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On semigroups with minimal left ideals and without minimal right ideals.

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Introduction. Let S be a semigroup, *i.e.* a non-vacuous set closed under an associative binary operation. If a non-vacuous subset A of S has the property $SA\subset A$ or $AS\subset A$, A is called a *left ideal* or a *right ideal* of S, respectively. If a non-vacuous subset A of S is a left ideal and at the same time a right ideal, A is called simply an *ideal* of S. A minimal ideal of S is determined uniquely, if it exists. (The term 'minimal' is always used in the sense of the ordering by set inclusion.) The minimal ideal of S is called the *kernel* of S. A semigroup S may or may not have the kernel. Concerning this situation A. H. Clifford has shown the following result:

LEMMA 1. *If a semigroup S contains at least one minimal left ideal, then it has the kernel K, and K is the union of all the minimal left ideals of S.* (A. H. Clifford [2])

In the following discussion, we treat exclusively a semigroup S having at least one minimal left ideal. Following the above lemma, in this case S has the kernel, which we denote always by K.

The structure of the kernel K of such semigroups S has been determined by D. Rees [3] and A. H. Clifford [2], when S have also a minimal right ideal. We shall consider in this paper semigroups S which contain minimal left ideals but no minimal right ideals.

In §1, we shall prove that a semigroup S containing a minimal left ideal contains also a minimal right ideal, if and only if the kernel K of S contains an idempotent (Theorem 1). In the rest of the paper, we shall consider therefore only the case where K has no idempotent.

Now, M. Teissier [6] considered the semigroup S in which the equation $xa=b$ is always solvable in x and has no idempotent. In this case, S itself becomes clearly a minimal left ideal of S and coincides with the kernel K. Teissier then constructed a certain semigroup, in generalizing a construction given by R. Baer and F. Levi [1], and proved that every semigroup S of the above-mentioned type is isomorphic to a subsemigroup of this semigroup, provided that the left cancellation law holds.

In the present paper, we shall again generalize the construction of Teissier, and obtain semigroups which we shall call B-L-semigroups (§ 3).

In § 2, we shall prove some lemmas in preparation to §§ 3,4. In the final § 4, we shall prove that the kernel K of our semigroup S is always isomorphic to a subsemigroup of a B-L-semigroup, under the condition that $ax=ay$ for all $a\in K$ implies $x=y$ (Theorem 2). The case, where this additional condition is satisfied, will be treated as '*Case* 1' and the general case as '*Case* 2'. Finally, we shall consider as '*Case* 3', the case where the left cancellation law holds.

§ **1.** First, we refer two lemmas which are needed in this note, but can be proved easily.

Lemma 2. *Two different minimal left ideals of S are disjoint.*

Lemma 3. *If L, L' are minimal left ideals, and $a\in L$, then $L=La=L'a=Ka=Sa$.*

Now we shall prove

Theorem 1. *Let S be a semigroup which has at least one minimal left ideal. In order that S has at least one minimal right ideal, it is necessary and sufficient that the kernel K of S has at least one idempotent.*

Proof. If S has at least one minimal left ideal and at least one minimal right ideal, then K is completely simple without zero (A. H. Clifford [2]) and therefore K has at least one idempotent. Conversely, let S be a semigroup which has at least one minimal left ideal and such that its kernel K has at least one idempotent e. Then by Lemma 1, there is a minimal left ideal $L\ni e$, and by Lemma 3 we have $L=Se$. Now, let f be any idempotent of K such that $ef=fe=f$. Then $Sf=Sfe\subset Se=L$. But since L is a minimal left ideal, we have $Sf=L$. Hence there is an element $s\in S$ such that $sf=e$. Then $ef=(sf)f=sf=e$. Therefore $f=ef=e$. This shows that e is a primitive idempotent. Therefore K is completely simple without zero and S has a minimal right ideal (D. Rees [4] and R.P. Rich [5]).

§ **2.** Hereafter we consider a semigroup S with at least one minimal left ideal whose kernel K has no idempotent. Besides, since we shall discuss exclusively the structure of K, all the elements which will appear will belong to K unless otherwise mentioned. We denote the family of all the minimal left ideals of S as $\{L_\lambda;\lambda\in\Lambda\}$.

Lemma 4. *For an element a, there is no element d such that $ad=a$.*

Proof. Suppose there is such an element d. If L, L' are the minimal left ideals of S, containing a and d respectively, we have $ad\in SL'\subset L'$. But as $ad=a$ and $a\in L$, we have $L=L'$. Hence $d\in L'=L=Ka$, and so there is an element k such that $d=ka$. But then $d=ka=kad=d^2$ and d is an idempotent, which contradicts our assumption.

LEMMA 5. *If $ax=ay$, and L is a minimal left ideal containing a, then $px=py$ for every element $p\in L$.*

PROOF. By assumption, $a, p\in L$. Hence $p\in Ka$, and therefore there is an element k such that $p=ka$. Therefore $px=kax=kay=py$.

For two elements x, y of K, we write $x\equiv y\ (\lambda)$ when $p_\lambda x=p_\lambda y$ for every element $p_\lambda\in L_\lambda$. By Lemma 5, $ax=ay$ for one element $a\in L_\lambda$, implies $x\equiv y\ (\lambda)$. This relation $x\equiv y\ (\lambda)$ is evidently an equivalence relation. Therefore by this relation, K is decomposed into mutually disjoint classes. These classes are called *λ-classes* and we denote the family of all the λ-classes in K as $K(\lambda)$. λ-class containing an element a, *i.e.* $\{x; x\equiv a\ (\lambda)\}$ is denoted as $\overline{a_\lambda}$.

LEMMA 6. *Each λ-class is contained in some minimal left ideal.*

PROOF. Suppose $x\equiv y\ (\lambda)$ and $x\in L_\alpha, y\in L_\beta$. Then by definition, $ax=ay$ for every $a\in L_\lambda$. But $ax\in L_\alpha$, $ay\in L_\beta$, and therefore $L_\alpha=L_\beta$. Hence x and y belong to the same minimal left ideal.

According to Lemma 6, the decomposition of K into $K(\lambda)$ is a subdivision of the decomposition of K into minimal left ideals. The family of all the λ-classes in a minimal left ideal L_α is denoted as $L_\alpha(\lambda)$.

LEMMA 7. *Let $x\not\equiv y\ (\lambda)$. Then for any $a\in L_\lambda$ and $\mu\in\Lambda$, we have $ax\not\equiv ay\ (\mu)$.*

PROOF. Suppose $ax\equiv ay\ (\mu)$. Then for any $b\in L_\mu$, we have $bax=bay$. But $a\in L_\lambda$ implies $ba\in L_\lambda$ and then $x\equiv y(\lambda)$. q. e. d.

Now for a fixed $a\in L_\lambda$, we consider a mapping $\overline{x_\lambda}\to ax$. By the definition of λ-class, it is evident that this mapping of the λ-class into K is well defined irrespective of the choice of element x_λ in the λ-class. Besides, according to Lemma 7, the mapping $\overline{x_\lambda}\to\overline{(ax)_\mu}$ is the one-one mapping of $K(\lambda)$ into $K(\mu)$. We denote this mapping as $a_\mu{}^\lambda$, *i.e.* $a_\mu{}^\lambda(\overline{x_\lambda})=\overline{(ax)_\mu}$. Since $x\in L_\alpha$ implies $ax\in L_\alpha$, $a_\mu{}^\lambda$ maps $L_\alpha(\lambda)$ one-one into $L_\alpha(\mu)$ in particular.

For a set M, we denote the cardinal number of M as $\overline{\overline{M}}$.

LEMMA 8. $\overline{\overline{L_\alpha(\lambda)}}=\overline{\overline{L_\alpha(\mu)}}$.

PROOF. For $a\in L_\lambda$, $a_\mu{}^\lambda$ maps $L_\alpha(\lambda)$ one-one into $L_\alpha(\mu)$. Hence $\overline{\overline{L_\alpha(\lambda)}}\leqq\overline{\overline{L_\alpha(\mu)}}$. By considering $b_\lambda{}^\mu$ for $b\in L_\mu$, we obtain the inverse inequality.

LEMMA 9. *$xk=y$ implies $x\not\equiv y\ (\lambda)$ for every $\lambda\in\Lambda$.*

PROOF. Suppose that $x\equiv y\ (\lambda)$. Then $ax=ay$ for $a\in L_\lambda$. Hence $(ax)k=ay=ax$, which contradicts to Lemma 4.

LEMMA 10. *$L_\alpha(\lambda)$ and $L_\alpha(\mu)-a_\mu{}^\lambda(L_\alpha(\lambda))$ are both infinite sets, where $a\in L_\lambda$.*

PROOF. For $a\in L_\lambda$, there exists an element $e_1\in L_\alpha$ such that $e_1a=a$, since $L_\alpha a=L_\lambda$. Then, since $L_\alpha e_1=L_\alpha$, there exists $e_2\in L_\alpha$ such that $e_2e_1=e_1$. Continuing in this fashion, we obtain a sequence of elements $\{e_i\}$ of L_α, such that $e_{i+1}e_i=e_i$ for all $i=1, 2, 3, \cdots$. Then we can easily prove that for any two integers i, j such that $i<j$, we have $e_j e_i = e_i$, and for any integer i, we have $e_i(ax)=ax$. Therefore, by Lemma 9, $\overline{(ax)_\mu}, \overline{(e_1)_\mu}, \overline{(e_2)_\mu}, \cdots$ are all different. Hence for any i, $\overline{(e_i)_\mu} \notin a_\mu{}^\lambda(L_\alpha(\lambda))$, *i.e.* $\in L_\alpha(\mu) - a_\mu{}^\lambda(L_\alpha(\lambda))$. This shows clearly $L_\alpha(\mu) - a_\mu{}^\lambda(L_\alpha(\lambda))$ is an infinite set. Besides, since $\overline{\overline{L_\alpha(\lambda)}} = \overline{\overline{L_\alpha(\mu)}} \geqq \overline{\overline{L_\alpha(\mu) - a_\mu{}^\lambda(L_\alpha(\lambda))}}$, $L_\alpha(\lambda)$ is an infinite set.

Lemma 11. *For $a \in L_\lambda$ and $b \in L_\mu$, set $c = ba$. Then $c \in L_\lambda$, and for any $\nu \in \Lambda$, we have $c_\nu{}^\lambda = b_\nu{}^\mu a_\mu{}^\lambda$.*

Proof. $c \in L_\lambda$ is evident. And by definition, for any $\overline{x_\lambda} \in K(\lambda)$, we have $b_\nu{}^\mu a_\mu{}^\lambda(\overline{x_\lambda}) = b_\nu{}^\mu(\overline{(ax)_\mu}) = \overline{(bax)_\nu} = \overline{(cx)_\nu}$. Therefore $b_\nu{}^\mu a_\mu{}^\lambda = c_\nu{}^\lambda$.

Lemma 12. *For $a \in L_\lambda$ and $b \in L_\mu$, set $c = ba$. Then for any $\alpha, \nu \in \Lambda$, we have $\overline{\overline{L_\alpha(\nu) - c_\nu{}^\lambda(L_\alpha(\lambda))}} = \overline{\overline{L_\alpha(\mu) - a_\mu{}^\lambda(L_\alpha(\lambda))}} + \overline{\overline{L_\alpha(\nu) - b_\nu{}^\mu(L_\alpha(\mu))}}$.*

Proof. We decompose the family of ν-classes of $L_\alpha(\nu) - c_\nu{}^\lambda(L_\alpha(\lambda))$ into two disjoint subfamilies, one consisting of the ν-classes which are images of the mapping of $b_\nu{}^\mu$, and the other consisting of the ν-classes which are not. Since $b_\nu{}^\mu$ is one-one and by Lemma 11 $c_\nu{}^\lambda(L_\alpha(\lambda)) = b_\nu{}^\mu(a_\mu{}^\lambda(L_\alpha(\lambda)))$, the first subfamily is $b_\nu{}^\mu(L_\alpha(\mu) - a_\mu{}^\lambda(L_\alpha(\lambda)))$, which has the same cardinal number as $L_\alpha(\mu) - a_\mu{}^\lambda(L_\alpha(\lambda))$ since $b_\nu{}^\mu$ is one-one. The second subfamily is clearly $L_\alpha(\nu) - b_\nu{}^\mu(L_\alpha(\mu))$. Hence we get the result required.

Lemma 13. *$\overline{\overline{L_\alpha(\mu) - a_\mu{}^\lambda(L_\alpha(\lambda))}}$ is determined by α only, irrespective of the choice of $\lambda, \mu \in \Lambda$ and the choice of $a \in L_\lambda$.*

Proof. First we prove that $\overline{\overline{L_\alpha(\mu) - a_\mu{}^\lambda(L_\alpha(\lambda))}}$ is independent of $\mu \in \Lambda$, $a \in L_\lambda$, *i.e.* for any $\mu, \nu \in \Lambda$ and $a, b \in L_\lambda$, $\overline{\overline{L_\alpha(\mu) - a_\mu{}^\lambda(L_\alpha(\lambda))}} = \overline{\overline{L_\alpha(\nu) - b_\nu{}^\lambda(L_\alpha(\lambda))}}$. Take $c \in L_\mu$ such that $b = ca$. Such c really exists since $L_\lambda = L_\mu a$. Then by Lemma 12, we have $\overline{\overline{L_\alpha(\nu) - b_\nu{}^\lambda(L_\alpha(\lambda))}} = \overline{\overline{L_\alpha(\mu) - a_\mu{}^\lambda(L_\alpha(\lambda))}} + \overline{\overline{L_\alpha(\nu) - c_\nu{}^\mu(L_\alpha(\mu))}} \geqq \overline{\overline{L_\alpha(\mu) - a_\mu{}^\lambda(L_\alpha(\lambda))}}$. By taking an element $d \in L_\nu$ such that $a = db$, we get the inverse inequality.

Now we consider the general case, *i.e.* we prove that $\overline{\overline{L_\alpha(\mu) - a_\mu{}^\lambda(L_\alpha(\lambda))}} = \overline{\overline{L_\alpha(\kappa) - b_\kappa{}^\nu(L_\alpha(\nu))}}$ for any $\lambda, \mu, \nu, \kappa \in \Lambda$ and $a \in L_\lambda$, $b \in L_\nu$. Let $c = ba$, then $c \in L_\lambda$, and by Lemma 12, we get an inequality $\overline{\overline{L_\alpha(\kappa) - c_\kappa{}^\lambda(L_\alpha(\lambda))}} \geqq \overline{\overline{L_\alpha(\kappa) - b_\kappa{}^\nu(L_\alpha(\nu))}}$. But since $a, c \in L_\lambda$, by the result proved above $\overline{\overline{L_\alpha(\mu) - a_\mu{}^\lambda(L_\alpha(\lambda))}} = \overline{\overline{L_\alpha(\kappa) - c_\kappa{}^\lambda(L_\alpha(\lambda))}}$. Therefore we get an inequality $\overline{\overline{L_\alpha(\mu) - a_\mu{}^\lambda(L_\alpha(\lambda))}} \geqq \overline{\overline{(L_\alpha(\kappa' - b_\kappa{}^\nu(L_\alpha(\nu))}}$. The inverse inequality is obtained similarly.

§3. Let A, Λ be arbitrary sets and let $\{K_{\lambda\alpha};\ \lambda \in \Lambda, \alpha \in A\}$ be an arbitrary family of mutually disjoint sets indexed by all the elements of $\Lambda \times A$, such that, for fixed α, $K_{\lambda\alpha}$ has the same infinite cardinal number $\mathfrak{m}_\alpha$. And let $\mathfrak{m}_\alpha'$ be an arbitrarily given but fixed infinite cardinal number $\leqq \mathfrak{m}_\alpha$. We denote for each λ, $K(\lambda) = \bigcup_\alpha K_{\lambda\alpha}$ and $K^*(\lambda) = K(\lambda) \cup \{\theta_\lambda\}$, where θ_λ is an 'imaginary point'. Then it is clear that $\overline{\overline{K^*(\lambda)}}=\overline{\overline{K(\lambda)}}=\sum_{\alpha}\mathfrak{m}_\alpha$ which is independent of λ.

Let us consider a one-one mapping f_μ^λ of $K^*(\lambda)$ into $K(\mu)$ such that for each α, $f_\mu^\lambda(K_{\lambda\alpha})\subset K_{\mu\alpha}$ and $\overline{\overline{K_{\mu\alpha}-f_\mu^\lambda(K_{\lambda\alpha})}}=\mathfrak{m}_\alpha'$. That such a mapping really exists, can be proved easily. The set of all such mappings is denoted as $\mathfrak{F}_\mu^\lambda$. Next, we consider an element $f^\lambda=\{f_\mu^\lambda;\mu\in\Lambda\}$ of the direct product $\prod_\mu\mathfrak{F}_\mu^\lambda$, such that the set $K_{\mu\alpha}$ which contains $f_\mu^\lambda(\theta_\lambda)$ has the second index α which is independent of μ. The set of all such elements is denoted as $\mathfrak{F}(\lambda)$. Finally, we denote $\mathfrak{F}=\bigcup_{\lambda\in\Lambda}\mathfrak{F}(\lambda)$.

For two elements $f^\lambda, g^\mu\in\mathfrak{F}$, we define a composition $g^\mu\circ f^\lambda$ as follows: if $f^\lambda=\{f_\nu^\lambda;\nu\in\Lambda\}$, $g^\mu=\{g_\nu^\mu;\nu\in\Lambda\}$, then $g^\mu\circ f^\lambda=\{g_\nu^\mu f_\mu^\lambda;\nu\in\Lambda\}$ where $g_\nu^\mu f_\mu^\lambda$ is the resultant mapping.

It is not difficult to show that with this composition $\mathfrak{F}$ is a semigroup having minimal left ideals, that its kernel coincides with $\mathfrak{F}$ and that the kernel has no idempotent. We call this semigroup *B-L-semigroup.*

§**4.** Now, we return to a semigroup S with at least one minimal left ideal and whose kernel has no idempotent. We denote the family of all minimal left ideals of S as in the preceding paragraphs by $\{L_\lambda(\lambda\in\Lambda)\}$.

Case 1. We assume that "*$ax=ay$ for all $a\in K$ implies $x=y$*".

In this case, we obtain the following lemma:

LEMMA 14. *$x\equiv y\ (\lambda)$ for all $\lambda\in\Lambda$ implies $x=y$.*

PROOF. For any $a\in K$, a belongs to some $L_\lambda(\lambda\in\Lambda)$. Then by definition $ax=ay$. Accordingly $x=y$.

Now let us define a B-L-semigroup as follows:

We consider A,Λ in §3 both as the set Λ of index of the minimal left ideals of S. We consider $K_{\lambda\alpha}$ in §3 as $L_\alpha(\lambda)$ *i.e.* the set of all λ-classes contained in L_α. It is evident that $\{K_\alpha(\lambda);\alpha,\lambda\in\Lambda\}$ are the family of mutually disjoint sets, and by Lemmas 8 and 10, for fixed $\alpha, K_\alpha(\lambda)$ has an infinite cardinal number $\mathfrak{m}_\alpha$ which is independent of $\lambda\in\Lambda$. Finally, we consider $\mathfrak{m}_\alpha'$ in §3 as $\overline{\overline{L_\alpha(\mu)-a_\mu^\lambda(L_\alpha(\lambda))}}$ for $\lambda,\mu\in\Lambda$ and $a\in L_\lambda$. By Lemma 13 $\mathfrak{m}_\alpha'$ is well defined by α, irrespective of the choice of $\lambda,\mu\in\Lambda$ and $a\in L_\lambda$, and it is clear that $\mathfrak{m}_\alpha'\leqq\mathfrak{m}_\alpha$. Under the above determination, a B-L-semigroup is defined.

For any $a\in K$, when $a\in L_\lambda$, we consider $a^*{}_\mu^\lambda$ as the mapping of $K^*(\lambda)$ into $K(\mu)$ as follows:

$$a^*{}_\mu^\lambda(x)=a_\mu^\lambda(x)\quad\text{for}\quad x\in K(\lambda);\quad a^*{}_\mu^\lambda(\theta_\lambda)=\overline{a_\mu}.$$

As a_μ^λ is one-one and by Lemma 9, $\overline{(ax)_\mu}\neq\overline{a_\mu}$, $a^*{}_\mu^\lambda$ is a one-one mapping. Moreover, $a^*{}_\mu^\lambda(K_{\lambda\alpha})=a_\mu^\lambda(L_\alpha(\lambda))\subset L_\alpha(\mu)=K_{\mu\alpha}$ and $\overline{\overline{K_{\mu\alpha}-a^*{}_\mu^\lambda K_{(\lambda\alpha)}}}=\overline{\overline{L_\alpha(\mu)-a_\mu^\lambda(L_\alpha(\lambda))}}$

$=\mathfrak{m}_{\alpha}'$, by definition. Hence $a^*{}_{\mu}{}^{\lambda}\in\mathfrak{F}_{\mu}{}^{\lambda}$. Besides, $a^*{}_{\mu}{}^{\lambda}(\theta_{\lambda})=\overline{a}_{\mu}\in L_{\lambda}(\mu)$ where λ is the index of L_{λ} which contains a and therefore independent of μ. Accordingly $a^*=\{a^*{}_{\mu}{}^{\lambda}; \mu\in\Lambda\}\in\mathfrak{F}(\lambda)\subset\mathfrak{F}$.

Now we make correspond $a\in K$ to $a^*\in\mathfrak{F}$.

This correspondence is one-one. For, let a and b be two distinct elements of K. When a and b belong to the distinct minimal left ideals L_{λ} and L_{μ} respectively, then $a^*\in\mathfrak{F}(\lambda)$, $b^*\in\mathfrak{F}(\mu)$ and $a^*\neq b^*$ since $\lambda\neq\mu$. Next we suppose that $a, b\in L_{\lambda}$. Then, by Lemma 14, there exists $\mu\in\Lambda$ for which $a\not\equiv b\,(\mu)$. Then $a^*{}_{\mu}{}^{\lambda}(\theta_{\lambda})=\overline{a}_{\mu}\neq\overline{b}_{\mu}=b^*{}_{\mu}{}^{\lambda}(\theta_{\lambda})$. Therefore $a^*\neq b^*$.

Moreover, the above correspondence $a\to a^*$ is isomorphism, i.e. $(ba)^*=b^*\circ a^*$. For, if $a\in L_{\lambda}$ and $b\in L_{\mu}$, we have by Lemma 11, $(ba)_{\nu}{}^{\lambda}=b_{\nu}{}^{\mu}a_{\mu}{}^{\lambda}$ for any $\nu\in\Lambda$. Hence, for $x\in K(\lambda)$, we have $(ba)^*{}_{\nu}{}^{\lambda}(x)=(ba)_{\nu}{}^{\lambda}(x)=b_{\nu}{}^{\mu}a_{\mu}{}^{\lambda}(x)=b^*{}_{\nu}{}^{\mu}a^*{}_{\mu}{}^{\lambda}(x)$ and $(ba)^*{}_{\nu}{}^{\lambda}(\theta_{\nu})=\overline{(ba)}_{\nu}=b^*{}_{\nu}{}^{\mu}(\overline{a}_{\mu})=b^*{}_{\nu}{}^{\mu}a^*{}_{\mu}{}^{\lambda}(\theta_{\lambda})$. Therefore $(ba)^*=b^*\circ a^*$.

Thus we obtain the following theorem:

THEOREM 2. *Let S be a semigroup which has at least one minimal left ideal and whose kernel K has no idempotent. Besides, we assume that if $ax=ay$ for all $a\in K$ then $x=y$. Then K is isomorphic with a subsemigroup of a B-L-semigroup.*

Case 2. Let us consider the general case. For $x, y\in K$, we write $x\sim y$ if and only if $ax=ay$ for all $a\in K$. Then $x\sim y$ is evidently an equivalence relation. We denote the set $\{x; x\sim a\}=[a]$. Moreover this equivalence relation is a congruence relation i.e. $a\sim b$ and $c\sim d$ imply $ac\sim bd$. In fact, for any k, we have $ka=kb$, and so $kac=kad=kbd$. Accordingly, we can define the product $[a][b]$ as $[ab]$, which is independent of the choice of the element in the equivalence class. This multiplication among the equivalence classes clearly yields a semigroup. We denote this semigroup by $\overline{K}$.

This semigroup $\overline{K}$ satisfies the conditions assumed in Case 1. For, let $[a][x]=[a][y]$ for all $[a]$, then for any a, $[ax]=[ay]$. Therefore $bax=bay$ for all a, b. Taking especially b as an element such that $ba=a$, we conclude that $ax=ay$ for all a, which shows that $[x]=[y]$.

Thus we can apply Theorem 2 to $\overline{K}$.

Case 3. Let us assume the cancellation law: "*$ax=ay$ implies $x=y$*".

In this case each λ-class consists of a single element. It is shown in the same way as in Case 1, that B-L-semigroups in § 3, in which the index-set Λ consists of a single element, are semigroups which satisfy the assumption of Case 3, and conversely that every semigroup which satisfies the assumption of Case 3 is isomorphic to a subsemigroup of such a B-L-semigroup.

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