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Homeomorphy classification of total spaces of sphere bundles over spheres.

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Introduction.

The homeomorphy problem, i. e. the problem to determine whether two given topological spaces are homeomorphic or not, has been approached from various directions. Although we are as yet far from the general solution of this problem, even for the case where the given spaces are differentiable manifolds, the recent developments of the homotopy theory and of characteristic classes seem to form significant contributions to the homeomorphy theory of differentiable manifolds.

The present paper attempts a step in the homeomorphy theory of differentiable manifolds. We shall consider here namely total spaces of 3-sphere bundles over the 4-sphere and those of 7-sphere bundles over the 8-sphere, and shall explicitly give homeomorphic maps between some of these spaces. We shall see that these spaces have the same Pontrjagin classes (with respect to differentiable structures defined naturally), and also that, conversely, the spaces under our consideration (under some restrictions, see Theorems 3.4 and 3.5) which have the same homotopy type and the same Pontrjagin classes are homeomorphic. These results would offer some interesting facts to the problem of "topological invariance of Pontrjagin classes".

As an application we shall obtain the *homotopy* classification of the sphere bundles over spheres which have no cross section. And also we are able to generalize Milnor's result (Milnor [3]), that is, we shall obtain further examples of 7-dimensional and 15-dimensional manifolds which are homeomorphic but not diffeomorphic.

Notations and terminologies used in the paper are made clear in section 1. Section 2 contains a remark on Pontrjagin classes of sphere bundles over spheres computed in the previous paper (Tamura [5]), and Pontrjagin classes of the Cayley projective plane are obtained as a corollary.

In section 3, main part of this paper, we construct maps which induce homeomorphisms between total spaces of 3-sphere bundles over the 4-sphere and between total spaces of 7-sphere bundles over the 8-sphere. The total spaces are first covered by curves, and a map between total spaces is naturally defined by a map between these curves. Moreover a relation between homeomorphy and Pontrjagin classes is considered.

In section 4, homotopy classifications of 3-sphere bundles over the 4-sphere and 7-sphere bundles over the 8-sphere which have no cross section, are given. Section 5 is devoted to define the invariant λ (Milnor [3]) for 7-dimensional (resp. 15-dimensional) manifolds in a slightly generalized form, in order that we can apply it to the manifolds whose 4-dimensional (resp. 8-dimensional) cohomology groups are cyclic groups of finite order.

7-dimensional and 15-dimensional topological manifolds which possess different differentiable structures are exposed in section 6, using invariant λ.

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1. Preliminaries.

In this paper, we use the same notations as in the previous paper (Tamura [5]).

Let $\mathfrak{B}=\{B, p, S^q, S^r, SO(r+1)\}$ be fibre bundles over the q-sphere S^q with total space B, r-sphere S^r as fibre and the rotation group $SO(r+1)$ as structural group.

As is well-known, we have

$$\pi_3(SO(4))\approx Z+Z, \qquad \pi_7(SO(8))\approx Z+Z.$$

(Z means as usual the additive group of integers, Z_n the group Z mod n.)

And generators $\{\rho\}$, $\{\sigma\}$ of $\pi_3(SO(4))$ are given respectively by

$$\rho(u)v=uvu^{-1}, \qquad \sigma(u)v=uv \ ;$$

where u and v denote quaternions with norm 1 as usual.

In the case of $\pi_7(SO(8))$, we can write, by the recent result [6] (by H. Toda, Y. Saito and I. Yokota), generators $\{\bar{\rho}\}$, $\{\bar{\sigma}\}$ of $\pi_7(SO(8))$ as follows:

$$\bar{\rho}(x)y=xyx^{-1}, \qquad \bar{\sigma}(x)y=xy \ ;$$

where x and y denote Cayley numbers with norm 1 as usual.

Now we define bundles $\mathfrak{B}_{m,n}^{(4,3)}$ and $\mathfrak{B}_{m,n}^{(8,7)}$ as follows:

$$\mathfrak{B}_{m,n}^{(4,3)}=\{B_{m,n}^{(4,3)}, p, S^4, S^3, SO(4)\},$$

$$\mathfrak{B}_{m,n}^{(8,7)}=\{B_{m,n}^{(8,7)}, p, S^8, S^7, SO(8)\},$$

where $\mathfrak{B}_{m,n}^{(4,3)}$ and $\mathfrak{B}_{m,n}^{(8,7)}$ have the characteristic map $m\rho+n\sigma$ and $m\bar{\rho}+n\bar{\sigma}$ respectively.

Let E^4 be the 4-dimensional open cell which is defined by interior points of the unit circle of the space of quaternions and S^3 the boundary of E^4.

For the north pole x_1 and the south pole x_2 of S^4, we denote $S^4-\{x_1\}$ and $S^4-\{x_2\}$ by V and V' respectively.

$B^{(4,3)}_{m,n}$ has an open covering constituted by $p^{-1}(V)$ and $p^{-1}(V')$ which are homeomorphic to $E^4\times S^3$. (In this paper, *homeomorphic* always means 'homeomorphic onto.') Consequently we can define a coordinate on $p^{-1}(V)$ and $p^{-1}(V')$ by $E^4\times S^3$ in natural manner, and we denote it by (u, v) and $(u, v)'$ respectively, where u and v have norm $\|u\|<1$ and $\|v\|=1$. Then, by the definition of $\mathfrak{B}^{(4,3)}_{m,n}$, (u, v) and $((1-\|u\|)u/\|u\|, u^{m+n}\,v\,u^{-m}/\|u\|^n)'$ represent the same point on $B^{(4,3)}_{m,n}$ for $u\neq 0$. Since the transition function defined above is differentiable, a differentiable structure is defined on $B^{(4,3)}_{m,n}$. (In this paper, the word *differentiable* always means 'differentiable of class C^∞'.) We denote with $M^{(4,3)}_{m,n}$ the manifold thus defined. (All *manifolds* considered in this paper are C^∞-differentiable, orientable and compact.)

We obtain similar notions for $\mathfrak{B}^{(8,7)}_{m,n}$ by replacing quaternions by Cayley numbers and use corresponding notations for them.

2. A remark on Pontrjagin classes.

We denote by p_i i-th Pontrjagin class as usual.

Pontrjagin classes of $\mathfrak{B}^{(4,3)}_{m,n}$, $\mathfrak{B}^{(8,7)}_{m,n}$, $M^{(4,3)}_{m,n}$ and $M^{(8,7)}_{m,n}$ were computed in the previous paper (Tamura [5]). That is

$$p_1(\mathfrak{B}^{(4,3)}_{m,n})=\pm 2(2m+n)\alpha_4\,, \qquad p_1(M^{(4,3)}_{m,n})=\pm 4m\,\beta_4\,,$$

where α_4 and β_4 are generators of $H^4(S^4)\approx Z$ and $H^4(M^{(4,3)}_{m,n})\approx Z_n$ respectively.

In the case of $\mathfrak{B}^{(8,7)}_{m,n}$ and $M^{(8,7)}_{m,n}$, we can express, by the result about the generators of $\pi_7(SO(8))$ stated in section 1, Theorems 4.4. (i) and 6.2. (i) in the previous paper as follows:

THEOREM 2.1. $\qquad p_2(\mathfrak{B}^{(8,7)}_{m,n})=\pm 6(2m+n)\alpha_8\,,$
where α_8 is a generator of $H^8(S^8)\approx Z$.

THEOREM 2.2. $\qquad p_2(M^{(8,7)}_{m,n})=\pm 12m\,\beta_8\,,$
where β_8 is a generator of $H^8(M^{(8,7)}_{m,n})\approx Z_n$.

As an application of Theorem 2.1, we compute Pontrjagin classes of the Cayley projective plane W.

Let $\overline{\mathfrak{B}}^{(8,7)}_{0,1}=\{\overline{B}^{(8,7)}_{0,1}, p, S^8, \sigma^8, SO(8)\}$ be the bundle associated with $\mathfrak{B}^{(8,7)}_{0,1}$, where σ^8 is the 8-cell of closed interior of S^7. Differentiable structure is defined on $\overline{B}^{(8,7)}_{0,1}$ in natural manner as on $B^{(8,7)}_{0,1}$ and we obtain the 16-dimensional manifold $N^{(8,7)}_{0,1}$ with boundary $M^{(8,7)}_{0,1}$. It is easily verified that $N^{(8,7)}_{0,1}$ is the Cayley projective plane with a 16-cell removed.

By the Whitney product theorem of Pontrjagin classes and Theorem 2.1, we obtain

$$p_2(W)=6\iota,$$

where ι is a generator of $H^8(W)\approx Z$.

Since naturally $p_1(W)=p_3(W)=0$, we can compute $p_4(W)$ by $p_2(W)$ and the index theorem of Hirzebruch (Hirzebruch [2]):

$$\tau(W)=<\nu,\ \frac{1}{3^4\cdot5^2\cdot7}(381p_4(W)-71p_3(W)p_1(W)-19p_2^2(W)$$
$$+22p_2(W)p_1^2(W)-3p_1^4(W))>,$$

where ν is the generator of $H_{16}(W;Z)$ determined by an orientation of W. Therefore we obtain following (Hirzebruch [1]):

THEOREM 2.3. *Non-zero Pontrjagin classes of the Cayley projective plane W are*

$$p_0(W)=1, \qquad p_2(W)=6\iota, \qquad p_4(W)=39\iota^2,$$

where ι is a generator of $H^8(W)$.

3. Homeomorphy classification of $B^{(4,3)}_{m,n}$ and $B^{(8,7)}_{m,n}$.

First we consider $B^{(4,3)}_{m,n}$.

Let a be a point on S^3, i. e. a quaternion of norm $||a||=1$. For $0\leqq t\leqq 1$, we denote $ta\in\overline{E^4}$ by $[a]_t$. Obviously $[a]_0=0$, $[a]_1=a$ and $\{[a]_t;a\in S^3, 0\leqq t<1\}=E^4$. Let S^2 be the great circle of S^3 determined by quaternions u with $\mathfrak{R}(u)=0$. (We denote with $\mathfrak{R}(u)$ the real part of quaternion u.)

Let b be a point on S^2 and b_t the point on the arc $\widehat{1,b,-1}$ of great circle in the distance πt from -1 on $\widehat{1,b,-1}$. Obviously $b_0=-1$, $b_1=1$ and $\{b_t;b\in S^2, 0\leqq t\leqq 1\}=S^3$.

Now let S^3_1 be the 3-sphere $p^{-1}(x_1)$, where x_1 is a pole of the base space S^4 of $\mathfrak{B}^{(4,3)}_{m,n}$ defined in section 1. S^3_1 has the coordinate induced by one of $p^{-1}(V')$ defined in section 1.

Let E^4_1 be the 4-dimensional open cell $\{([a]_t,1);a\in S^3, 0\leqq t<1\}$ contained in $p^{-1}(V)$, which can be identified with E^4.

Then, since $([a]_t,1)$ and $((1-t)[a]_t/t, a^n)'$ represent the same point on $B^{(4,3)}_{m,n}$ for $0<t<1$, the point a on the boundary of E^4 can be regarded as representing the point a^n on S^3_1.

Next we decompose $p^{-1}(V)-E^4_1$ by curves as follows.

For $0\leqq s\leqq 1$, $a\in S^3$ and $b\in S^2$, we denote by $l(a,[b]_s)$ the subset $\{([a]_t,b_{st});0\leqq t<1\}$ and by $l([a]_s,b)$ the subset $\{([a]_{st},b_t);0\leqq t<1\}$ of $p^{-1}(V)$. Obviously $l(a,[b]_s)$ and $l([a]_s,b)$ are curves having no common point except $(0,-1)$ and the set of all these curves covers $p^{-1}(V)-E^4_1$.

Let $\bar{l}(a,[b]_s)$ (resp. $\bar{l}([a]_s,b)$) be the closure of $l(a,[b]_s)$ (resp. $l([a]_s,b)$) in $B^{(4,3)}_{m,n}$. Then the set of all $\bar{l}(a,[b]_s)$ and $\bar{l}([a]_s,b)$ covers $B^{(4,3)}_{m,n}$ and $\bar{l}(a,[b]_s)$

$-l(a,[b]_s)$ (resp. $\bar{l}([a]_s,b)-l([a]_s,b)$) are contained in $E_1^4\cup S_1^3$.

We denote by $(l(a,[b]_s),t)$ (resp. $(l([a]_s,b),t)$) the point $([a]_t,b_{st})$ (resp. $([a]_{st},b_t)$) for $0\leqq t<1$, and by $(l(a,[b]_s),1)$ (resp. $(l([a]_s,b),1)$) the point $\bar{l}(a,[b]_s)-l(a,[b]_s)$ (resp. $\bar{l}([a]_s,b)-l([a]_s,b)$).

Now we proceed to construct maps which will induce homeomorphisms between $B_{m,n}^{(4,3)}$ and $B_{m',n}^{(4,3)}$.

(i) First we shall assume $n=1$.

As was remarked above, E_1^4 in $B_{m,n}^{(4,3)}$ can be identified with E^4 in the space of quaternions. In this case $n=1$, this identification can be continuously extended to the closure $\overline{E_1^4}$ of E_1^4 and the boundary S_1^3 of E_1^4 is identified with the boundary S^3 of E^4.

For $0\leqq s\leqq 1$, $a\in S^3$, $b\in S^2$ and $0\leqq t\leqq 1$, we define the points $f_m(a,[b]_s)(t)$ and $f_m([a]_s,b)(t)$ of $B_{m,1}^{(4,3)}$ as follows;

$$0\leqq t\leqq 1/4,\qquad f_m([a]_s,b)(t)=(l([a]_s,b),t)\qquad (s\neq 1),$$

$$1/4\leqq t\leqq 1,\qquad f_m([a]_s,b)(t)=(l([a]_{\frac{4}{3}s(1-t)},b),t)\qquad (s\neq 1),$$

$$0\leqq t\leqq 1/4,\qquad f_m(a,[b]_s)(t)=(l(a,[b]_s),t),$$

$$1/4\leqq t\leqq 1/2,\qquad f_m(a,[b]_s)(t)=\left(l(a,[b]_s),\ \frac{1}{4}+3(1-s)\left(t-\frac{1}{4}\right)\right),$$

$$1/2\leqq t\leqq 3/4,\qquad f_m(a,[b]_0)(t)=(l(a,[b]_0),1),$$

$$f_m(a,[b]_s)(t)=(l(a^{m+1}b_s b^{-1}_{s+4(1-s)(t-\frac{1}{2})}a^{-m},$$

$$[(a^{m+1}b_s b^{-1}_{s+4(1-s)(t-\frac{1}{2})}a^{-m})^{-m}a^m ba^{-m}(a^{m+1}b_s b^{-1}_{s+4(1-s)(t-\frac{1}{2})}a^{-m})^m]_{s+4(1-s)(t-\frac{1}{2})}),$$

$$1-\frac{3}{4}s)\qquad (s\neq 0),$$

$$3/4\leqq t\leqq 1,\qquad f_m(a,[b]_0)(t)=(l([-a]_{4(1-t)},b),1),$$

$$f_m(a,[b]_s)(t)=\left(l([a^{m+1}b_s a^{-m}]_{4(1-t)},(a^{m+1}b_s a^{-m})^{-m}a^m ba^{-m}(a^{m+1}b_s a^{-m})^m),\right.$$

$$\left.1-\frac{3}{4}s+3s\left(t-\frac{3}{4}\right)\right)\qquad (s\neq 0).$$

Let $L^{(m)}(a,[b]_s)$ and $L^{(m)}([a]_s,b)$ be the subsets of $B_{m,1}^{(4,3)}$ defined by

$$L^{(m)}(a,[b]_s)=\{f_m(a,[b]_s)(t);\,0\leqq t\leqq 1\},$$

$$L^{(m)}([a]_s,b)=\{f_m([a]_s,b)(t);\ 0\leqq t\leqq 1\}\qquad (s\neq 1).$$

Then $L^{(m)}(a,[b]_s)$ and $L^{(m)}([a]_s,b)$ are curves joining $(0,-1)$ and $(0,1)$, where $(0,-1)$ and $(0,1)$ mean coordinates of $p^{-1}(V)$ defined in section 1. Obviously they cover $B_{m,1}^{(4,3)}$ and any two of them have no common point except $(0,-1)$ and $(0,1)$. Moreover they depend on a, b and s continuously.

Therefore we can regard $B_{m,1}^{(4,3)}$ as topological 7-sphere, whose poles are

$(0, -1)$ and $(0, 1)$ and whose circles of longitude are given by $L^{(m)}(a, [b]_s)$ and $L^{(m)}([a]_s, b)$.

In the case of $B^{(4,3)}_{0,1}$, it is clear that, in modifying these circles of longitude in an obvious manner, we can make them coincide with those of $M^{(4,3)}_{0,1}$ which is, as is well-known, diffemorphic to the 7-sphere with natural differentiable structure.

Now we define the map

$$g_m : B^{(4,3)}_{0,1} \to B^{(4,3)}_{m,1}$$

by

$$g_m(l([a]_s, b), t) = (l([a]_s, b), t) \qquad (0 \leqq t \leqq 1/4),$$

$$g_m(l([a]_{\frac{4}{3}s(1-t)}, b), t) = (l([a]_{\frac{4}{3}s(1-t)}, b), t) \qquad (1/4 \leqq t \leqq 1),$$

$$g_m(l(a, [b]_s), t) = (l(a, [b]_s), t) \qquad (0 \leqq t \leqq 1/4),$$

$$g_m\left(l(a, [b]_s), \frac{1}{4} + 3(1-s)\left(t - \frac{1}{4}\right)\right) = \left(l(a, [b]_s), \frac{1}{4} + 3(1-s)\left(t - \frac{1}{4}\right)\right)$$
$$(1/4 \leqq t \leqq 1/2),$$

$$g_m\left(l(ab_s b^{-1}_{s+4(1-s)(t-\frac{1}{2})}, [b]_{s+4(1-s)(t-\frac{1}{2})}), 1 - \frac{3}{4}s\right)$$
$$= \Big(l(a^{m+1} b_s b^{-1}_{s+4(1-s)(t-\frac{1}{2})} a^{-m},$$
$$[(a^{m+1} b_s b^{-1}_{s+4(1-s)(t-\frac{1}{2})} a^{-m})^{-m} a^m b a^{-m} (a^{m+1} b_s b^{-1}_{s+4(1-s)(t-\frac{1}{2})} a^{-m})^m]_{s+4(1-s)(t-\frac{1}{2})},$$
$$1 - \frac{3}{4}s\Big) \qquad (s \neq 0,\ 1/2 \leqq t \leqq 3/4),$$

$$g_m(l[-a]_{4(1-t)}, b), 1) = (l([-a]_{4(1-t)}, b), 1) \qquad (3/4 \leqq t \leqq 1),$$

$$g_m\left(l([ab_s]_{4(1-t)}, b),\ 1 - \frac{3}{4}s + 3s\left(t - \frac{3}{4}\right)\right)$$
$$= \Big(l([a^{m+1} b_s a^{-m}]_{4(1-t)}, (a^{m+1} b_s a^{-m})^{-m} a^m b a^{-m} (a^{m+1} b_s a^{-m})^m),$$
$$1 - \frac{3}{4}s + 3s\left(t - \frac{3}{4}\right)\Big) \qquad (s \neq 0,\ 3/4 \leqq t \leqq 1).$$

Clearly g_m is one to one and bicontinuous map. However, g_m is not a diffeomorphic map, unless $m=0$ or -1. Moreover if $m \neq 0$, g_m is not a bundle map $\mathfrak{B}^{(4,3)}_{0,1} \to \mathfrak{B}^{(4,3)}_{m,1}$.

It is to be noticed that

$$g_m(p^{-1}(V)) = p^{-1}(V),$$
$$g_m(p^{-1}(V')) = p^{-1}(V').$$

Moreover, if $g_m((u, v))=(u', v')$ (resp. $g_m((u, v)')=(u', v')'$), we have

$$\Re(u)=\Re(u'), \qquad ||u||=||u'||.$$

REMARK. g_m thus defined may be useful to make clear the properties of the various differentiable structures defined on the 7-sphere.

(ii) Next we consider $B^{(4,3)}_{nr,n}$.

We define the map

$$h^{(r)}_n : B^{(4,3)}_{nr,n} \to B^{(4,3)}_{r,1}$$

by

$$h^{(r)}_n((0, v))=(0, v),$$
$$h^{(r)}_n((0, v)')=(0, v)',$$
$$h^{(r)}_n((u, v))=(u^n/||u||^{n-1}, v) \qquad (u\neq 0),$$
$$h^{(r)}_n((u, v)')=(u^n/||u||^{n-1}, v)' \qquad (u\neq 0).$$

Obviously $h^{(r)}_n$ is well defined and continuous.

Let (u_1, v_1) (resp. $(u_2, v_2)'$) be a point of $B^{(4,3)}_{0,n}$ such that

$$\Re(u_1^n)\neq u_1^n \quad (\text{resp. } \Re(u_2^n)\neq u_2^n),$$

that is, $h^{(0)}_n((u_1, v_1))$ (resp. $h^{(0)}_n((u_2, v_2)')$) lies neither on $\bar{l}(\pm 1, [b]_s)$ nor on $\bar{l}([\pm 1]_s, b)$.

Then, by the definition of g_r, $g_r h^{(0)}_n((u_1, v_1))=(u_1', v_1')$ (resp. $g_r h^{(0)}_n((u_2, v_2)')=(u_2', v_2')'$) satisfies

$$\Re(u_1')\neq u_1' \quad (\text{resp. } \Re(u_2')\neq u_2').$$

Therefore there exists unique (u_1'', v_1') (resp. $(u_2'', v_2')'$) such that

$$\Re(u_1'')=\Re(u_1) \qquad (\text{resp. } \Re(u_2'')=\Re(u_2)),$$
$$h^{(r)}_n((u_1'', v_1'))=(u_1', v_1') \qquad (\text{resp. } h^{(r)}_n((u_2'', v_2')')=(u_2', v_2')').$$

We define the map $g^{(n)}_r$ by

$$g^{(n)}_r((u_1, v_1))=(u_1'', v_1'),$$
$$g^{(n)}_r((u_2, v_2)')=(u_2'', v_2')'.$$

Then we have the commutative diagram

$$\begin{tikzcd} B^{(4,3)}_{0,n}-RB^{(4,3)}_{0,n} \arrow[r, "g^{(n)}_r"] \arrow[d, "h^{(0)}_n"] & B^{(4,3)}_{nr,n}-RB^{(4,3)}_{nr,n} \arrow[d, "h^{(r)}_n"] \\ B^{(4,3)}_{0,1} \arrow[r, "g_r"] & B^{(4,3)}_{r,1} \end{tikzcd}$$

where $RB^{(4,3)}_{0,n}$ (resp. $RB^{(4,3)}_{nr,n}$) denotes the subset $\{x; x\in B^{(4,3)}_{0,n}, h^{(0)}_n(x)\in\bar{l}(\pm 1, [b]_s) \cup \bar{l}([\pm 1]_s, b)\}$ (resp. $\{x; x\in B^{(4,3)}_{nr,n}, h^{(r)}_n(x)\in\bar{l}(\pm 1, [b]_s)\cup\bar{l}([\pm 1]_s, b)\}$) of $B^{(4,3)}_{0,n}$ (resp. $B^{(4,3)}_{nr,n}$).

It is clear that $g_r^{(n)}$ is homeomorphic.

Let x be a point of $RB_{0,n}^{(4,3)}$. Then since $\overline{B_{0,n}^{(4,3)}-RB_{0,n}^{(4,3)}}=B_{0,n}^{(4,3)}$, there exists a sequence $\{x_1, x_2, \cdots, x_i, \cdots\}$ $(x_i \in B_{0,n}^{(4,3)}-RB_{0,n}^{(4,3)})$ such that

$$\lim_{i\to\infty} x_i = x .$$

We define[1] $g_r^{(n)}(x)$ by

$$g_r^{(n)}(x)=\lim_{i\to\infty} g_r^{(n)}(x_i) .$$

Then by the definition of g_r, we can easily verify that $g_r^{(n)}(x)$ is uniquely determined independently from the choice of a sequence. $g_r^{(n)}(x)$ thus defined is a homeomorphic map between $B_{0,n}^{(4,3)}$ and $B_{nr,n}^{(4,3)}$.

Naturally the following commutative diagram

$$\begin{tikzcd} B_{0,n}^{(4,3)} \arrow[r, "g_r^{(n)}"'] \arrow[d, "h_n^{(0)}"] & B_{nr,n}^{(4,3)} \arrow[d, "h_n^{(r)}"] \\ B_{0,1}^{(4,3)} \arrow[r, "g_r"'] & B_{r,1}^{(4,3)} \end{tikzcd}$$

holds.

(iii) General case.

Clearly $g_r^{(n)}$ satisfies

$$g_r^{(n)}(p^{-1}(V))=p^{-1}(V) ,$$
$$g_r^{(n)}(p^{-1}(V'))=p^{-1}(V') .$$

Now we define the map

$${}^m g_r^{(n)} : B_{m,n}^{(4,3)} \to B_{m+nr,n}^{(4,3)}$$

by

$${}^m g_r^{(n)}((0, v))=(0, v) ,$$
$${}^m g_r^{(n)}((u, v)')=((u', v')') ,$$

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1) We can define $g_r^{(n)}(x)$ $(x\in RB_{0,n}^{(4,3)})$ also as follows.

Let $x=(u, v)$ be a point of $p^{-1}(V)\cap p^{-1}(V')\subset B_{0,n}^{(4,3)}$ and let $h_n^{(0)}(x)$ lie on $L^{(0)}([a]_s, b)$ or on $L^{(0)}(a, [b]_s)$. Then a depends continuously on x by the definitions of $L^{(0)}([a]_s, b)$ and $L^{(0)}(a, [b]_s)$.

Now we define $g_r^{(n)}(x)$ $(x\in RB_{0,n}^{(4,3)})$ by

$$g_r^{(n)}((u, v))=(a^r u a^{-r}, a^r v a^{-r}) \qquad (u\neq 0),$$
$$g_r^{(n)}((0, v))=(0, v),$$
$$g_r^{(n)}((0, v)')=(0, v)'.$$

It is easy to verify that this definition of $g_r^{(n)}(x)$ $(x\in RB_{0,n}^{(4,3)})$ coincides with the definition in the text. The continuity of $g_r^{(n)}: B_{0,n}^{(4,3)}\to B_{nr,n}^{(4,3)}$ at the point of $RB_{0,n}^{(4,3)}$ follows directly from the second definition.

where u' and v' are quaternions satisfying

$$g_r^{(n)}((u, v)')=(u', v')'.$$

Then it is clear that ${}^m g_r^{(n)}$ is one to one and bicontinuous[2].

Thus we have proved the following theorem:

THEOREM 3.1. *If* $m\equiv\pm m' \bmod n$, $B_{m,n}^{(4,3)}$ *and* $B_{m',\pm n}^{(4,3)}$ (*i. e.* $M_{m,n}^{(4,3)}$ *and* $M_{m',\pm n}^{(4,3)}$) *are homeomorphic.*

In the case of $B_{m,n}^{(8,7)}$, we have only to replace quaternions by Cayley numbers to obtain the following theorems:

THEOREM 3.2. *If* $m\equiv\pm m' \bmod n$, $B_{m,n}^{(8,7)}$ *and* $B_{m',\pm n}^{(8,7)}$ (*i. e.* $M_{m,n}^{(8,7)}$ *and* $M_{m',\pm n}^{(8,7)}$) *are homeomorphic.*

THEOREM 3.3. $B_{m,\pm 1}^{(8,7)}$ $(m=0, \pm 1, \pm 2, \cdots)$ *are topological* 15-*spheres.*

REMARK. As is well-known, by the equivalence of bundles, $B_{m,n}$ and $B_{m+n,-n}$ are homeomorphic. And moreover by the weak equivalence of bundles, $B_{m,n}$ and $B_{-m,-n}$ are naturally homemorphic.

As was noticed in section 2, non zero Pontrjagin classes of $M_{m,n}^{(4,3)}$ and $M_{m,n}^{(8,7)}$ are

$$p_1(M_{m,n}^{(4,3)})=\pm 4m\,\beta_4 \pmod n\,, \qquad p_2(M_{m,n}^{(8,7)})=\pm 12m\,\beta_8 \pmod n\,.$$

So the manifolds which were proved to be homeomorphic by Theorems 3.1 and 3.2, have the same Pontrjagin classes. In particular we state some of our results as follows:

THEOREM 3.4. *If n is odd, manifolds* $M_{m,n}^{(4,3)}$ *which have the same homotopy type* (*see section* 4) *and the same Pontrjagin class, are homeomorphic.*

THEOREM 3.5. *If n satisfies* $(12, n)=1$, *manifolds* $M_{m,n}^{(8,7)}$ *which have the same homotopy type* (*see section* 4) *and the same Pontrjagin classes, are homeomorphic.*

On the other hand, in the case of manifolds which have different Pontrjagin classes, attempts to prove them to be homeomorphic, even to have the same homotopy type, meet with great difficulties. Moreover we shall see in section 6 that Pontrjagin classes of differentiable structures defined on the spaces which possess different differentiable structures, are always vanishing in our cases. These facts seem to suggest that Pontrjagin classes would play important rôle in the homeomorphy theory of manifolds or even that they might probably be topologically invariant.

4. Homotopy classification of $B_{m,n}^{(4,3)}$ and $B_{m,n}^{(8,7)}$.

We have proved in the previous paper that if $m=m' \bmod 12$, $B_{m,n}^{(4,3)}$ and

2) Notice that if (u, v) is a point in a sufficiently small neighbourhood of a point x in $p^{-1}(x_2)-(0, 1)$, $g_r^{(n)}$ satisfies

$$g_r^{(n)}((u, v))=(u, v).$$

$B^{(4,3)}_{m',n}$ have the same homotopy type, and also that if $m \equiv m'$ mod 120, $B^{(8,7)}_{m,n}$ and $B^{(8,7)}_{m',n}$ have the same homotopy type (Tamura [5], Theorem 2.2. (i), Theorem 2.3. (i)).

Combining these and Theorems 3.1 and 3.2, we obtain easily:

THEOREM 4.1. *If* $m \equiv \pm m'$ mod $(12, n)$, $B^{(4,3)}_{m,n}$ *and* $B^{(4,3)}_{m',\pm n}$ *have the same homotopy type.*

THEOREM 4.2. *If* $m \equiv \pm m'$ mod $(120, n)$, $B^{(8,7)}_{m,n}$ *and* $B^{(8,7)}_{m',\pm n}$ *have the same homotopy type.*

In particular, we have:

THEOREM 4.3. *If n satisfies* $(n, 12)=1$, $B^{(4,3)}_{m,n}$ $(m=0, \pm 1, \pm 2, \cdots)$ *have the same homotopy type.*

THEOREM 4.4. *If n satisfies* $(n, 120)=1$, $B^{(8,7)}_{m,n}$ $(m=0, \pm 1, \pm 2, \cdots)$ *have the same homotopy type.*

REMARK. We cannot expect the same conclusions as Theorems 4.3 and 4.4 for arbitrary n. For example, $B^{(4,3)}_{1,3}$ and $B^{(4,3)}_{0,3}$ have different homotopy types. Suppose, in fact, that they have the same homotopy type. Since

$$p_1(M^{(4,3)}_{1,3}) = \pm \alpha_4 \pmod 3, \qquad p_1(M^{(4,3)}_{0,3}) = 0 \pmod 3,$$

it contradicts to the homotopy invariance of Pontrjagin classes mod 3.

5. The invariant λ.

We consider here 7-dimensional manifolds and 15-dimensional manifolds which satisfy certain conditions.

First we define $P^7(n)$ manifold as follows.

DEFINITION 1[3]. If 7-dimensional closed oriented manifold M^7 satisfies following conditions, M^7 is called a $P^7(n)$ manifold.

(i) $H^3(M^7; Z)=0$, $H^4(M^7; Z) \approx Z_n$ $(n \neq 0)$.

(ii) There exists an 8-dimensional manifold N^8 with boundary M^7 such that $p_1(N^8)$ is divisible by n. Such N^8 will be called a manifold associated with the $P^7(n)$ manifold M^7.

Now let M^7 be a $P^7(n)$ manifold and let N^8 be a manifold associated

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3) We can replace the condition (ii) by the following (ii)$'$.

(ii)$'$ $\qquad p_1(M^7)=0$.

The existence of N^8 in condition (ii) is assured by a well-known result $\Omega^7=0$, where Ω^7 denotes the homogeneous part of the 7th degree of Thom algebra Ω. By (i) and (ii)$'$, it is easily verified that $p_1(N^8)$ is divisible by n, making use of the Whitney product theorem of Pontrjagin classes. Thus (ii)$'$ implies (ii). The converse can also be proved.

However, we can not replace the condition (ii) in Definition 3 by $p_2(M^{15})=0$, say, since the homogeneous part of the 15th degree of Ω is not zero.

with M^7.

An orientation $\nu \in H_8(N^8, M^7; Z)$ is determined by the relation $\partial \nu = \mu$, where μ is the generator of $H_7(M^7; Z)$ determined by the orientation of M^7.

Define a quadratic form over the group $H^4(N^8, M^7; R)$ by the formula $\alpha \to <\nu, \alpha^2>$, where R is the real number field. Let $\tau(N^8)$ be the index of this form.

By Definition 1 and the exact sequence

$$\cdots \to H^3(M^7; Z) \to H^4(N^8, M^7; Z) \xrightarrow{i} H^4(N^8; Z) \to H^4(M^7; Z) \to \cdots,$$

there exists unique element $i^{-1} p_1(N^8)$ of $H^4(N^8, M^7; Z)$.

If we replace here Z by real number field R, i becomes isomorphism onto i_R,

$$i_R : H^4(N^8, M^7; R) \approx H^4(N^8; R).$$

Then we define Pontrjagin number $q(N^8)$ by

$$q(N^8) = <\nu, (i^{-1} p_1)^2>.$$

DEFINITION 2. For a $P^7(n)$ manifold M^7, we define $\lambda(M^7)$ by

$$\lambda(M^7) = 2n^2 q(N^8) - \tau(N^8) \qquad \text{mod } 7.$$

Let N_1^8, N_2^8 be two manifolds associated with M^7. Then $C^8 = N_1^8 \cup N_2^8$ is a closed 8-dimensional manifold. Choose that orientation ν for C^8 which is consistent with the orientation ν_1 of N_1^8 and $-\nu_2$ of N_2^8. Let $q(C^8)$ be the Pontrjagin number $<\nu, p_1^2(C^8)>$. Then, using the index theorem (Hirzebruch [2]), we have

$$\tau(C^8) = <\nu, \frac{1}{45}(7p_2(C^8) - p_1^2(C^8))>,$$

and therefore

$$2q(C^8) - \tau(C^8) \equiv 0 \qquad (\text{mod } 7).$$

In the same way as in Milnor [3] lemma 1, we can prove that

$$\tau(C^8) = \tau(N_1^8) - \tau(N_2^8),$$

$$q(C^8) = n^2 q(N_1^8) - n^2 q(N_2^8),$$

and

$$2n^2 q(N_1^8) - \tau(N_1^8) \equiv 2n^2 q(N_2^8) - \tau(N_2^8) \qquad (\text{mod } 7).$$

Therefore $\lambda(M^7)$ is a diffeomorphy invariant of M^7.

REMARK. In the case of $P^7(1)$ manifold, our definition coincides with Milnor's.

Next we consider 15-dimensional manifolds.

DEFINITION 3. If 15-dimensional closed oriented manifold M^{15} satisfies following conditions, M^{15} is called a $P^{15}(n)$ manifold.

(i) $H^3(M^{15}; Z)=H^4(M^{15}; Z)=H^7(M^{15}; Z)=H^{11}(M^{15}; Z)=H^{12}(M^{15}; Z)=0$, $H^8(M^{15}; Z)\approx Z_n \quad (n\neq 0)$.

(ii) There exists a 16-dimensional manifold N^{16} with boundary M^{15} such that $p_2(N^{16})$ is divisible by n. Such N^{16} will be called a manifold associated with the $P^{15}(n)$ manifold M^{15}.

Now let M^{15} be a $P^{15}(n)$ manifold and let N^{16} be a manifold associated with M^{15}.

An orientation $\nu\in H_{16}(N^{16}, M^{15}; Z)$ is determined by the relation $\partial\nu=\mu$, where μ is the generator of $H_{15}(M^{15}; Z)$ determined by the orientation of M^{15}.

Define a quadratic form over the group $H^8(N^{16}, M^{15}; R)$ by the formula $\alpha\to<\nu, \alpha^2>$ and let $\tau(N^{16})$ be the index of this form.

For $j=1, 2, 3$, by Definition 3 and the exact sequence

$$\cdots\to H^{4j-1}(M^{15}; Z)\to H^{4j}(N^{16}, M^{15}; Z)\xrightarrow{i} H^{4j}(N^{16}; Z)\to H^{4j}(M^{15}; Z)\to\cdots,$$

there exists unique element $i^{-1}p_j(N^{16})$ of $H^{4j}(N^{16}, M^{15}; Z)$.

If we replace Z by R, i becomes isomorphism onto i_R,

$$i_R: H^8(N^{16}, M^{15}; R)\approx H^8(N^{16}; R).$$

Then for integers $j_1, \cdots, j_k$ such that $0<j_1\leqq j_2\leqq\cdots\leqq j_k$ and $j_1+j_2+\cdots+j_k=4$, we define Pontrjagin number $q_{(j_1,j_2,\cdots,j_k)}(N^{16})$ by

$$q_{(j_1,j_2,\cdots,j_k)}(N^{16})=<\nu, \quad (i^{-1}p_{j_1}(N^{16}))(i^{-1}p_{j_2}(N^{16}))\cdots(i^{-1}p_{j_k}(N^{16}))>.$$

Let N_1^{16}, N_2^{16} be two manifolds associated with M^{15}. Then $C^{16}=N_1^{16}\cup N_2^{16}$ is a closed 16-dimensional manifold. Choose that orientation ν for C^{16} which is consistent with the orientation ν_1 of N_1^{16} and $-\nu_2$ of N_2^{16}. Let $q_{(j_1,j_2,\cdots,j_k)}(C^{16})$ be Pontrjagin number $<\nu, p_{j_1}(C^{16})p_{j_2}(C^{16})\cdots p_{j_k}(C^{16})>$.

Using the index theorem (Hirzebruch [2]), we have

$$\tau(C^{16})=<\nu, \frac{1}{3^4\cdot 5^2\cdot 7}(381p_4(C^{16})-71p_3(C^{16})p_1(C^{16})-19p_2^2(C^{16})$$
$$+22p_2(C^{16})p_1^2(C^{16})-3p_1^4(C^{16}))>$$

and therefore

$$78\tau(C^{16})+71q_{(1,3)}(C^{16})+19q_{(2,2)}(C^{16})-22q_{(1,1,2)}(C^{16})+3q_{(1,1,1,1)}(C^{16})\equiv 0 \pmod{381}.$$

In the same way as for M^7, we can prove that

$$\tau(C^{16})=\tau(N_1^{16})-\tau(N_2^{16}),$$
$$q_{(1,1,2)}(C^{16})=nq_{(1,1,2)}(N_1^{16})-nq_{(1,1,2)}(N_2^{16}),$$
$$q_{(2,2)}(C^{16})=n^2q_{(2,2)}(N_1^{16})-n^2q_{(2,2)}(N_2^{16}),$$

$$q_{(j_1,j_2,\cdots,j_k)}(C^{16})=q_{(j_1,j_2,\cdots,j_k)}(N_1^{16})-q_{(j_1,j_2,\cdots,j_k)}(N_2^{16})$$

$$(j_1, j_2, \cdots, j_k)\neq\begin{cases}(1,1,2)\\(2,2)\end{cases}.$$

And so

$$78\tau(N_1^{16})+71q_{(1,3)}(N_1^{16})+19n^2 q_{(2,2)}(N_1^{16})-22n\, q_{(1,1,2)}(N_1^{16})+3q_{(1,1,1,1)}(N_1^{16})$$
$$\equiv 78\tau(N_2^{16})+71q_{(1,3)}(N_2^{16})+19n^2 q_{(2,2)}(N_2^{16})-22n\, q_{(1,1,2)}(N_2^{16})+3q_{(1,1,1,1)}(N_2^{16}) \mod 381.$$

Therefore we can define a diffeomorphy invariant λ for a $P^{15}(n)$ manifold M^{18} as follows.

Definition 4. For a $P^{15}(n)$ manifold M^{15}, we define $\lambda(M^{15})$ by

$$\lambda(M^{15})=78\tau(N^{16})+71q_{(1,3)}(N^{16})+19n^2 q_{(2,2)}(N^{16})-22n\, q_{(1,1,2)}(N^{16})+3q_{(1,1,1,1)}(N^{16}) \mod 381,$$

where N^{16} is a 16-dimensional manifold associated with M^{15}.

6. Manifolds which are homeomorphic and not diffeomorphic.

Now we consider bundles $\mathfrak{B}_{m,n}^{(4,3)}$ and $\mathfrak{B}_{m,n}^{(8,7)}$.

We associate to $\mathfrak{B}_{m,n}^{(4,3)}$ (resp. $\mathfrak{B}_{m,n}^{(8,7)}$), the bundle $\bar{\mathfrak{B}}_{m,n}^{(4,3)}=\{\bar{B}_{m,n}^{(4,3)}, p, S^4, \sigma^4, SO(4)\}$ (resp. $\bar{\mathfrak{B}}_{m,n}^{(8,7)}=\{\bar{B}_{m,n}^{(8,7)}, p, S^8, \sigma^8, SO(8)\}$) with 4-cell σ^4 (resp. 8-cell σ^8) of closed interior of S^3 (resp. S^7) as fibre. Differentiable structure is defined on $B_{m,n}^{(4,3)}$ and on $B_{m,n}^{(8,7)}$ in natural way, and we obtain an 8-dimensional manifold $N_{m,n}^{(4,3)}$ with boundary $M_{m,n}^{(4,3)}$ and a 16-dimensional manifold $N_{m,n}^{(8,7)}$ with boundary $M_{m,n}^{(8,7)}$.

Non trivial cohomology groups of $N_{m,n}^{(4,3)}$ and $N_{m,n}^{(8,7)}$ are

$$H^4(N_{m,n}^{(4,3)}; Z)\approx Z, \qquad H^8(N_{m,n}^{(8,7)}; Z)\approx Z.$$

We denote by γ_4 and γ_8 generators of $H^4(N_{m,n}^{(4,3)}; Z)$ and $H^8(N_{m,n}^{(8,7)}; Z)$ respectively.

Making use of the Whitney product theorem of Pontrjagin classes, Pontrjagin classes of $N_{m,n}^{(4,3)}$ and $N_{m,n}^{(8,7)}$ are easily computed by Pontrjagin classes of $\mathfrak{B}_{m,n}^{(4,3)}$ and $\mathfrak{B}_{m,n}^{(8,7)}$ (see section 2). They are zero except

$$p_1(N_{m,n}^{(4,3)})=\pm 2(2m+n)\gamma_4,$$
$$p_2(N_{m,n}^{(8,7)})=\pm 6(2m+n)\gamma_8.$$

Hence by Definitions 1 and 3 in section 5, we have

$M_{nr,n}^{(4,3)}$ $(r=0, \pm 1, \pm 2, \cdots)$ are $P^7(n)$ manifolds,

$M_{nr,2n}^{(4,3)}$ $(r=0, \pm 1, \pm 2, \cdots)$ are $P^7(2n)$ manifolds,

$M_{nr,4n}^{(4,3)}$ $(r=0, \pm 1, \pm 2, \cdots)$ are $P^7(4n)$ manifolds,

$M_{nr,n}^{(8,7)}$ $(r=0, \pm 1, \pm 2, \cdots)$ are $P^{15}(n)$ manifolds,

$$M^{(8,7)}_{nr,2n}\ (r=0,\pm 1,\pm 2,\cdots\ )\ \text{are}\ P^{15}(2n)\ \text{manifolds},$$
$$M^{(8,7)}_{nr,3n}\ (r=0,\pm 1,\pm 2,\cdots\ )\ \text{are}\ P^{15}(3n)\ \text{manifolds},$$
$$M^{(8,7)}_{nr,4n}\ (r=0,\pm 1,\pm 2,\cdots\ )\ \text{are}\ P^{15}(4n)\ \text{manifolds},$$
$$M^{(8,7)}_{nr,6n}\ (r=0,\pm 1,\pm 2,\cdots\ )\ \text{are}\ P^{15}(6n)\ \text{manifolds},$$
$$M^{(8,7)}_{nr,12n}\ (r=0,\pm 1,\pm 2,\cdots\ )\ \text{are}\ P^{15}(12n)\ \text{manifolds}.$$

That is, $M^{(4,3)}_{m,n}$ *(resp.* $M^{(8,7)}_{m,n}$*) is a* $P^7(n)$ *(resp.* $P^{15}(n)$*) manifold, if Pontrjagin classes vanish.*

And their invariant λ are:

$$\lambda(M^{(4,3)}_{nr,n})\equiv n^2(2r+1)^2-1 \pmod 7,$$
$$\lambda(M^{(4,3)}_{n+2nr,2n})\equiv 2n^2(r+1)^2-1 \pmod 7,$$
$$\lambda(M^{(4,3)}_{nk+4nr,4n})\equiv 2n^2(4r+2+k)^2-1 \pmod 7,$$
$$\lambda(M^{(8,7)}_{nr,n})\equiv 78(1-n^2(2r+1)^2) \pmod{381},$$
$$\lambda(M^{(8,7)}_{n+2nr,2n})\equiv 78(1-16n^2(r+1)^2) \pmod{381},$$
$$\lambda(M^{(8,7)}_{nk+3nr,3n})\equiv 78(1-n^2(6r+2k+3)^2 \pmod{381},$$
$$\lambda(M^{(8,7)}_{nk+4nr,4n})\equiv 78(1-n^2(8r+2k+4)^2) \pmod{381},$$
$$\lambda(M^{(8,7)}_{nk+6nr,6n})\equiv 78(1-n^2(12r+2k+6)^2) \pmod{381},$$
$$\lambda(M^{(8,7)}_{nk+12nr,12n})\equiv 78(1-n^2(24r+2k+12)^2) \pmod{381}.$$

Since, by Theorems 3.1 and 3.2, $B_{m+nr,n}$ are homeomorphic to $B_{m,n}$, we can regard $M_{m+nr,n}$ as a manifold defined on $B_{m,n}$.

Hence we obtain following theorems:

THEOREM 6.1. *If* $n\not\equiv 0$ *mod* 7, *each of the following spaces possesses different differentiable structures.*

$$B^{(4,3)}_{0,n}, \qquad B^{(4,3)}_{n,2n}, \qquad B^{(4,3)}_{n,4n}, \qquad B^{(4,3)}_{3n,4n}.$$

THEOREM 6.2. *If* $n\not\equiv 0$ *mod* 127, *each of the following spaces possesses different differentiable structures.*

$$B^{(8,7)}_{0,n},\quad B^{(8,7)}_{n,2n},\quad B^{(8,7)}_{n,3n},\quad B^{(8,7)}_{2n,3n},\quad B^{(8,7)}_{n,4n},\quad B^{(8,7)}_{3n,4n},$$
$$B^{(8,7)}_{n,6n},\quad B^{(8,7)}_{5n,6n},\quad B^{(8,7)}_{n,12n},\quad B^{(8,7)}_{5n,12n},\quad B^{(8,7)}_{7n,12n},\quad B^{(8,7)}_{11n,12n}.$$

REMARK. If $n\neq\pm n'$, $B^{(4,3)}_{rn,n}$ and $B^{(4,3)}_{r'n',n'}$ (resp. $B^{(8,7)}_{rn,n}$ and $B^{(8,7)}_{r'n',n'}$) are not the same topological space, because they have different homotopy types. On the other hand, in case $n=\pm n'$ (for example, for $B^{(4,3)}_{0,2n}$ and $B^{(4,3)}_{n,2n}$), we do not know whether they are the same topological space or not.

Since $B^{(8,7)}_{0,1}$ is a topological 15-sphere (Theorem 3.3), we have in particular (Shimada [4], Tamura [5] Remark 4.6.):

THEOREM 6.3. *15-sphere possesses differentiable structures different from natural one.*

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