdash x \subset y)).$$

We define $\mathrm{Max}(a)$ by

$$\iota z (\exists x \exists y (\mathrm{ord}(x) \wedge \mathrm{ord}(y) \wedge a = {<}x, y{>} \wedge (x \subset y \vdash z = y) \wedge (y \in x \vdash z = x)))$$

and $\max(a, b)$ by $\mathrm{Max}({<}a, b{>})$. Then we have

$$\mathrm{ord}(a), \mathrm{ord}(b) \rightarrow \mathrm{ord}(\max(a, b))$$

$$\mathrm{ord}(a),\ b \in a \rightarrow a = \max(a, b)$$

$$\mathrm{ord}(b),\ a \in b \rightarrow b = \max(a, b)$$

$$\mathrm{ord}(a) \rightarrow a = \max(a, a)$$

In the same way, we can define N, Iq, Eq, and δ satisfying the following conditions (cf. [2], Chapter II, §1 for their definition):

$$\mathrm{ord}(a),\ 0 \in a \rightarrow N(a) = 0$$

$$\rightarrow N(0) = 1$$

$\mathrm{ord}(b),\ a\in b\rightarrow \mathrm{Iq}(a,b)=0$

$\mathrm{ord}(a),\ \mathrm{ord}(b)\rightarrow \mathrm{ord}(\mathrm{Iq}(a,b))$

$\mathrm{ord}(a),\ \mathrm{ord}(b),\ \mathrm{Iq}(a,b)=0\rightarrow a\in b$

$\mathrm{ord}(a)\rightarrow \mathrm{Eq}(a,a)=0$

$\mathrm{ord}(a),\ \mathrm{ord}(b)\rightarrow \mathrm{ord}(\mathrm{Eq}(a,b))$

$\mathrm{ord}(a),\ \mathrm{ord}(b),\ \mathrm{Eq}(a,b)=0\rightarrow a=b$

$\mathrm{ord}(a)\rightarrow \delta(a')=a$

where a' means $a\cup\{a\}$.

$\mathrm{ord}(a)\rightarrow \mathrm{ord}(\delta(a))$

$\mathrm{ord}(a),\ \mathrm{ord}(b),\ a\in b\rightarrow \delta(a)\subset\delta(b)$

$a\in\omega\rightarrow (\delta(a))'=a,\ a=0.$

$\mathrm{cl}(f)$ is defined as $\forall x(\mathrm{ord}(x)\vdash\mathrm{ord}(f(x)))$ if the type of f is (0), and $\forall x\forall y(\mathrm{ord}(x)\wedge\mathrm{ord}(y)\vdash\mathrm{ord}(f(x,y)))$ if the type of f is (0, 0).

We define $\sup(f,a)$ by

$$\iota z(\mathrm{ord}(z)\wedge(\mathrm{cl}(f)\wedge\mathrm{ord}(a)\vdash(\forall x(x\in a\vdash f(x)\in z)$$
$$\wedge\forall x(\mathrm{ord}(x)\wedge\forall y(y\in a\vdash f(y)\in x)\vdash z\subset x)))).$$

We see easily the following sequences:

$\rightarrow \mathrm{ord}(\sup(f,a))$

$\mathrm{cl}(f),\ \mathrm{ord}(a),\ b\in a\rightarrow f(b)\in\sup(f,a)$

$\mathrm{cl}(f),\ \mathrm{ord}(a),\ \mathrm{ord}(b),\ \forall x(x\in a\vdash f(x)\in b)\rightarrow \sup(f,a)\subset b.$

In the same way we can define Min(Minimum), Con(Contraction), S(Sum) and mg(minimum gap) so as to satisfy the following conditions:

$\mathrm{cl}(f)\rightarrow \mathrm{ord}(\mathrm{Min}(f))$

$\mathrm{cl}(f),\ \mathrm{ord}(a),\ a\in f\rightarrow f(\mathrm{Min}(f))=0\wedge\mathrm{Min}(f)\subset a$

$\mathrm{cl}(f),\ \mathrm{ord}(a),\ \mathrm{ord}(b)\rightarrow \mathrm{ord}(\mathrm{Con}(f,a,b))$

$\mathrm{cl}(f),\ \mathrm{ord}(a),\ b\in a\rightarrow \mathrm{Con}(f,a,b)=f(b)$

$\mathrm{cl}(f),\ \mathrm{ord}(a),\ \mathrm{ord}(b),\ a\subset b\rightarrow \mathrm{Con}(f,a,b)=0$

$\mathrm{cl}(f),\ \mathrm{cl}(g),\ \mathrm{cl}(h),\ \mathrm{ord}(a)\rightarrow \mathrm{ord}(S(f,g,h,a))$

$\mathrm{cl}(f),\ \mathrm{cl}(g),\ \mathrm{cl}(h),\ \mathrm{ord}(a),\ f(a)=0\rightarrow S(f,g,h,a)=g(a)$

$\mathrm{cl}(f),\ \mathrm{cl}(g),\ \mathrm{cl}(h),\ \mathrm{ord}(a),\ 0\in f(a)\rightarrow S(f,g,h,a)=h(a)$

$\mathrm{cl}(f),\ \mathrm{ord}(a)\rightarrow \mathrm{ord}(\mathrm{mg}(f,a))$

$\mathrm{cl}(f),\ \mathrm{ord}(a),\ b\in a,\ f(b)=\mathrm{mg}(f,a)\rightarrow$

$\mathrm{cl}(f),\ \mathrm{ord}(a),\ \mathrm{ord}(b),\ \forall x(x\in a\vdash\neg f(x)=b)\rightarrow \mathrm{mg}(f,a)\subset b.$

$\mathrm{cl}(\boldsymbol{f})$ is defined as $\forall f\forall x(\mathrm{cl}(f)\wedge\mathrm{ord}(x)\vdash\mathrm{ord}(\boldsymbol{f}(f,x)))$ if the type of $\boldsymbol{f}$ is ((0), 0).

We use $A(\boldsymbol{f},a,b)$ for the abbreviation of

$$\mathrm{ord}(a)\wedge\exists x(\forall u\forall v\forall w(<u,v>\in x\wedge<u,w>\in x\vdash v=w\wedge\mathrm{ord}(v))$$
$$\wedge\forall y(y\subset a\wedge\mathrm{ord}(y)\vdash\exists z(<y,z>\in x))$$
$$\wedge\forall y\forall z(y\subset a\wedge\mathrm{ord}(y)\wedge<y,z>\in x\vdash$$
$$z=\boldsymbol{f}(\{u\}\mathrm{Con}(\{v\}\mathrm{Fon}(\{w\}B(<w,x>),v),y,u),y))$$
$$\wedge<a,b>\in x).$$

We define $T_0(\boldsymbol{f}, x, y)$ as $\boldsymbol{f}(\{u\}\mathrm{Con}(\{v\}\mathrm{Fon}(\{w\}B(<w, x>), v), y, u), y)$. Then clearly we can define the functionals A_1, A_2, A_3 and A_4 satisfying the following formula:

$$\begin{aligned} A(\boldsymbol{f}, a, b) \leftrightarrow\ & <a, b> \in A_1 \\ & \wedge \exists x(<a, b, x> \in A_2 \wedge \forall y \forall z(<a, b, x, y, z> \in A_3 \rightarrow z = T_0(\boldsymbol{f}, x, y)) \\ & \wedge <a, b, x> \in A_4). \end{aligned}$$

Now, we can define $\{x\}T_1(\boldsymbol{f}, x)$ which satisfies the following formula under Γ_3:

$$\begin{aligned} z = T_0(\boldsymbol{f}, x, y) & \leftrightarrow \forall u(u \in z \leftrightarrow u \in T_0(\boldsymbol{f}, x, y)) \\ & \leftrightarrow \forall u(u \in z \leftrightarrow <x, y, u> \in \{v\}\mathrm{L}(\{w_1, w_2\}T_0(\boldsymbol{f}, w_1, w_2), v)) \\ & \leftrightarrow <x, y, z> \in \{u\}T_1(\boldsymbol{f}, u). \end{aligned}$$

From this it follows that we can define $\{x\}T_2(\boldsymbol{f}, x)$ such that the following formula holds under Γ_3:

$$A(\boldsymbol{f}, a, b) \leftrightarrow <a, b> \in \{x\}T_2(\boldsymbol{f}, x).$$

We define $\mathrm{Rec}(\boldsymbol{f}, a)$ (read as "the value for the argument a of the function recursively generated by $\boldsymbol{f}$) as $\mathrm{Fon}(\{x\}T_2(\boldsymbol{f}, x), a)$. Then we have the following sequences:

$$\mathrm{cl}(\boldsymbol{f}),\ \mathrm{ord}(a) \rightarrow \mathrm{ord}(\mathrm{Rec}(\boldsymbol{f}, a))$$

$$\mathrm{cl}(\boldsymbol{f}),\ \mathrm{ord}(a) \rightarrow \mathrm{Rec}(\boldsymbol{f}, a) = \boldsymbol{f}(\{u\}\mathrm{Con}(\{v\}\mathrm{Rec}(\boldsymbol{f}, v), a, u), a).$$

$R(a, b, c, d)$ is defined as

$$\begin{aligned} & \mathrm{ord}(a) \wedge \mathrm{ord}(b) \wedge \mathrm{ord}(c) \wedge \mathrm{ord}(d) \\ & \wedge (\max(a, b) < \max(c, d) \vee (\max(a, b) = \max(c, d) \wedge (b < d \vee (b = d \wedge a < c)))) \end{aligned}$$

where $a < b$ means $a \in b$ if $\mathrm{ord}(a)$ and $\mathrm{ord}(b)$. We have easily the following sequences:

$$\begin{aligned} & \mathrm{ord}(a),\ \mathrm{ord}(b),\ \mathrm{ord}(c),\ \mathrm{ord}(d) \\ & \qquad \rightarrow R(a, b, c, d),\ R(c, d, a, b),\ a = c \wedge b = d\ ; \\ & R(a, b, c, d),\ R(c, d, a, b) \rightarrow\ ; \\ & \exists x \exists y(\mathrm{ord}(x) \wedge \mathrm{ord}(y) \wedge <x, y> \in f) \\ & \qquad \rightarrow \exists x \exists y(\mathrm{ord}(x) \wedge \mathrm{ord}(y) \wedge <x, y> \in f \\ & \qquad \wedge \forall u \forall v(\mathrm{ord}(u) \wedge \mathrm{ord}(v) \wedge <u, v> \in f \rightarrow (x = u \wedge y = v) \vee R(x, y, u, v))). \end{aligned}$$

We define $J(a)$ by

$$\begin{aligned} \iota r(& \exists x \exists y \exists z(\mathrm{ord}(x) \wedge \mathrm{ord}(y) \wedge a = <x, y> \\ & \wedge \forall u \forall v(R(u, v, x, y) \vee (x = u \wedge y = v) \rightarrow \exists s(\mathrm{ord}(s) \wedge <u, v, s> \in z) \\ & \wedge \forall s(s \in r \rightarrow \exists u \exists v(\mathrm{ord}(u) \wedge \mathrm{ord}(v) \wedge <u, v, s> \in z) \\ & \wedge \forall u \forall v \forall s \forall u_0 \forall v_0 \forall t(<u, v, s> \in z \wedge <u_0, v_0, t> \in z \\ & \qquad \rightarrow (R(u, v, u_0, v_0) \leftrightarrow s \in t)) \\ & \wedge <0, 0, 0> \in z \wedge <x, y, r> \in z) \wedge \mathrm{ord}(r)) \end{aligned}$$

and define $j(a, b)$ by $J(<a, b>)$ which corresponds to $J_0'<ab>$ in Gödel's [1]. We have easily the following sequences:

$$\mathrm{ord}(a),\ \mathrm{ord}(b) \rightarrow \mathrm{ord}(j(a, b))\ ;$$

$R(a, b, c, d) \to j(a, b) \in j(c, d)$;

$j(a, b),\ j(c, d),\ \mathrm{ord}(a),\ \mathrm{ord}(b),\ \mathrm{ord}(c),\ \mathrm{ord}(d) \to R(a, b, c, d)$;

$\mathrm{ord}(a) \to \exists x \exists y(a = j(x, y))$.

We define $g_1(a)$ by $\iota x \exists y(a = j(x, y) \wedge \mathrm{ord}(x))$ and $g_2(a)$ by $\iota x \exists y(a = j(y, x) \wedge \mathrm{ord}(x))$. Then we have easily the following sequences:

$\mathrm{ord}(a) \to \mathrm{ord}(g_1(a))$;

$\mathrm{ord}(a) \to \mathrm{ord}(g_2(a))$;

$\mathrm{ord}(a) \to j(g_1(a), g_2(a)) = a$;

$\mathrm{ord}(a),\ \mathrm{ord}(b) \to g_1(j(a, b)) = a$;

$\mathrm{ord}(a),\ \mathrm{ord}(b) \to g_2(j(a, b)) = b$.

By the usual method we have the following sequence:

$$\mathrm{ord}(a) \to \exists x(\mathrm{ord}(x) \wedge \forall y(\forall u(\mathrm{ord}(u) \vdash \exists v(\mathrm{ord}(v) \wedge \langle u, v \rangle \in y)) \wedge \forall u \forall v \forall w(\langle u, v \rangle \in y \wedge \langle u, w \rangle \in y \vdash v = w) \vdash \exists z(z \in x \wedge \forall u \neg(\langle u, z \rangle \in y \wedge u \in a)))).$$

Let us denote this sequence shortly by

$\mathrm{ord}(a) \to \exists x B(x, a)$.

Further we define $\chi(a)$ by $B(b, a) \wedge \forall x(B(x, a) \vdash b \subset x)$. We see clearly

$\forall f \forall x(\mathrm{ord}(x) \wedge \mathrm{cl}(f) \vdash \mathrm{mg}(f, x) < \chi(x))$

and

$\forall x \forall y(\mathrm{ord}(x) \wedge \mathrm{ord}(y) \wedge \forall f(\mathrm{cl}(f) \vdash \mathrm{mg}(f, x) < y) \vdash \chi(x) \subset y)$.

By the restriction theory, [4], § 7, we have then the following proposition.

Proposition 11. *If Γ_3 is consistent, then Γ_0 and the following axioms are consistent.*

$\forall f \forall x \forall y(y < x \vdash f(y) < \sup(f, x))$,

$\forall f \forall x \forall y(\forall z(z < x \vdash f(z) < y) \vdash \sup(f, x) \leqq y)$,

$\forall f \forall x \forall y(\mathrm{mg}(f, x) = f(y) \vdash y \geqq x)$,

$\forall f \forall x \forall y(\forall z(y = f(z) \vdash z \geqq x) \vdash \mathrm{mg}(f, x) \leqq y)$,

$\forall f \forall x(\mathrm{mg}(f, x) < \chi(x))$

$\forall x \forall y(\forall f(\mathrm{mg}(f, x) < y) \vdash \chi(x) \leqq y)$.

Γ_0 and these axioms are simply denoted by Γ_{00}. Γ_{00} and the axioms 2.1, 2.2, 2.3 and 2.4 are denoted by $\tilde{\Gamma}_{00}$. Then we see easily the following proposition by [5].

Proposition 12. *If Γ_0 or Γ_1 is consistent, then both $\tilde{\Gamma}_{00}$ and $\tilde{\Gamma}_3$ are consistent.*

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