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On the genus of the alternating knot, I.

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F. Frankel and L. Pontrjagin [2] and H. Seifert [5] have given methods of construction of an orientable closed surface spanning a given knot i. e. having a given knot as a boundary. Seifert [5] has defined the genus $G(k)$ of the knot k as the minimum of the genera of orientable closed surfaces spanning k, whose existences are assured by [2] and [5]. Now let d be the degree of the Alexander polynomial of k. Seifert has proved that we have always

$$\frac{d}{2} \leqq G(k) \tag{1}$$

where the equality holds, if k is a torus knot, but there are also cases where the equality does not hold. (There are namely knots, whose Alexander polynomials are 1 and which are not equivalent to circles.)

In this paper, we shall show that the equality holds in (1) in certain classes of alternating knots (Theorem 1.1). For example, "alternierender Brezelknoten" of type $(p_1, p_2, \cdots, p_{2n+1})$, p_i being odd, i. e. alternating knots, whose projections have p_i crossing points on each arm and divide the plane into $\sum_{i=1}^{2n+1} p_i+2$ regions, of which $2n+2$ are "black", belong to these classes. It will be shown, at the same time, that for an alternating knot k of our classes, the orientable closed surface spanning k, whose genus is just equal to $G(k)$, is obtained by Seifert's construction.

§ 1. Main theorem.

Let k be a knot[1] and let K be an image of a regular projection[2] of k onto the plane E and let K be oriented by the orientation induced by that of k. Let K have n double points $c_1, c_2, \cdots, c_n$, called the *crossing points*. One of the two segments through a crossing point c_i passes under the other. It is called the *lower* segment at c_i and the other the *upper* segment. The

1) A knot means a polygonal simple closed (oriented) curve in Euclidean three dimensional space E^3.

2) See [3].

segments[3] of K connecting two consecutive crossing points are called *sides* of K. K divides E into $n+2$ regions $r_0, r_1, \cdots, r_{n+1}$, where we assume that r_0 is always an unbounded region. We can classify these regions into two classes, called "black" and "white" for convenience' sake, in such a way that each side is always a common boundary of a black and a white region, where r_0 belongs to a black class.

Let us assign to each crossing point c_i the *incidence number* $I(c_i)$, where $I(c_i)=+1$ or -1 according as the smaller rotation to make the lower segment coincide with the upper segment, the orientation of the segments being taken into account, is carried out in the black or in the white region (Fig. 1).

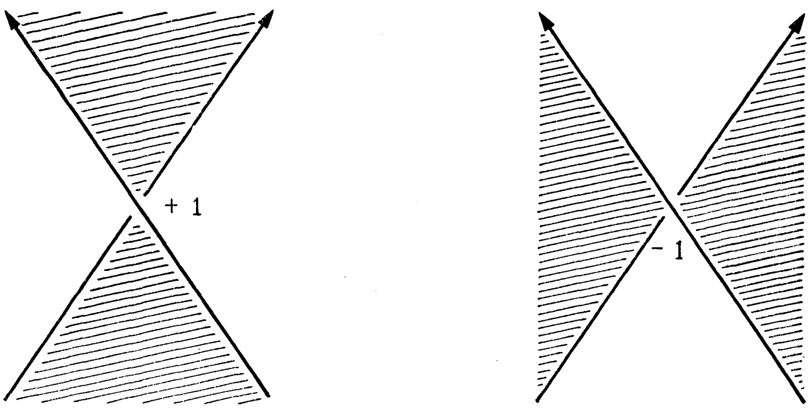

Fig. 1. (The parts drawn by the oblique lines represent the black regions)

Then the main theorem of our paper is the following

THEOREM 1.1. *For any alternating knot with a constant incidence number, the genus is exactly equal to one half of the degree of its Alexander polynomial.*

As a corollary of this theorem we have the following

COROLLARY 1.2. *Let k_1 and k_2 be alternating knots with constant incidence numbers. Then the degree of the Alexander polynomial of a product knot k_0 of k_1 and k_2 is exactly equal to double of the genus of k_0, where k_0 may not be an alternating knot and may not be of constant incidence numbers.*

COROLLARY 1.3. *The knots k_0, k_1, k_2 being as in Cor. 1.2, the genus of k_0 is equal to the sum of the genera of k_1 and k_2.*

REMARK. It was already shown by H. Schubert in [4] that the genus of the product knot is always equal to the sum of the genera of factors.

§ 2. Alexander polynomial and the genus of a knot.

Let us remember the definition of the Alexander polynomials defined in [1]. As in § 1 let us assume that there are n crossing points $c_1, c_2, \cdots, c_n$

3) Hereafter, a segment means generally a polygonal line.

in K and that K divides E into $n+2$ regions $r_0, r_1, \cdots, r_{n+1}$ and that these regions are classified into two classes, black and white.

To each region r_i an integer $I(r_i)$, called an *index* of r_i, is assigned. At each crossing point c_i, just four corners of four regions r_j, r_k, r_l and r_m, let us say, meet. Two corners among these four corners are marked with *dots* [1].

Now for each crossing point c_i, we shall write the following linear equation

$$c_i(r) = xr_j - xr_k + r_l - r_m = 0\,,$$

where c_i-corners[4] of r_j and r_k are dotted. We may assume, hereafter, that j, k, l and m are different from one another.[5]

Consider the matrix M, called the *L-matrix*, of the coefficients of these equations. M has n rows and $n+2$ columns, each row corresponding to a crossing point and each column corresponding to a region. If we denote the determinant of the square matrix obtained from M by striking out two columns corresponding to a pair of regions with consecutive indices p and $p+1$, by $\Delta_{p(p+1)}$, it follows[6]

$$\Delta_{p(p+1)} = \pm x^{r-p}\, \Delta_{r(r+1)}\,. \tag{2.1}$$

The *G.C.M.* of these determinants, freed from the factor x, is the *Alexander polynomial* of k. According to Alexander [1], we can assume that the signs of all the elements distinct from zero in the L-matrix M are positive, i.e. either x or 1.

Let us compute the genus of an orientable surface spanning k after the manner of H. Seifert [5].

Let us divide K into some loops,[7] called *standard loops*, in the same way as in [5]. Suppose that K is divided into m standard loops. Then the genus $G(k)$ of k is limited by[8]

$$G(k) \leqq \frac{n-m+1}{2}\,. \tag{2.2}$$

LEMMA 2.1. *For any alternating knot with a constant incidence number $I(c_i)$, the number m of the standard loops is either the number of the white or of the black regions according as $I(c_i)>0$ or $I(c_i)<0$.*

PROOF. We shall only prove Lemma in the case where $I(c_i)>0$. We shall prove that a standard loop L corresponds to a white region. To do

4) c_i-corner of r_j means the corner of r_j meeting at c_i.

5) In fact, it is impossible that $j=k$, or $k=l$, or $l=m$, or $m=j$. If $i=k$, we can transform K into K' which does not contain such a crossing point c_i. See [3].

6) See [1].

7) A loop means a simple closed curve.

8) See [5].

this we shall show that L will bound a white region W. Suppose that a point P moves positively along $\dot{W}$,[9] looking W on the left. When P arrives at a crossing point c_i, suppose it is always on the upper segment at c_i. Then the lower segment must be crossing under the upper segment from right to left, as $I(c_i)>0$. Thus P must turn to the left, and hence P must move positively along the boundary of a white region W', seeing it on the left again. It will be evident that $W=W'$. Thus P makes a round on $\dot{W}$, seeing W on the left. Consequently L bounds W. Furthermore it will be easily shown that two different standard loops do not bound the same white region.

If we assume that when P arrives at a crossing point, it is always on the lower segment, then we can prove Lemma in the same way as above.

In the same way, it will be proved that if $I(c_i)<0$, a standard loop will bound a black region. q. e. d.

§ 3. L_0-matrix.

By Lemma 2.1 we can see that it is sufficient to prove Theorem 1.1 in the case where $I(c_i)>0$. Consequently we shall suppose, hereafter, that

(A) $I(c_i)>0$ for all i.

Hence the number m of standard loops is equal to the number of the white regions.

LEMMA 3.1. *Under the assumption (A) the elements distinct from zero in the columns corresponding to the white regions are all x's or all 1's.*

PROOF. It is sufficient to prove that the corners of a white region are either all dotted or all undotted. The proof of this fact is, however, contained in the proof of Lemma 2.1, taking notice of the dots of the corners. q. e. d.

On account of this Lemma we can replace the L-matrix M by the matrix M_0, whose elements distinct from zero in the columns corresponding to the white regions are all equal to 1. M_0 will be called the *L_0-matrix.*

LEMMA 3.2. *Under the assumption (A) all the indices of the black regions are constant, say p, and then the indices of the white regions are either $p-1$ or $p+1$.*

PROOF. Let two black regions B_1 and B_2, and two white regions W_1 and W_2, be four regions whose corners meet at a crossing point c_i. Among these four regions the c_i-corners of two regions, of which one is the black and the other the white, are dotted. Suppose that the c_i-corner of B_1 is

9) A dot over the symbol denotes the set of boundary points.

dotted. If the c_i-corner of W_1 is dotted, then the lower segment is oriented as we see W_1 and B_1 on the left. Since $I(c_i)=1$, the upper segment must be oriented as we see W_1 and B_2 on the left. Hence it follows $I(W_1)=p+1$, $I(W_2)=p-1$ and $I(B_2)=p$. Similarly if the c_i-corner of W_2 is dotted, then it follows $I(W_1)=p-1$, $I(W_2)=p+1$ and $I(B_2)=p$. In the case where the c_i-corner of B_2 is dotted, it will be shown in the same way that we have the same result. q. e. d.

From the proof of this Lemma, it follows

LEMMA 3.3. *The index of the white region with dotted corners is $p+1$ and the index of the other white region is $p-1$, provided that the index of the black region is p.*

From this Lemma it follows

LEMMA 3.4. *The elements distinct from zero in either column of two columns of the L_0-matrix M_0, which are corresponding to two regions with consecutive indices, are all 1's.*

Consequently, the following Lemma will be easily shown from Lemmas 3.2, 3.3 and 3.4.

LEMMA 3.5. *Any determinant $\Delta^0_{(p-1)p}$ or $\Delta^0_{p(p+1)}$ of the square matrix obtained from M_0 by striking out two columns corresponding to two regions with consecutive indices is uniquely determined, except for the sign.*

Hence, hereafter, we shall consider only $\Delta^0_{p(p+1)}$.

LEMMA 3.6.[10] *Under the assumption (A) there exist $2q$ $(q>0)$ crossing points on the boundary of any black region B and the corners adjacent to the dotted (or undotted) corner of the black region are undotted (or dotted).*

PROOF. Suppose that $\dot{B}$ and the boundary of a white region W have a side s in common. Let us denote the end points of s by c_i and c_j. If c_i-corner and c_j-corner of B are both dotted, then either one of c_i-corner or c_j-corner of W is undotted and the other is dotted, which contradicts to Lemma 3.1. If two corners of B are both undotted, then c_i-corner of B' and c_j-corner of B'' are dotted, where B' and B'' are black regions meeting with B at c_i and c_j respectively. Then it is impossible that c_i-corner and c_j-corner of W are both dotted or both undotted. This is a contradiction. q. e. d.

§ 4. *L*-correspondence.

Consider the terms of the largest and the smallest degrees in the determinant $\Delta^0_{p(p+1)}$. Since $\Delta^0_{p(p+1)}$ is the determinant of the degree n and the elements of $m-1$ columns are either 0 or 1, it is the polynomial of the degree $n-m+1$ at most.

10) That the converse is also true, is pointed out by Prof. H. Terasaka.

Now let us assign to each crossing point c_i one of the four regions meeting at it such that

(C) Each one of the $n+2$ regions except certain two regions r_α and r_β with consecutive indices corresponds to one and only one of the crossing point lying on its boundary.

Such a correspondence will be called an *L^q-correspondence* if $n-m+1$ crossing points c_i correspond to $n-m+1$ black regions, of which the corners of the q black regions at the corresponding crossing points are dotted. Then we have clearly

Lemma 4.1. *An L^q-correspondence corresponds to a term x^q or $-x^q$ in $\Delta^0_{p(p+1)}$.*

Lemma 4.2. *Let σ be an L^{n-m+1}-correspondence*[11] *such that each crossing point corresponds to one and only one of the $n+2$ regions except for a pair of two adjacent regions r_α and r_β, and let τ be another L^{n-m+1}-correspondence which is obtained from σ by changing the correspondences in some crossing points. Denoting the terms in $\Delta^0_{p(p+1)}$ corresponding to σ and τ by εx^{n-m+1} and $\bar{\varepsilon} x^{n-m+1}$ respectively, it follows*

$$\varepsilon=\bar{\varepsilon},$$

where $\varepsilon, \bar{\varepsilon}=\pm 1$.

Proof. We can suppose that the columns of $\Delta^0_{p(p+1)}$ have been arranged so that i-th column corresponds to a black region B_i $(i=1,2,\cdots,n-m+1)$ and j-th column corresponds to a white region $W_{j-n+m-1}$ $(j=n-m+2,\cdots,n)$.

Let us suppose that c_{j_λ} corresponds to B_λ $(\lambda=1,\cdots,n-m+1)$ and c_{j_ν} corresponds to $W_{\nu-n+m-1}$ $(\nu=n-m+2,\cdots,n)$ in σ. Then we can write

$$\varepsilon=\mathrm{sgn}\begin{pmatrix} 1 & 2 & \cdots & n \\ j_1 & j_2 & \cdots & j_n \end{pmatrix}^{12)}.$$

In τ, if c_{k_λ} and c_{k_ν} correspond to B_λ and $W_{\nu-n+m-1}$ respectively, we can write

$$\bar{\varepsilon}=\mathrm{sgn}\begin{pmatrix} 1 & 2 & \cdots & n \\ k_1 & k_2 & \cdots & k_n \end{pmatrix}.$$

Hence it is sufficient to prove that

$$\mathrm{sgn}\,\zeta=\mathrm{sgn}\begin{pmatrix} j_1 & j_2 & \cdots & j_n \\ k_1 & k_2 & \cdots & k_n \end{pmatrix}=1.$$

Let ζ be represented as the product of r cycles $\zeta_1, \zeta_2, \cdots, \zeta_r$, which are mutually disjoint. Since $\mathrm{sgn}\,\zeta=(\mathrm{sgn}\,\zeta_1)\,(\mathrm{sgn}\,\zeta_2)\cdots(\mathrm{sgn}\,\zeta_r)$, it is sufficient to show that $\mathrm{sgn}\,\zeta_i=1$ for every i.

Let $\zeta_i=(s_1\cdots s_k)$. Now let us assign a chain L, called an *L-chain*, to ζ_i as follows. Take a point, called a *center*, in each region and fix it. Since

11) It will be shown in § 5 that there exists such a σ.

12) Sgn $P=1$ or -1 according as P is an even or an odd permutation.

both c_{s_1} and c_{s_2} lie on the boundary of a region r_{i_1}, say, we can join c_{s_1} and c_{s_2} with the center a_{i_1} of r_{i_1} by a segment l_1, in r_{i_1}. l_1 will be oriented in the direction from c_{s_1} to c_{s_2} through a_{i_1}. In the same way, we can join c_{s_2} and c_{s_3} with the center a_{i_2} of r_{i_2} by a segment l_2 in r_{i_2}, and so forth. We set $L=\bigcup_{i=1}^{k} l_i$. L is a loop. If L contains the centers of black regions, then we shall transform L into L_0 as follows. Suppose the interior[13] of L does not contain r_α and r_β. If L contains an oriented segment joining c_λ with c_μ through the center b of a black region B, denoted by $c_\lambda b c_\mu$, we replace it by a chain of the segments $c_\lambda w_1 c_\nu \cup c_\nu w_2 c_\xi \cup \cdots \cup c_\zeta w_l c_\mu$, where $c_\lambda, \cdots, c_\mu$ are the crossing points such that, a point P moving positively or negatively from c_λ to c_μ along $\dot{B}$ according as the orientation of E induced by L is positive or negative[14], passes $c_\lambda, \cdots, c_\mu$ in this order, and where $w_1, \cdots, w_l$ are the centers of the white regions which have the sides $c_\lambda c_\nu, \cdots, c_\zeta c_\mu$ with $\dot{B}$ in common. Thus we obtain a figure F. Let us transform F into L_0 with two following operations. (a) If F contains $c_i w_j c_k \cup c_k w_j c_m$, then we shall replace it by $c_i w_j c_m$. (b) If F contains $c_i w_j c_k \cup c_k w_j c_i$, we shall take it away. Thus F is transformed into a loop L_0.

Here we shall prove the following two facts.

LEMMA 4.3. *Let p_0 and q_0 be the numbers of the black and the white regions in the interior L_0^0 (or the exterior) of L_0 respectively and let s_0 be the number of the crossing points in L_0^0. Then*

$$s_0=p_0+q_0-1\,.$$

PROOF. Let t be the number of the centers of the white regions on L_0. Since t is equal to the number of the crossing points on L_0, $L_0^0 \cup L_0$ is divided into s_0+2t points, $2s_0+3t$ segments and p_0+q_0+t faces by the crossing points, the centers and the sides. Hence Euler's characteristic χ of $L_0^0 \cup L_0$ is given by

$$\chi=s_0+2t-(2s_0+3t)+p_0+q_0+t=-s_0+p_0+q_0\,.$$

On the other hand $\chi=1$, since $L_0^0 \cup L_0$ is homeomorphic to an 2-dimensional closed cell. Thus we have $s_0=p_0+q_0-1$. q. e. d.

LEMMA 4.4. *Let p_1, q_1 and s_1 be the numbers of the black, the white regions and the crossing points in the interior of L. Then denoting the number of the centers of the (white) regions lying on L_0 by k_0, it follows*

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13) We may call either one of two sets E_1 and E_2 into which E is divided by L, the interior and the other the exterior. But hereafter, we assume that the interior of L, or generally a loop, means the bounded set among E_1 and E_2.

14) If the exterior of L does not contain r_α and r_β, P will move along $\dot{B}$ in the inverse direction.

$$k_0=k+\sum_{i=1}^{p_0-p_1}(2\lambda_i-1)-(s_0-s_1)-(q_0-q_1), \tag{4.1}$$

where λ_i are positive integers.

PROOF. The number[15] of the centers of the regions on F is given by $k+\sum_{i=1}^{p_0-p_1}(2\lambda_i-1)$, since it increases by $2\lambda_i-1$ per a black region which is contained in the interior of L_0 and is not contained in the interior of L. But the number of the centers of the regions on L_0 is first decreased by s_0-s_1 by the operation (a) and again it is decreased by q_0-q_1 by the operation (b). Thus we have (4.1). q. e. d.

Now in our case it follows $s_1=p_1+q_1$ by the definition. Hence it follows from Lemmas 4.3 and 4.4

$$\begin{aligned} k_0&=k+\sum_{i=1}^{p_0-p_1}(2\lambda_i-1)-(s_0-s_1)-(q_0-q_1)\\ &=k+2\sum_{i=1}^{p_0-p_1}\lambda_i-(p_0-p_1)-(p_0+q_0-1-p_1-q_1)-(q_0-q_1)\\ &\equiv k+1 \pmod 2\,. \end{aligned}$$

While $k_0\equiv 0 \pmod 2$, as shown from the fact that if c-corner of a white region X is dotted (or undotted), then c-corner of the white region X' which is opposite to X over c is undotted (or dotted). Hence we obtain $k\equiv 1 \pmod 2$, i. e. $\operatorname{sgn}\zeta_i=1$. q. e. d.

§ 5. Proof of theorem.

The subset G (or H) of E obtained by connecting the centers of all the black regions (or all the white regions) with the crossing points lying on their boundaries will be called the *graph* (or the *dual graph*) of K. The segments of G (or H) connecting two consecutive centers of the regions are called *sides* of G (or H). There is only one crossing point on each side. Denote by M_k the regions into which E is divided by G. M_k contains clearly only one white region. We can suppose that the indices k are so arranged that $\bigcup_{\lambda=1}^{r}\dot{M}_\lambda\cap\dot{M}_{r+1}$ contains at least one side on $\dot{M}_{r+1}$ for $r=1,2,\cdots,n-m+1$.

Now let us prove the existence of an L^{n-m+1}-correspondence. To do this let us assign in the following way to each crossing point one of the $n+2$ regions except a pair of a white region r_α, contained in M_1, and a black region r_β adjacent to r_α.

15) A center lying on the part $c_i w_j c_k \cup c_k w_j c_m$ or $c_i w_j c_k \cup c_k w_j c_i$ of F is counted doubly.

First we shall assign $n-m+2$ black regions except r_β to $n-m+1$ crossing points. Let $\dot{M}_1$ consist of t sides $m_1, m_2, \cdots, m_t$, where m_i denotes the side connecting the center of B_i with that of B_{i+1} through a crossing point c_{j_i} for $i=1, 2, \cdots, t$. (We put $B_{t+1}=B_1$.) We assume here that B_i does not coincide with B_j for any i, j, $(i \neq j)$. It is easily seen that if $B_i=B_j$ for some i, j, then k will be a product knot. We shall consider this case in the next section. We can assume without loss of generality that r_β is the black region B_1. Now, from the definition of the graph either of the c_1-corner or the c_t-corner of B_1 is dotted. Let the c_1-corner of B_1 be dotted. Then, since the c_1-corner of B_2 is undotted, the c_2-corner of B_2 is dotted. In general the c_i-corner of B_i is dotted. Hence we shall assign B_i to c_i for $i=2, \cdots, t$. If the c_t-corner of B_1 is dotted, we shall assign B_i to c_{i+1}. Next let us suppose that each of the black regions except r_β, whose center is on $\bigcup_{j=1}^{h} \dot{M}_j$, corresponds to one and only one crossing point such that the corner of this region at the corresponding crossing point is dotted. Then we shall assign the regions whose centers are on $\dot{M}_{h+1}$ to the crossing point as follows. Let $\dot{M}_{h+1}$ consist of s sides $m_1', m_2', \cdots, m_s'$ and let $m_1', m_2', \cdots, m'_{h_1}, m'_{h_2}, m'_{h_2+1}, \cdots, m'_{h_3}, \cdots, m'_{h_{\lambda-1}}, m'_{h_{\lambda-1}+1}, \cdots, m'_{h_\lambda}$ be contained in $\bigcup_{j=1}^{h} \dot{M}_j$, where m_j' denotes the side connecting the center of B_j' with that of B'_{j+1} through a crossing point c_j'. Then either of the c'_{h_1}-corner or the c'_{h_1+1}-corner of B_{h_1+1} is dotted. If c'_{h_1}-corner of B_{h_1+1} is dotted, then we shall assign B_{h_1+1} to c_{h_1}. In general, we shall assign B_{h_1+i+1} to c_{h_1+i} for $0 \leqq i \leqq h_2-h_1-2$. If c_{h_1+1}-corner of B_{h_1+1} is dotted, then we shall assign B_{h_1+i+1} to c_{h_1+i+1}. In the same way we shall assign B_{h_l+j+1} to c_{h_l+j} or c_{h_l+j+1} for $l=2, \cdots, \lambda$ and $j=0, 1, \cdots, h_{l+1}-h_l-2$; $h_{\lambda+1}=s$.

Thus we obtain a correspondence such that each of the black regions except r_β corresponds to a crossing point on its boundary, where the corner of each region at corresponding point is dotted.

Finally we shall assign all white regions except r_α to the crossing points. To do this we shall consider a subset M of G, called the *semi-graph* with respect to the correspondence of the black regions. M is defined as a subset of G obtained from G by striking out the sides, where the crossing points on these sides do not correspond to any black regions. Then we have

LEMMA 5.1. *M is a tree, i.e. M is connected and does not contain a loop.*

PROOF. Set $M^h = M \cap \bigcup_{j=1}^{h} \dot{M}_j$, for $h=1, 2, \cdots, n-m+2$. Then it is obvious that $M^1=\dot{M}_1-m_1$ or $M^1=\dot{M}_1-m_t$ according as the c_1-corner of B_1 is dotted or undotted and that M^1 is a tree. Furthermore it follows from the definition of M that if M^h is connected then M^{h+1} is connected. Hence we shall see that $M=M^{n-m+2}$ will be connected by the induction. To prove the latter half of Lemma let us compute the Euler's characteristic χ of M. Since M is divided into $2(n-m+1)+1$ points and $2(n-m+1)$ segments by $n-m+1$ crossing points and $n-m+2$ centers of the black regions on M, we have $\chi=1$. Hence M does not contain a loop. q. e. d.

Now let N be the subset of the dual graph H obtained by striking out from H the sides meeting with M. Then

Lemma 5.2. *N is a tree.*

Proof. If N is decomposed into two components N_1 and N_2, where $N_1 \cap N_2 = \phi$, the sides $h_1, h_2, \cdots, h_t$ of N connecting N_1 with N_2 are meeting with the sides $g_1, g_2, \cdots, g_t$ of M respectively. Then $g_1 \cup \cdots \cup g_t$ is a loop, which contradicts to Lemma 5.1. Furthermore N does not contain a loop. For, if N contains a loop T, then the interior of T contains at least one black region B. Since M contains the center of B, $M \cap T \neq \phi$. Thus $M \cap N \neq \phi$, which is a contradiction. q. e. d.

Now we shall assign the white regions to the crossing points by means of N. Let $n_1, \cdots, n_\lambda$ be all the sides of N connecting the center of r_α with the centers of the white regions $W_1, \cdots, W_\lambda$ through the crossing points $c_{l_1}, \cdots, c_{l_\lambda}$ respectively. Then we shall assign W_i to c_{l_i}. Next, to the crossing points c_{p_j} on the sides n_j' of N, except n_i through the centers of W_i, we shall assign the regions W_j' which are opposite to W_i over c_{p_j}. Thus we shall obtain a correspondence such that each white region except r_α will correspond to one and only one crossing point on its boundary. For, we see from definition of N that each white region corresponds to a crossing point and moreover we see that if two crossing point correspond to one white region, then N would contain a loop. Thus we obtain

Lemma 5.3. *There is an L^{n-m+1}-correspondence as stated in Lemma 4.2.*

Similarly it follows

Lemma 5.4. *There is an L^0-correspondence σ_0 such that each crossing point corresponds to one and only one of the $n+2$ regions except a certain pair of two adjacent regions.*

Proof. σ_0 will be constructed as follows. If a crossing point c corresponds to a black region B in an L^{n-m+1}-correspondence, then we assign to c a black region B', which is opposite to B over c. Since c-corner of B' is undotted, we shall obtain a correspondence such that $n-m+1$ crossing points c_i correspond to $n-m+1$ black regions whose c_i-corners are undotted. For the rest it will be shown in the same way as in the proof of Lemma 5.3, as the analogue of Lemma 4.2 holds for an L^0-correspondence. q. e. d.

Lemma 5.5. *If K is of m standard loops, the Alexander polynomial of k is a polynomial of degree $n-m+1$.*

Proof. It follows from Lemma 4.2, 5.3 and 5.4.

PROOF OF THEOREM 1.1.

Denote the genus of k by $G(k)$. If K is of m standard loops, then $G(k) \leqq \frac{n-m+1}{2}$. Thus $2G(k) \leqq n-m+1=d$, where d denotes the degree of the Alexander polynomial of k. On the other hand $d \leqq 2G(k)$. Hence it follows $d=2G(k)$. Thus the proof of Theorem 1.1 is completed.

§ 6. Proof of corollary 1.2.

It is well known[16] that the Alexander polynomial of k_0 is the product of those of k_1 and k_2. Hence $d_0=d_1+d_2$, where d_i denote the degrees of the Alexander polynomials of k_i. Let K_i be the images of the regular projections of k_i onto E. From the assumption, there is a circle C on E such that C meets with K_0 at only two points P and Q, where P and Q are not crossing points and these lie on two sides of the boundary of a region r_k. C divides E into two parts E_1 and E_2, and $s=C \cap r_k$ divides r_k into two regions r_k' and r_k''. Let E_1 and E_2 contain r_k' and r_k'' respectively. Let $(E_1 \cap K_0) \cup s=K_1'$ and $(E_2 \cap K_0) \cup s=K_2'$. Since K_i' are equivalent to K_i, we shall write K_i instead of K_i'. Denoting the number of the crossing points and that of the standard loops of K_i by n_i and m_i respectively, the genera $G(k_i)$ are given by

$$G(k_i)=\frac{n_i-m_i+1}{2} \qquad \text{for } i=1,2. \tag{6.1}$$

Now it is obvious that

$$n_0=n_1+n_2. \tag{6.2}$$

To compute m_0, let us classify the regions into which E is divided by K_i, into two classes, called *black* and *white*, where the unbounded region always belongs to the black class. Then it is easy to show that

$$m_0=m_1+m_2-1. \tag{6.3}$$

Hence it follows from (6.1), (6.2) and (6.3) that

$$\begin{aligned} 2G(k_0) &\leqq n_0-m_0+1 \\ &=n_1+n_2-(m_1+m_2-1)+1 \\ &=(n_1-m_1+1)+(n_2-m_2+1) \\ &=2G(k_1)+2G(k_2) \\ &=d_1+d_2 \\ &=d_0. \end{aligned}$$

16) For example, see [1].

Since $d_0 \leqq 2G(k_0)$, we have $d_0 = 2G(k_0)$. q. e. d

Corollary 1.3 is immediately obtained from Corollary 1.2.

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