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The Gauss-Bonnet Theorem for V-manifolds.

By Ichiro SATAKE

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Introduction.

The purpose of this paper is to generalize the Gauss-Bonnet theorem (established by Allendoerfer and Weil [1]) to the case of V-manifolds. The notion of a V-manifold has been introduced by the author in a previous short paper [7].

The outline of this paper is as follows. In § 1 we shall give fundamental concepts concerning V-manifolds and V-bundles. Our principal idea is the following: while an ordinary manifold can be considered as an inverse injective limit of Euclidean spaces, a V-manifold is considered as that of Euclidean spaces allowing a finite group of automorphisms. More precisely speaking, let $\{\tilde{U}_\alpha\}_{\alpha\in A}$ be a system of Euclidean spaces (of the same dimension), A being a directed system such that for any $\alpha, \beta\in A$ there exists a $\gamma\in A$ with $\gamma\leqq\alpha$, $\gamma\leqq\beta$ and assume that for any $\alpha, \beta\in A$, $\alpha\leqq\beta$ we have 'an injection' $\lambda_{\beta\alpha}$ from $\tilde{U}_\alpha$ into $\tilde{U}_\beta$ such that we have $\lambda_{\gamma\alpha}=\lambda_{\gamma\beta}\circ\lambda_{\beta\alpha}$ for $\alpha\leqq\beta\leqq\gamma$. Then the injective limit $M=\bigcup_\alpha\varphi_\alpha(\tilde{U}_\alpha)$ of the inverse system $\{\tilde{U}_\alpha, \lambda_{\beta\alpha}\}$, φ_α denoting the canonical injection $\tilde{U}_\alpha\to M$, is a manifold. ($\varphi_\alpha(\tilde{U}_\alpha)$ being homeomorphic to $\tilde{U}_\alpha$, M is locally homeomorphic to a Euclidean space.) Now a slight modification of the above definition will lead to a V-manifold. Namely replacing the word 'an injection' by 'a finite number of injections' we obtain a V-manifold $M=\bigcup\varphi_\alpha(\tilde{U}_\alpha)$, which is locally homeomorphic to $\varphi_\alpha(\tilde{U}_\alpha)\approx G_\alpha\backslash\tilde{U}_\alpha$, G_α being a finite group of automorphisms of $\tilde{U}_\alpha$, composed of $\lambda_{\alpha\alpha}$. Similar considerations can be applied also to the definition of V-bundles. Namely a V-bundle B can be considered as an injective limit (allowing a finite group of automorphisms) of an (inverse) system of direct products of a Euclidean space $\tilde{U}_\alpha$ and a fixed manifold F (called fibre) with respect to injections of a special form. As examples of V-bundles the notions of tangent vectors and differential forms will be introduced at the end of this section.

In § 2 a short resumé of Riemannian geometry on a V-manifold is given. After these preparations it is quite easy to see that Chern's proof [3] of the Gauss-Bonnet theorem can be transferred almost literally to our case. But in order to make clear the point in which our proof differs from Chern's and also the meanings of the notations used, we shall restate the main arguments of his proof in § 3. In the Gauss-Bonnet formula for a V-manifold M, thus obtained, appears the Euler characteristic $\chi_V(M)$ of M as a V-manifold in place of the ordinary Euler characteristic. This number will be of some interest, because, on the one hand, it is also related with Hopf's formula [4] on vector fields over M and, on the other hand, it appears in the theory of automorphic functions of one variable (e. g. in the Riemann-Roch formula concerning the dimension of the space of automorphic forms).

Finally in § 4 we shall show that the Gauss-Bonnet formula can be also applied to Siegel's modular variety $\mathfrak{V}_n = \boldsymbol{M}_n \backslash \mathfrak{H}_n$, $\mathfrak{H}_n$ being the space of all symmetric complex matrices $Z = X + iY$ of degree n with the imaginary parts $Y > 0$ and $\boldsymbol{M}_n$ Siegel's modular group of degree n operating on $\mathfrak{H}_n$. This case was treated already by Siegel in his 'Symplectic Geometry' [9], which gave a motive to this investigation. Siegel considered the Gauss-Bonnet formula only for the case of $\mathfrak{G} \backslash \mathfrak{H}_n$, where $\mathfrak{G} \backslash \mathfrak{H}_n$ is compact and $\mathfrak{G}$ is a discontinuous group of symplectic transformations without fixed point. Our results will complete his work in the sense that we extend his formula to the case of compact $\mathfrak{G} \backslash \mathfrak{H}_n$, where $\mathfrak{G}$ allows fixed points, as well as to the non-compact case $\mathfrak{V}_n = \boldsymbol{M}_n \backslash \mathfrak{H}_n$. We conclude this paper by an application of the Gauss-Bonnet formula for $\mathfrak{V}_n$ to the determination of the least common multiple N_n of the orders of isotropy subgroups of $\boldsymbol{M}_n / \{\pm E_{2n}\}$. The Gauss-Bonnet formula for $\mathfrak{V}_n$ gives a lower estimation of N_n, while an upper estimation of it will be obtained by a method of Minkowski [6].

§ 1. V-manifolds and V-bundles.

1. Definition of V-manifold. Let M be a Hausdorff space. A (C^∞-) *local uniformizing system* (abbreviated in the following as l. u. s.) $\{\tilde{U}, G, \varphi\}$ for an open set U in M is by definition a collection of the following objects:

$\tilde{U}$: a connected open set in $\boldsymbol{R}^m$ (m-dimensional Euclidean space),

G : a finite group of C^∞-automorphisms of $\tilde{U}$, with the set of fixed points of dimension $\leqq m-2$,

φ : a continuous map from $\tilde{U}$ onto U such that $\varphi\circ\sigma=\varphi$ for all $\sigma\in G$, inducing a homeomorphism from the quotient space $G\backslash\tilde{U}$ onto U.

Let $\{\tilde{U}, G, \varphi\}$, $\{\tilde{U}', G', \varphi'\}$ be l. u. s. for U, U', respectively, and let $U\subset U'$. By a (C^∞-) *injection* $\lambda:\{\tilde{U}, G, \varphi\}\rightarrow\{\tilde{U}', G', \varphi'\}$ we mean a C^∞-isomorphism λ from $\tilde{U}$ onto an open subset of $\tilde{U}'$ such that $\varphi=\varphi'\circ\lambda$. Every $\sigma\in G$ can be then considered as an injection of $\{\tilde{U}, G, \varphi\}$ into itself. Also if $\lambda:\{\tilde{U}, G, \varphi\}\rightarrow\{\tilde{U}', G', \varphi'\}$, $\lambda':\{\tilde{U}', G', \varphi'\}\rightarrow\{\tilde{U}'', G'', \varphi''\}$ are injections, $\lambda'\circ\lambda$ becomes an injection $\{\tilde{U}, G, \varphi\}\rightarrow\{\tilde{U}'', G'', \varphi''\}$. Hence if λ is an injection $\{\tilde{U}, G, \varphi\}\rightarrow\{\tilde{U}', G', \varphi'\}$ and $\sigma'\in G'$, then $\sigma'\circ\lambda$ becomes also an injection $\{\tilde{U}, G, \varphi\}\rightarrow\{\tilde{U}', G', \varphi'\}$. Conversely we have the following

LEMMA 1. *Let λ, μ be two injections $\{\tilde{U}, G, \varphi\}\rightarrow\{\tilde{U}', G', \varphi'\}$. Then there exists a uniquely determined $\sigma'\in G'$ such that $\mu=\sigma'\circ\lambda$.*

PROOF. Let $\tilde{p}\in\tilde{U}$. As we have $\varphi'(\mu(\tilde{p}))=\varphi(\tilde{p})=\varphi'(\lambda(\tilde{p}))$, there exists a $\sigma'\in G'$ such that $\mu(\tilde{p})=\sigma'(\lambda(\tilde{p}))$. Choosing $\lambda(\tilde{p})$ not to be a fixed point of G', the automorphism $\sigma'\in G'$ is uniquely determined. As the set of non-fixed points of G' in $\lambda(\tilde{U})$ is, by the above assumption, connected and everywhere dense in $\lambda(\tilde{U})$, the relation $\mu(\tilde{p})=\sigma'(\lambda(\tilde{p}))$ holds for all $\tilde{p}\in\tilde{U}$. Hence we have $\mu=\sigma'\circ\lambda$ with a uniquely determined $\sigma'\in G'$, q. e. d.

It follows, in particular, that if λ is an injection $\{\tilde{U}, G, \varphi\}\rightarrow\{\tilde{U}', G', \varphi'\}$ and $\sigma\in G$, there corresponds uniquely a $\sigma'\in G'$ such that $\lambda\circ\sigma=\sigma'\circ\lambda$. The correspondence $\sigma\rightarrow\sigma'$ becomes clearly an isomorphism from G into G'.

LEMMA 2. *Let λ be an injection $\{\tilde{U}, G, \varphi\}\rightarrow\{\tilde{U}', G', \varphi'\}$. If $\sigma'(\lambda(\tilde{U}))\cap\lambda(\tilde{U})\neq\phi$ with $\sigma'\in G'$, then $\sigma'(\lambda(\tilde{U}))=\lambda(\tilde{U})$ and σ' belongs to the image of the isomorphism $G\rightarrow G'$ defined above.*

PROOF. Assume that $\sigma'(\lambda(\tilde{U}))\cap\lambda(\tilde{U})\neq\phi$. Then there exist $\tilde{p}, \tilde{q}\in\tilde{U}$ such that $\sigma'\circ\lambda(\tilde{p})=\lambda(\tilde{q})$. Then, since $\varphi(\tilde{p})=\varphi(\tilde{q})$, we have $\tau(\tilde{p})=\tilde{q}$ with some $\tau\in G$. Let τ' be the element of G' corresponding to τ (i. e. one such that $\lambda\circ\tau=\tau'\circ\lambda$). Then we have $\sigma'(\lambda(\tilde{p}))=\tau'(\lambda(\tilde{p}))$. Choosing $\lambda(\tilde{p})$ not to be a fixed point of G', we have $\sigma'=\tau'$ and hence $\sigma'(\lambda(\tilde{U}))=\tau'(\lambda(\tilde{U}))=\lambda(\tau(\tilde{U}))=\lambda(\tilde{U})$, q. e. d.

It follows, in particular, that if $\lambda:\{\tilde{U}, G, \varphi\}\rightarrow\{\tilde{U}', G', \varphi'\}$ is an injection and if $\varphi(\tilde{U})=\varphi'(\tilde{U}')$, we have $\sigma'(\lambda(\tilde{U}))=\lambda(\tilde{U})$ for all $\sigma'\in G'$. (For otherwise $\tilde{U}'=\bigcup_{\sigma'\in G'}\sigma'(\lambda(\tilde{U}))$ would be disconnected.) Hence the C^∞-isomorphism $\lambda: \tilde{U}\to\tilde{U}'$ and the associated isomorphism $G\to G'$ become onto, and λ^{-1} becomes also an injection $\{\tilde{U}', G', \varphi'\}\to\{\tilde{U}, G, \varphi\}$. In this case, we call two l. u. s. $\{\tilde{U}, G, \varphi\}$, $\{\tilde{U}', G', \varphi'\}$ *equivalent*.

After these preparations we shall give the definition of V-manifold.

DEFINITION 1. A (C^∞-) *V-manifold* is a composite concept formed of a (Hausdorff) topological space M and a family $\mathfrak{F}$ (called a *defining family* for a V-manifold) of (C^∞-) l. u. s. for open subsets in M satisfying the following conditions.

(I) Every point p of M is contained in at least one $\mathfrak{F}$-uniformized open set (i. e. an open set U for which there exists l. u. s. $\{\tilde{U}, G, \varphi\}$ in $\mathfrak{F}$ such that $\varphi(\tilde{U})=U$). If p is contained in two $\mathfrak{F}$-uniformized open sets U_1, U_2, then there exists an $\mathfrak{F}$-uniformized open set U_3 such that $p\in U_3\subset U_1\cap U_2$.

(II) If $\{\tilde{U}, G, \varphi\}$, $\{\tilde{U}', G', \varphi'\}$ are l. u. s. in $\mathfrak{F}$ such that $\varphi(\tilde{U})\subset\varphi'(\tilde{U}')$, then there exists always a (C^∞-) injection $\lambda: \{\tilde{U}, G, \varphi\}\to\{\tilde{U}', G', \varphi'\}$. ($\lambda$ is uniquely determined up to $\sigma'\in G'$, by Lemma 1.)

By what we have mentioned above, it follows from (II) that two l. u. s. in $\mathfrak{F}$ for one and the same open set in M are always equivalent. Also if $\{\tilde{U}, G, \varphi\}$, $\{\tilde{U}', G', \varphi'\}$, $\{\tilde{U}'', G'', \varphi''\}$ are l. u. s. in $\mathfrak{F}$ such that $\varphi(\tilde{U})\subset\varphi'(\tilde{U}')\subset\varphi''(\tilde{U}'')$, then an injection $\{\tilde{U}, G, \varphi\}\to\{\tilde{U}'', G'', \varphi''\}$ is given by a composite of injections $\{\tilde{U}, G, \varphi\}\to\{\tilde{U}', G', \varphi'\}\to\{\tilde{U}'', G'', \varphi''\}$.

Two defining families $\mathfrak{F}, \mathfrak{F}'$ are said to be *directly equivalent*, if $\mathfrak{F}, \mathfrak{F}'$ are both contained in a defining family (satisfying (I), (II)), and *equivalent*, if there exists a chain of defining families $\mathfrak{F}_i$ ($1\leqq i\leqq r$) such that $\mathfrak{F}_1=\mathfrak{F}$, $\mathfrak{F}_r=\mathfrak{F}'$ and that $\mathfrak{F}_i, \mathfrak{F}_{i+1}$ ($1\leqq i\leqq r-1$) are directly equivalent. Equivalent families are regarded as defining one and the same V-manifold structure on M.[1)][2)]

1) Thus a V-manifold is, strictly speaking, a composite concept of a topological space M and an equivalent class of defining families. But in the following we consider a V-manifold M with a fixed defining family $\mathfrak{F}$ (i. e. a 'coordinate V-manifold' $(M, \mathfrak{F})$).

2) It can be proved that for any defining family $\mathfrak{F}$, there exists a defining family $\bar{\mathfrak{F}}$ such that $\bar{\mathfrak{F}}\supset\mathfrak{F}$ and that any open set in M which is simply connected and is contained in an $\mathfrak{F}$-uniformized open set is $\bar{\mathfrak{F}}$-uniformized.

Let M be a V-manifold and $p \in M$. Take a l. u. s. $\{\tilde{U}, G, \varphi\} \in \mathfrak{F}$ such that $p \in \varphi(\tilde{U})$ and choose $\tilde{p} \in \tilde{U}$ such that $\varphi(\tilde{p})=p$. Then it can easily be seen by Lemma 2 that the structure of the isotropy subgroup $G_{\tilde{p}}$ of G at $\tilde{p}$ does not depend on the choice of $\tilde{U}$ and $\tilde{p}$ (and hence of $\mathfrak{F}$), and is uniquely determined by p. Hence we call sometimes $G_{\tilde{p}}$ simply the 'isotropy group of p.'

An ordinary C^∞-manifold is nothing other than a C^∞-V-manifold for which the isotropy group of each point reduces to the unit group. In general, let M be a V-manifold and consider the set S composed of all 'singular points' of M, i. e. the points of M with non-trivial isotropy groups. Let $p \in S$ and $\{\tilde{U}, G, \varphi\}$, $\tilde{p}$, $G_{\tilde{p}}$ be as above. Then taking a suitable coordinate system around $\tilde{p}$, $G_{\tilde{p}}$ becomes a finite group of linear transformations. (Let $\{u^1, \cdots, u^m\}$ be a coordinate system around $\tilde{p}$ and consider the system $v^i = \dfrac{1}{N_{G_{\tilde{p}}}} \sum a_{ij}(\sigma^{-1}) u^j \circ \sigma$, where $a_{ij}(\sigma) = \left[\dfrac{\partial u^i \circ \sigma}{\partial u^j}\right]_{\tilde{p}}$, $N_{G_{\tilde{p}}} = [G_{\tilde{p}} : 1]$.) Hence $\varphi^{-1}(S)$ is expressed locally by a finite union of linear submanifolds of $\tilde{U}$. This means that S is a V-subvariety of dimension $\leqq m-2$ of M.[3)]

The isotropy group of an 'ordinary point' of S, i. e. a point where S has locally a structure of V-manifold, depends only on the irreducible component S_i of S containing that point and is called sometimes the isotropy group of S_i. Clearly $M-S$ is an ordinary C^∞-manifold (connected if M is connected).

It can be also proved that if $\tilde{M}$ is a C^∞-manifold and $\mathfrak{G}$ is a properly discontinuous group of C^∞-automorphisms of $\tilde{M}$, then the quotient space $\mathfrak{G} \backslash \tilde{M}$ possesses a canionical V-manifold structure.

REMARK. If we replace the word "a connected open set in R^m" in the above definition of l. u. s. by "a connected C^∞-manifold", we obtain an equivalent definition of V-manifold. This modification of the definition of l. u. s. will be used in 3.

2. C^∞-V-manifold map.

DEFINITION 2. Let $(M_1, \mathfrak{F}_1)$, $(M_2, \mathfrak{F}_2)$ be two V-manifolds. We

3) A C^∞-V-subvariety X of M is a (closed) subset of M such that for any l. u. s. $\{\tilde{U}, G, \varphi\} \in \mathfrak{F}$ for U, $\varphi^{-1}(X \cap U)$ is a (G-invariant) C^∞-subvariety of $\tilde{U}$ in the usual sense. A V-subvariety X is decomposed into a locally finite union of irreducible subvarieties X_i, called the 'irreducible components' of X.

mean by a (C^∞-) *V-manifold map h* from $(M_1, \mathfrak{F}_1)$ into $(M_2, \mathfrak{F}_2)$ a system of mappings $\{h_{\tilde{U}_1}\}$ ($\{\tilde{U}_1, G_1, \varphi_1\} \in \mathfrak{F}_1$) as follows:

(i) There is a correspondence $\{\tilde{U}_1, G_1, \varphi_1\} \to \{\tilde{U}_2, G_2, \varphi_2\}$ from $\mathfrak{F}_1$ into $\mathfrak{F}_2$ such that for any $\{\tilde{U}_1, G_1, \varphi_1\} \in \mathfrak{F}_1$ we have a C^∞-map $h_{\tilde{U}_1}$ from $\tilde{U}_1$ into $\tilde{U}_2$.

(ii) Let $\{\tilde{U}_1, G_1, \varphi\}$, $\{\tilde{U}_1', G_1', \varphi_1'\} \in \mathfrak{F}_1$, $\{\tilde{U}_2, G_2, \varphi_2\}$, $\{\tilde{U}_2', G_2', \varphi_2'\} \in \mathfrak{F}_2$ be the corresponding l. u. s. (in the sense of (i)) and let $\varphi_1(\tilde{U}_1) \subset \varphi_1'(\tilde{U}_1')$. Then for any injection $\lambda_1: \{\tilde{U}_1, G_1, \varphi_1\} \to \{\tilde{U}_1', G_1', \varphi_1'\}$ there exists an injection $\lambda_2: \{\tilde{U}_2, G_2, \varphi_2\} \to \{\tilde{U}_2', G_2', \varphi_2'\}$ such that

$$\lambda_2 \circ h_{\tilde{U}_1} = h_{\tilde{U}_1'} \circ \lambda_1 .$$

It follows from (i), (ii) that there exists uniquely a continuous map h from M_1 into M_2 such that for any $\{\tilde{U}_1, G_1, \varphi_1\} \in \mathfrak{F}_1$, and for the corresponding $\{\tilde{U}_2, G_2, \varphi_2\} \in \mathfrak{F}_2$, we have

$$\varphi_2 \circ h_{\tilde{U}_1} = h \circ \varphi_1 .$$

h is called a *C^∞-map* $M_1 \to M_2$ defined by a C^∞-V-manifold map $h = \{h_{\tilde{U}_1}\}$: $(M_1, \mathfrak{F}_1) \to (M_2, \mathfrak{F}_2)$.[4] (We use thus the same notation for a C^∞-V-manifold map and the corresponding C^∞-map. But no confusion will arise.)

It is possible to define an equivalence relation between V-manifold maps quite similarly as in **1**. Then the C^∞-map $M_1 \to M_2$ defined by a V-manifold map $(M_1, \mathfrak{F}_1) \to (M_2, \mathfrak{F}_2)$ depends only on the equivalence class of the latter.

Considering $\boldsymbol{R}$ (the set of all real numbers) as a V-manifold defined by a single l. u. s. $\{\boldsymbol{R}, \{1\}, 1\}$, we define a *$C^\infty$-function* on a V-manifold M as a C^∞-map $M \to \boldsymbol{R}$.

3. V-bundle. Let M, B be two V-manifolds with a C^∞-map $\pi: B \to M$. Let further F be a C^∞-manifold and $\boldsymbol{G}$ be a Lie group operating on F as a (C^∞-) group of transformations. (We don't assume $\boldsymbol{G}$ to be effective.)

DEFINITION 3. A pair of defining families $(\mathfrak{F}, \mathfrak{F}^*)$, $\mathfrak{F}$ being a defining family of M and $\mathfrak{F}^*$ that of B[5], is called a pair of defining

4) The notion of C^∞-map thus defined is inconvenient in the point that a composite of two C^∞-maps defined in a different choice of defining families is not always a C^∞-map.

5) Here the l. u. s. in $\mathfrak{F}$, $\mathfrak{F}^*$ (or at least those in $\mathfrak{F}^*$) are understood in the modified sense as was stated in Remark at the end of **1**.

families for a (coordinate) V-bundle (B, M, π, F, G), if it satisfies the following conditions:

(i) There exists a one-to-one correspondence $\{\tilde{U}, G, \varphi\} \leftrightarrow \{\tilde{U}^*, G^*, \varphi^*\}$ between $\mathfrak{F}$ and $\mathfrak{F}^*$ such that $\tilde{U}^* = \tilde{U} \times F$ and denoting by $\pi_{\tilde{U}^*}$ the projection $\tilde{U}^* \rightarrow \tilde{U}$, we have

$$\pi \circ \varphi^* = \varphi \circ \pi_{\tilde{U}^*}.$$

(ii) Let $\{\tilde{U}, G, \varphi\}$, $\{\tilde{U}^*, G^*, \varphi^*\}$; $\{\tilde{U}', G', \varphi'\}$, $\{\tilde{U}^{*\prime}, G^{*\prime}, \varphi^{*\prime}\}$ be two pairs of corresponding l. u. s. in $(\mathfrak{F}, \mathfrak{F}^*)$ and let $\varphi(\tilde{U}) \subset \varphi'(\tilde{U}')$. Then $\varphi^*(\tilde{U}^*) \subset \varphi^{*\prime}(\tilde{U}^{*\prime})$ and there exists a one-to-one correspondence $\lambda \leftrightarrow \lambda^*$ between injections $\lambda: \{\tilde{U}, G, \varphi\} \rightarrow \{\tilde{U}', G', \varphi'\}$ and $\lambda^*: \{\tilde{U}^*, G^*, \varphi^*\} \rightarrow \{\tilde{U}^{*\prime}, G^{*\prime}, \varphi^{*\prime}\}$ such that for $(\tilde{p}, q) \in \tilde{U}^* = \tilde{U} \times F$ we have

$$\lambda^*(\tilde{p}, q) = (\lambda(\tilde{p}),\ g_\lambda(\tilde{p})q)$$

with $g_\lambda(\tilde{p}) \in G$. The mapping $g_\lambda: \tilde{U} \rightarrow G$ is a C^∞-map satisfying the relation

$$g_{\mu\lambda}(\tilde{p}) = g_\mu(\lambda(\tilde{p})) \cdot g_\lambda(\tilde{p}) \tag{1}$$

for any injections $\{\tilde{U}, G, \varphi\} \xrightarrow{\lambda} \{\tilde{U}', G', \varphi'\} \xrightarrow{\mu} \{\tilde{U}'', G'', \varphi''\}$. ((1) is satisfied automatically, if G acts on F effectively.)

A composite concept of B, M, π, F, G and a pair of defining families $(\mathfrak{F}, \mathfrak{F}^*)$ satisfying the above conditions is called a *(coordinate) V-bundle.* (For simplicity we call sometimes B a (coordinate) V-bundle.)

It follows, in particular, from (ii) that there exists a one-to-one correspondence $\sigma \leftrightarrow \sigma^*$ between G and G^* such that $\sigma^*(\tilde{p}, q) = (\sigma(\tilde{p}), g_\sigma(\tilde{p})q)$. This correspondence is clearly an isomorphism. Denoting by $G_{\tilde{p}}$ the isotropy subgroup of G at $\tilde{p}$ and by $G_{\tilde{p}}^*$ the corresponding subgroup of G^*, we can prove easily that for $p = \varphi(\tilde{p})$

$$\pi^{-1}(p) \approx G_p^* \backslash \tilde{p} \times F \approx \{g_\sigma(\tilde{p})\,;\ \sigma \in G_p\} \backslash F.$$

Thus a V-bundle B is not always a bundle in the usual sense over M.

Equivalence relation between pairs of defining families $(\mathfrak{F}, \mathfrak{F}^*)$ is defined quite similarly as in **1.** (A V-bundle is a composite concept formed of (B, M, π, F, G) and an equivalent class of pairs of defining families.) Especially it can be easily verified that two pairs of defining families $(\mathfrak{F}, \mathfrak{F}_1^*)$, $(\mathfrak{F}, \mathfrak{F}_2^*)$ are directly equivalent (i. e. $(\{\mathfrak{F}, \mathfrak{F}\}, \{\mathfrak{F}_1^*, \mathfrak{F}_2^*\})$ becomes also a pair of defining families), if and only if there exists a C^∞-map $\delta_{\tilde{U}}: \tilde{U} \rightarrow G$ such that

$$\varphi_1{}^*(\tilde{p}, q) = \varphi_2{}^*(\tilde{p}, \delta_{\tilde{U}}(\tilde{p})q)$$

for any $\{\tilde{U}\times F, G_1{}^*, \varphi_1{}^*\}\in\mathfrak{F}_1{}^*$, $\{\tilde{U}\times F, G_2{}^*, \varphi_2{}^*\}\in\mathfrak{F}_2{}^*$ corresponding to the same $\{\tilde{U}, G, \varphi\}\in\mathfrak{F}$ and that

$$g_\lambda^{(2)}(\tilde{p}) = \delta_{\tilde{U}'}(\lambda(\tilde{p}))g_\lambda^{(1)}(\tilde{p})\delta_{\tilde{U}}(\tilde{p})^{-1} \tag{2}$$

for any injection $\lambda: \{\tilde{U}, G, \varphi\}\to\{\tilde{U}', G', \varphi'\}$, $g_\lambda^{(1)}, g_\lambda^{(2)}$ denoting the map g_λ in (ii) corresponding to $\mathfrak{F}_1{}^*, \mathfrak{F}_2{}^*$, respectively. (The necessity of these conditions is evident. If, conversely, they are satisfied, define the injection $\{\tilde{U}\times F, G_1{}^*, \varphi_1{}^*\}$ $(\in\mathfrak{F}_1{}^*)\to\{\tilde{U}\times F, G_2{}^*, \varphi_2{}^*\}$ $(\in\mathfrak{F}_2{}^*)$ corresponding to the identical injection of $\{\tilde{U}, G, \varphi\}$ onto itself by the map $(\tilde{p}, q)\to(\tilde{p}, \delta_{\tilde{U}}(\tilde{p})q)$.)

Let $(B, M, \pi, F, \boldsymbol{G})$ be a V-bundle with a pair of defining families $(\mathfrak{F}, \mathfrak{F}^*)$. A V-manifold map $f=\{f_{\tilde{U}}\}: (M, \mathfrak{F})\to(B, \mathfrak{F}^*)$ is called a (C^∞-) *cross section* of this V-bundle, if the correspondence $\mathfrak{F}\to\mathfrak{F}^*$ in Definition 2 (i) is given by the correspondence in Definition 3 (i) and if $\pi_{\tilde{U}}\circ f_{\tilde{U}}=1$ (then clearly $\pi\circ f=1$). To give a cross section $f: (M, \mathfrak{F})\to(B, \mathfrak{F}^*)$ is therefore to give a cross section (in the usual sense) $f_{\tilde{U}}$ of each $\tilde{U}^*=\tilde{U}\times F$ such that for any injection $\lambda: \{\tilde{U}, G, \varphi\}\to\{\tilde{U}', G', \varphi'\}$ we have $f_{\tilde{U}'}\circ\lambda=\lambda^*\circ f_{\tilde{U}}$. (In particular, $f_{\tilde{U}}$ is G-invariant in the sense that $f_{\tilde{U}}\circ\sigma=\sigma^*\circ f_{\tilde{U}}$ for all $\sigma\in G$.)

A direct product V-bundle (i. e. a V-bundle for which $g_\lambda=1$ in Definition 3 (ii)) has clearly many cross sections. Conversely let $(B, M, \pi, \boldsymbol{G}, \boldsymbol{G})$ be a *principal V-bundle*, i. e. a V-bundle such that $F=\boldsymbol{G}$ and $g\in\boldsymbol{G}$ acts on $\boldsymbol{G}$ as a left transformation, and assume that it has a cross section $f=\{f_{\tilde{U}}\}$. Then, denoting $f_{\tilde{U}}(\tilde{p})=(\tilde{p}, \delta_{\tilde{U}}(\tilde{p}))$, we have

$$g_\lambda(\tilde{p}) = \delta_{\tilde{U}'}(\lambda(\tilde{p}))\delta_{\tilde{U}}(\tilde{p})^{-1}$$

for any injection $\lambda: \{\tilde{U}, G, \varphi\}\to\{\tilde{U}', G', \varphi'\}$. This means that this (coordinate) V-bundle is directly equivalent to a direct product V-bundle over the same (coordinate) V-manifold.

4. *V*-bundle map. The notion of V-bundle map can by defined quite similarly as that of V-manifold map. Namely we have

DEFINITION 4. Let $(B_1, M_1, \pi_1, F, \boldsymbol{G})$, $(B_2, M_2, \pi_2, F, \boldsymbol{G})$ be two V-bundles with defining families $(\mathfrak{F}_1, \mathfrak{F}_1{}^*)$, $(\mathfrak{F}_2, \mathfrak{F}_2{}^*)$, respectively. A system of mappings $h^*=\{h_{\tilde{U}_1}\}$ $(\{U_1, G_1, \varphi_1\}\in\mathfrak{F}_1)$ is called a (C^∞-)*V-bundle map* if the following conditions are satisfied:

(i) There exists a correspondence $\{\tilde{U}_1, G_1, \varphi_1\}\to\{\tilde{U}_2, G_2, \varphi_2\}$ from $\mathfrak{F}_1$ into $\mathfrak{F}_2$, such that for any $\{\tilde{U}_1, G_1, \varphi_1\}$ we have a C^∞-map $h_{\tilde{U}_1}{}^*$

from $\tilde{U}_1 \times F$ into $\tilde{U}_2 \times F$ and a C^∞-map $h_{\tilde{U}_1}$ from $\tilde{U}_1$ into $\tilde{U}_2$ such that

$$h^*_{\tilde{U}_1}(\tilde{p}, q) = (h_{\tilde{U}_1}(\tilde{p}), \ r_{\tilde{U}_1}(\tilde{p})q)$$

with $r_{\tilde{U}_1}(\tilde{p}) \in \boldsymbol{G}$. $r_{\tilde{U}_1}$ is a C^∞-map from $\tilde{U}_1$ into $\boldsymbol{G}$.

(ii) Let $\{\tilde{U}_1, G_1, \varphi_1\}$, $\{\tilde{U}_1', G_1', \varphi_1'\}$ be l. u. s. in $\mathfrak{F}_1$ such that $\varphi_1(\tilde{U}_1) \subset \varphi_1'(\tilde{U}_1')$ and $\{\tilde{U}_2, G_2, \varphi_2\}$, $\{\tilde{U}_2', G_2', \varphi_2'\}$ be the corresponding l. u. s. in $\mathfrak{F}_2$. Then for any injection $\lambda_1 : \{\tilde{U}_1, G_1, \varphi_1\} \to \{\tilde{U}_1', G_1', \varphi_1'\}$ there exists an injection $\lambda_2 : \{\tilde{U}_2, G_2, \varphi_2\} \to \{\tilde{U}_2', G_2', \varphi_2'\}$ such that

$$\lambda_2^* \circ h^*_{\tilde{U}_1} = h^*_{\tilde{U}_1'} \circ \lambda_1^* .$$

(Hence also $\lambda_2 \circ h_{\tilde{U}_1} = h_{\tilde{U}_1'} \circ \lambda_1$). We assume further that

$$g_{\lambda_2}(h_{\tilde{U}_1}(\tilde{p})) = r_{\tilde{U}_1'}(\lambda_1(\tilde{p})) g_{\lambda_1}(\tilde{p}) r_{\tilde{U}_1}(\tilde{p})^{-1} .$$

(This condition is satisfied automatically if $\boldsymbol{G}$ acts on F effectively.)

Thus $h^* = \{h_{\tilde{U}_1}^*\}$, $h = \{h_{\tilde{U}_1}\}$ are V-manifold maps from $(B_1, \mathfrak{F}_1^*)$ into $(B_2, \mathfrak{F}_2^*)$, from $(M_1, \mathfrak{F}_1)$ into $(M_2, \mathfrak{F}_2)$, respectively, and the corresponding C^∞-maps $h^* : B_1 \to B_2$, $h : M_1 \to M_2$ satisfy the relation $\pi_2 \circ h^* = h \circ \pi_1$.

If, in particular, $M_1 = M_2$, $\mathfrak{F}_1 = \mathfrak{F}_2$ and all $h_{\tilde{U}_1}$ are the identity, then it follows that all $h_{\tilde{U}_1}^*$ are onto, one-to-one and $h^{*-1} = \{h_{\tilde{U}_1}^{*-1}\}$ becomes also a (C^∞-) V-bundle map. In this case, we call these two (coordinate) V-bundles equivalent.[6]

The following theorem is an analogue of the existence theorem for ordinary fibre bundles. The proof is also quite similar as in the ordinary case.

THEOREM 1. *Let M be a V-manifold with a defining family $\mathfrak{F}$, F a (C^∞-) manifold and $\boldsymbol{G}$ a Lie group operating on F (as a C^∞-group of transformations). If we have a system of C^∞-maps $g_\lambda : \tilde{U} \to \boldsymbol{G}$ ($\{\tilde{U}, G, \varphi\} \in \mathfrak{F}$, λ being any injections: $\{\tilde{U}, G, \varphi\} \to \{\tilde{U}', G', \varphi'\}$) satisfying the condition* (1), *then we can construct a* (*coordinate*) *V-bundle B over M as described in Definition* 3. *If two systems of mappings $\{g_\lambda^{(1)}\}$, $\{g_\lambda^{(2)}\}$ satisfy the relation* (2), *then there exists uniquely a V-bundle map $h^* = \{h_{\tilde{U}_1}^*\}$ from the V-bundle corresponding to $\{g_\lambda^{(1)}\}$ onto the one corresponding to $\{g_\lambda^{(2)}\}$ such that the correspondence $\mathfrak{F} \to \mathfrak{F}$ in Definition 4 is the identity and that we have $h_{\tilde{U}}^*(\tilde{p}, q) = (\tilde{p}, \delta_{\tilde{U}}(\tilde{p})q)$ for $(\tilde{p}, q) \in \tilde{U} \times F$.* (*Thus $\{h_{\tilde{U}_1}^*\}$ defines an equivalence between these two* (*coordinate*) *V-bundles.*)

REMARK. We can define the notion of V-bundle, replacing in Definition 3 (i) the words " $\tilde{U}^* = \tilde{U} \times F$ " by " $\tilde{U}^*$ is a fibre bundle

6) In case $B_1 = B_2$, this notion of equivalence is somewhat different from that given in **3.**

over $\tilde{U}$ with the fibre F and the group G" and in (ii) "that for $(\tilde{p}, q) \in \tilde{U}^* = \tilde{U} \times F \cdots$" by "that λ^* is a bundle map $\tilde{U}^* \to \tilde{U}^{*\prime}$ inducing the map $\lambda: \tilde{U} \to \tilde{U}'$." (Of course, these words require more explicit indications in case G acts ineffectively on F.) We need this modification in § **2.1.**

5. Examples. Let $(M, \mathfrak{F})$ be a V-manifold. Assuming that every $\tilde{U}(\{\tilde{U}, G, \varphi\} \in \mathfrak{F})$ is contained in $\boldsymbol{R}^m$, we fix a coordinate system $\{u^1, \cdots, u^m\}$ in each $\tilde{U}$ once for all. Let $F = \boldsymbol{R}^m$ (vector space of dimension m over $\boldsymbol{R}$) and $G = GL(m, \boldsymbol{R})$ (group of all non-singular matrices of degree m.) For any injection $\lambda: \{\tilde{U}, G, \varphi\} \to \{\tilde{U}', G', \varphi'\}$ put

$$g_\lambda(\tilde{p}) = \left(\frac{\partial u'^i \circ \lambda}{\partial u^j}\right) \text{ (Jacobian matrix of } \lambda \text{ at } \tilde{p}),$$

$\{u^i\}$, $\{u'^i\}$ being the (fixed) coordinate systems in $\tilde{U}$, $\tilde{U}'$, respectively. Then the system g_λ, satisfying the condition of Theorem **1**, defines a V-bundle (T, M, π, F, G) with a pair of defining families $(\mathfrak{F}, \mathfrak{F}^*)$. This V-bundle is called the *tangent vector bundle* over M.

Let us note that the 'fibre' $\pi^{-1}(p)$ $(p \in M)$ is not always a vector space. Let $p \in \varphi(\tilde{U})$, $\{\tilde{U}, G, \varphi\} \in \mathfrak{F}$ and choose $\tilde{p} \in \tilde{U}$ such that $\varphi(\tilde{p}) = p$. Then, $\pi^{-1}(p) \approx \{g_\sigma(\tilde{p})\,;\ \sigma \in G_{\tilde{p}}\} \backslash \boldsymbol{R}^m$, $G_{\tilde{p}}$ denoting the isotropy subgroup of G at $\tilde{p}$. Now $\pi_{\tilde{U}}^{-1}(\tilde{p}) = \tilde{p} \times \boldsymbol{R}^m$ can be identified with $T_{\tilde{p}}$ (the tangent space to $\tilde{U}$ at $\tilde{p}$) by the correspondence

$$\tilde{p} \times \begin{pmatrix} x^1 \\ \vdots \\ x^m \end{pmatrix} \leftrightarrow X = \sum_i x^i \frac{\partial}{\partial u^i}.$$

Then, denoting by $T_{\tilde{p}}^{G_{\tilde{p}}}$ the linear subspace of $T_{\tilde{p}}$ formed of all $G_{\tilde{p}}$-invariant vectors (i. e. vectors invariant under $g_\sigma(\tilde{p})$ $(\sigma \in G_{\tilde{p}})$), we see that $\pi^{-1}(p)$ contains a vector space $T_p = \varphi^*(T_{\tilde{p}}^{G_{\tilde{p}}})$, which is independent of the choice of $\tilde{U}$ and $\tilde{p}$. An element of T_p is called a *tangent vector* to M at p.

A cross section $\mathfrak{X}$ of the V-bundle T is called a (contravariant) *vector field* over M. In the above notations, $\mathfrak{X}_{\tilde{U}}$ being a G-invariant cross-section of $\tilde{U}^* = \tilde{U} \times F$ (i. e. a G-invariant vector field over $\tilde{U}$ in the usual sense), we have $\mathfrak{X}_{\tilde{U}}(\tilde{p}) \in T_{\tilde{p}}^{G_{\tilde{p}}}$ and so $\mathfrak{X}(p) \in T_p$. Thus $\mathfrak{X}(p)$ being a tangent vector at p for any $p \in M$, the set of all vector fields over M forms a vector space.

More generally we can construct an (r, s) tensor bundle over M by means of the system $\{g_\lambda\}$:

$$g_\lambda(\tilde{p})=\underbrace{\left(\frac{\partial u'^i}{\partial u^j}\right)\times\cdots\times}_{s}\underbrace{\left(\frac{\partial u^j}{\partial u'^i}\right)\times\cdots}_{r}$$

$\times$ denoting the Kronecker product of matrices. (In this case, $F=\boldsymbol{R}^{m(r+s)}$, and $\boldsymbol{G}=GL(m, \boldsymbol{R})$ operating on F as an (r, s)-tensor representation.) We can also consider skew-symmetric or symmetric tensor bundles over M.

In particular, consider the skew-symmetric $(h, 0)$-tensor bundle over M. As in the case of the tangent vector bundle, $\pi^{-1}(p)$ $(p\in M)$ contains a vector space D_p^h, which is isomorphic to the space of $G_{\tilde{p}}$-invariant skew-symmetric $(h, 0)$ tensors at $\tilde{p}$ $(\varphi(\tilde{p})=p)$. D_p^1 can be regarded as a dual space of T_p. It should be noted that $\sum_{h=0}^m D_p^h$ is not always an exterior algebra over D_p^1. A cross section ω of D^h is called a *differential form* of degree h (or briefly h-form) on M. Since $\omega(p)\in D_p^h$ for any $p\in M$, the set of all h-forms over M forms a vector space. By definition, to give an h-form ω on M is to give a (G-invariant) h-form $\omega_{\tilde{U}}$ on each $\tilde{U}$ $(\{\tilde{U}, G, \varphi\}\in\mathfrak{F})$ such that it holds $\omega_{\tilde{U}}=\omega_{\tilde{U}'}\circ\lambda$ for any injection $\lambda: \{\tilde{U}, G, \varphi\}\to\{\tilde{U}', G', \varphi'\}$. Using these 'local expressions', we can define the operations $\wedge$ and d just as in the case of ordinary manifold. Also if $h: M\to M'$ is a V-manifold map we can define a form $\omega'\circ h$ on M for any form ω' on M'.

In case M is *orientable*, i. e. in case we can choose the coordinate systems $\{u^1,\cdots, u^m\}$ such that we have $\det\left(\frac{\partial u'^i\circ\lambda}{\partial u^j}\right)>0$ for any injection λ, we can define the *integral* $\int_M\omega$ of an n-form ω as follows. If the carrier of ω (i. e. the closure of $\{p; p\in M, \omega(p)\neq 0\}$) is contained in an $\mathfrak{F}$-uniformized open set $U=\varphi(\tilde{U})$, we put

$$\int_M\omega=\frac{1}{N_G}\int_{\tilde{U}}\omega_{\tilde{U}}\,, \tag{3}$$

N_G denoting the order of G. In general, assume that there is a locally finite family of C^∞-functions $\{f_i\}$ such that the carrier of f_i is contained in an $\mathfrak{F}$-uniformized open set V_i and that $\sum_i f_i=1$ on the carrier of ω. Then we define the integral of ω by

$$\int_M\omega=\sum_i\int_M f_i\omega\,,$$

if the summation on the right side converges absolutely for any such 'partition of unity' $\{f_i\}$. We can prove easily that this definition does not depend on the choice of $\{f_i\}$.

Finally let P be the principal V-bundle associated with T. (i.e. a V-bundle with $F=\boldsymbol{G}=GL(m, \boldsymbol{R})$, $g_\lambda(p)=\left(\frac{\partial u'^i\circ\lambda}{\partial u^j}\right)$.) In the same notations as above, $(\tilde{p}, (x^i_j))\in\tilde{U}\times GL(m, \boldsymbol{R})$ is in one-to-one correspondence with a 'frame' (i.e. a base of the tangent space) $(X_1,\cdots, X_m)$ at $\tilde{p}$ by the relation $X_j=\sum_i x^i_j\frac{\partial}{\partial u^i}$. Hence P is called a *frame bundle* over M. Since G^* has no fixed point in $\tilde{U}^*=\tilde{U}\times GL(m, \boldsymbol{R})$, P becomes a C^∞-manifold in the ordinary sense.

§ 2. Riemannian geometry on a V-manifold.

1. Riemannian metric on a V-manifold. Let $(M, \mathfrak{F})$ be a V-manifold with a Riemannian metric g. By definition, to give a Riemannian metric g on M is to give a Riemannian metric $g_{\tilde{U}}$ on each $\tilde{U}$ ($\{\tilde{U}, G, \varphi\}\in\mathfrak{F}$) such that for any injection $\lambda:\{\tilde{U}, G, \varphi\}\to\{\tilde{U}', G', \varphi'\}$ we have

$$g_{\tilde{U}}(\tilde{\mathfrak{X}}, \tilde{\mathfrak{Y}})=g_{\tilde{U}'}(\lambda(\tilde{\mathfrak{X}}), \lambda(\tilde{\mathfrak{Y}}))\,,$$

$\tilde{\mathfrak{X}}$, $\tilde{\mathfrak{Y}}$ being arbitrary vector fields on $\tilde{U}$ and $\lambda(\tilde{\mathfrak{X}})$, $\lambda(\tilde{\mathfrak{Y}})$ the corresponding vector fields on $\lambda(\tilde{U})\subset\tilde{U}'$. (In particular, each $g_{\tilde{U}}$ is G-invariant.)

By means of the Riemannian metric g we can define the *unit tangent vector bundle* T_0 and the *orthonormal frame bundle* P_0 over M (by the 'reduction' of the structure group from $GL(m, \boldsymbol{R})$ to $O(m)$ (group of all orthogonal matrices of degree m)). For instance, T_0 is defined as follows. For each $\{\tilde{U}, G, \varphi\}\in\mathfrak{F}$, let $\tilde{U}^*=T_0(\tilde{U})$ be the unit tangent vector bundle (in the usual sense) over $\tilde{U}$ (with $F=S^{m-1}$, $\boldsymbol{G}=O(m)$); $T_0(\tilde{U})$ can be considered as a subset of $T(\tilde{U})$ (the tangent vector bundle over $\tilde{U}$). For each injection $\lambda:\{\tilde{U}, G, \varphi\}\to\{\tilde{U}', G', \varphi'\}$, let λ^* be the corresponding bundle map $\tilde{U}^*\to\tilde{U}^{*\prime}$. Then defining φ^* as a restriction to $T_0(\tilde{U})$ of the corresponding map $T(\tilde{U})\to T$ and putting $T_0=\bigcup\varphi^*(\tilde{U}^*)$, we obtain a V-manifold T_0 with a defining family $\mathfrak{F}^*=\{\{\tilde{U}^*, G^*, \varphi^*\}\}$, $(T_0, \mathfrak{F}^*)$ has clearly a structure of V-bundle with $F=S^{m-1}$, $\boldsymbol{G}=O(m)$ over $(M, \mathfrak{F})$ in the modified sense as stated in Remark at the end of § **1,4.**

2. Basic forms, connection forms and curvature forms. Let $\{\tilde{U}, G, \varphi\} \in \mathfrak{F}$, $\tilde{p} \in \tilde{U}$ and $(X_1, \cdots, X_m)$ be an orthonormal frame at $\tilde{p}$ (i. e. a base of $T_{\tilde{p}}$ such that $g_{\tilde{U}}(X_i, X_j) = \delta_{ij}$). Let $\{u^1, \cdots, u^m\}$ be a coordinate system in $\tilde{U}$ and put $X_j = \sum_i x_j^i \frac{\partial}{\partial u^i}$, $(X_j^i) = (x_j^i)^{-1}$. Then the basic forms over $\tilde{U}$ are defined by

$$\theta_{\tilde{U}}{}^i = \sum_j X_j^i du^j \qquad (1 \leqq i \leqq m).$$

They are 1-forms on $P_0(\tilde{U})$ and the above formula is considered as giving their expression at $p^* = (\tilde{p}; X_1, \cdots, X_m)$. (If we consider them as covariant vectors at $\tilde{p}$, $(\theta^1, \cdots, \theta^m)$ is nothing other than a dual base of $(X_1, \cdots, X_m)$.)

If $\theta_{\tilde{U}}{}^i$, $\theta_{\tilde{U}'}{}^i$ are basic forms over $\tilde{U}$, $\tilde{U}'$, respectively and if λ is an injection $\{\tilde{U}, G, \varphi\} \rightarrow \{\tilde{U}', G', \varphi'\}$, then we have $\theta_{\tilde{U}}{}^i = \theta_{\tilde{U}'} \circ \lambda^*$. For let $\tilde{p}^* = (\tilde{p}; X_1, \cdots, X_m) \in P_0(\tilde{U})$, $\lambda^*(\tilde{p}^*) = (\lambda(\tilde{p}); \lambda(X_1) \cdots, \lambda(X_m))$, $\{u'^i\}$ be a coordinate system in $\tilde{U}'$ and $X_j' = \sum_i x_j'^i \frac{\partial}{\partial u'^i}$, $(X_j'^i) = (x_j'^i)^{-1}$. Then

$$\theta_{\tilde{U}'}{}^i \text{ at } \lambda^*(\tilde{p}^*) = \sum_j X_j'^i du'^j = \sum_{j,k} X_j'^i \frac{\partial u'^j}{\partial u^k} du^k$$
$$= \sum_j X_j^i du^j = \theta_{\tilde{U}}{}^i \text{ at } \tilde{p}^*.$$

This proves our assertion. (It follows, in particular, that $\theta_{\tilde{U}}{}^i$ $(1 \leqq i \leqq m)$ are G^*-invariant.) Hence the systems $\{\theta_{\tilde{U}}{}^i\}$ $(\{\tilde{U}, G, \varphi\} \in \mathfrak{F})$ define 1-forms θ^i on P_0, which we call the *basic forms* over M.

Quite similarly *connection forms* ω_{ij} $(1 \leqq i, j \leqq m)$ over M are defined by those $\omega_{ij\tilde{U}}$ over $\tilde{U}$. They are 1-forms on P_0 with the following characterizing properties:

$$d\theta^i = \sum_j \omega_{ij} \wedge \theta^j, \quad \omega_{ij} = -\omega_{ji}$$

and have the local expressions as follows:

$$\omega_{ij\tilde{U}} = -\sum_k X_k{}^i dx_j{}^k - \sum_{k,l,h} X_k{}^i \Gamma_{lh}^k du^l x_j{}^h. \tag{4}$$

Also *curvature forms* Ω_{ij} are defined by the 'structure equation':

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} + \Omega_{ij} \qquad (1 \leqq i, j \leqq m)$$

and have local expressions as follows:

$$\Omega_{ij\tilde{U}} = -\frac{1}{2}\sum_{p,q,k,l} x_i{}^p x_j{}^q R_{pqkl}\, du^k \wedge du^l . \tag{5}$$

As is well-known, we have the following relations

$$\sum_j \Omega_{ij} \wedge \theta^j = 0 ,$$

$$d\Omega_{ij} = \sum_k \omega_{ik} \wedge \Omega_{kj} - \sum_k \Omega_{ik} \wedge \omega_{kj} .$$

Denoting by R_A the 'right translation' of P_0 by $A=(a_{ij})\in O(m)$ and putting $A^{-1}=(A_{ij})$, we have easily

$$\theta^i \circ R_A = \sum_j A_{ij}\, \theta^j, \quad \omega_{ij}\circ R_A = \sum_{k,l} A_{ik}A_{jl}\, \omega_{kl} ,$$

$$\Omega_{ij}\circ R_A = \sum_{k,l} A_{ik}A_{jl}\, \Omega_{kl} .$$

3. Gaussian curvature. Now let m be even $=2p$. We put

$$\Omega = \frac{(-1)}{2^{2p}p!\,\pi^p} \sum_{(i_1,\cdots,i_m)} \varepsilon^{i_1\cdots i_m}\, \Omega_{i_1 i_2} \wedge \cdots \wedge \Omega_{i_{m-1} i_m} , \tag{6}$$

$(i_1,\cdots,i_m)$ running over all permutations of $(1,\cdots,m)$ and $\varepsilon^{i_1\cdots i_m}$ denoting the sign of $(i_1,\cdots,i_m)$. (In case m is odd, we put simply $\Omega=0$.) Clearly Ω is an m-form on P_0 invariant under the 'right translations' of $SO(m)$ (group of all orthogonal matrices of the determinant 1). Hence Ω can be considered as an m-form on M, if M is orientable (and one of its orientations is fixed), and as an m-form on the orientable covering of M (i.e. an m-form on M of the 2nd kind in the sense of de Rham), if M is not orientable.

Now, assuming for simplicity M to be orientable, we have

$$\Omega = \frac{2}{O_m} K\, dw . \tag{7}$$

where

$$O_m = \frac{2\pi^{\frac{m+1}{2}}}{\Gamma\left(\frac{m+1}{2}\right)} = \frac{2^{2p+1}p!\,\pi^p}{(2p)!} \quad \text{(volume of } m\text{-dimensional sphere } S^m) ,$$

$$K = \frac{1}{2^p(2p)!\,g} \sum_{\substack{(i_1,\cdots,i_m)\\(k_1,\cdots,k_m)}} \varepsilon^{i_1\cdots i_m} \varepsilon^{k_1\cdots k_m} R_{i_1 i_2 k_1 k_2} \cdots R_{i_{m-1} i_m k_{m-1} k_m}$$

(Gaussian curvature of M),

$$dw=\sqrt{g}\,du^1\cdots du^m \qquad \text{(volume element of } M\text{)},$$

$(u^1,\cdots,u^m)$, $(X_1,\cdots,X_m)$ being taken to be concordant with the (fixed) orientation of $\tilde{U}$ and $g\left(\frac{\partial}{\partial u^i},\frac{\partial}{\partial u^j}\right)=g_{ij}$, $g=\det(g_{ij})$. For

$$\begin{aligned}
&\sum \varepsilon^{i_1\cdots i_m}\,\Omega_{i_1i_2}\wedge\cdots\wedge\Omega_{i_{m-1}i_m}\\
&=\left(-\frac{1}{2}\right)^p\sum \varepsilon^{i_1\cdots i_m}\,x_{i_1}^{p_1}\cdots x_{i_m}^{p_m}R_{p_1p_2k_1k_2}\cdots R_{p_{m-1}p_mk_{m-1}k_m}\,du^{k_1}\cdots du^{k_m}\\
&=\left(-\frac{1}{2}\right)^p\det(x_j^{\,i})\sum \varepsilon^{p_1\cdots p_m}\,\varepsilon^{k_1\cdots k_m}\,R_{p_1p_2k_1k_2}\cdots R_{p_{m-1}p_mk_{m-1}k_m}\,du^1\cdots du^m\\
&=\frac{(-1)^p}{2^p\,g}\sum \varepsilon^{p_1\cdots p_m}\,\varepsilon^{k_1\cdots k_m}\,R_{p_1p_2k_1k_2}\cdots R_{p_{m-1}p_mk_{m-1}k_m}\,dw\,,
\end{aligned}$$

since $\det(x_j^{\,i})=\sqrt{g}^{\,-1}$. $\sum \varepsilon^{i_1\cdots i_m}\,\Omega_{i_1i_2}\wedge\cdots\wedge\Omega_{i_{m-1}i_m}$ is thus considered as an m-form on M. (7) follows now immediately.

§ 3. Chern's proof of Gauss-Bonnet formula.

1. Auxiliary forms on P_0'. In the following we assume, for the sake of simplicity, that M is orientable. Then the orthonormal frame bundle P_0 splits into two parts P_0', P_0'', which are principal V-bundles with $G=SO(m)$. If we fix an orientation of M once for all, we shall be concerned exclusively with one of them, say P_0'.

Together with the projection $(\tilde{p};X_1,\cdots,X_m)\in P_0'(\tilde{U})\to(\tilde{p};X_m)\in T_0(\tilde{U})$, P_0' can be also considered as a principal fibre bundle with $G=SO(m-1)$ (subgroup of $SO(m)$ leaving fixed X_m) over T_0. Any form on P_0' invariant under the 'right translations' of $SO(m-1)$ can be considered as a form on T_0.

Following an idea of Chern [3], consider the following forms on P_0':

$$\Phi_k=\sum_{(i_1,\cdots,i_{m-1})}\varepsilon^{i_1\cdots i_{m-1}}\,\Omega_{i_1i_2}\cdots\Omega_{i_{2k-1}i_{2k}}\,\omega_{i_{2k+1}m}\cdots\omega_{i_{m-1}m}\qquad\left(0\leqq k\leqq\left[\frac{m-1}{2}\right]\right),$$

(8)

$$\Psi_k=2(k+1)\sum_{(i_1,\cdots,i_{m-1})}\varepsilon^{i_1\cdots i_{m-1}}\,\Omega_{i_1i_2}\cdots\Omega_{i_{2k-1}i_{2k}}\,\Omega_{i_{2k+1}m}\,\omega_{i_{2k+2}m}\cdots\omega_{i_{m-1}m}$$

$$\left(0\leqq k\leqq\left[\frac{m}{2}\right]-1\right).$$

Being all $SO(m-1)$-invariant forms on P_0', they are considered as $((m-1)\text{-}, m\text{-})$ forms on T_0. We have

$$d\Phi_k = k\sum \varepsilon^{i_1\cdots i_{m-1}} d\Omega_{i_1i_2}\Omega_{i_3i_4}\cdots\Omega_{i_{2k-1}i_{2k}}\omega_{i_{2k+1}m}\cdots\omega_{i_{m-1}m}$$

$$+(m-2k-1)\sum \varepsilon^{i_1\cdots i_{m-1}}\Omega_{i_1i_2}\cdots\Omega_{i_{2k-1}i_{2k}}d\omega_{i_{2k+1}m}\cdots\omega_{i_{m-1}m}$$

$$=\Psi_{k-1}+\frac{m-2k-1}{2k+2}\Psi_k \qquad (\Psi_{-1}=0)\,.$$

It follows that

$$\Psi_k = d\Theta_k\,, \qquad \left(0\leqq k\leqq\left[\frac{m}{2}\right]-1\right)$$

$$\Theta_k=\sum_{\lambda=0}^{k}(-1)^{k-\lambda}\frac{(2k+1)\cdots(2\lambda+2)}{(m-2\lambda-1)\cdots(m-2k-1)}\Phi_\lambda\,.$$

If m is even $(=2p)$, we have $d\Theta_{p-1}=\Psi_{p-1}$ and

$$\Psi_{p-1}=m\sum \varepsilon^{i_1\cdots i_{m-1}}\Omega_{i_1i_2}\cdots\Omega_{i_{m-1}m}$$

$$=\sum \varepsilon^{i_1\cdots i_m}\Phi_{i_1i_2}\cdots\Omega_{i_{m-1}i_m}=(-1)^p2^{2p}p!\,\pi^p\Omega\,.$$

If m is odd $(=2q+1)$, we have $d\Theta_{q-1}=\Psi_{q-1}$ and also $d\Phi_q=\Psi_{q-1}$. Hence

$$d(\Theta_{q-1}-\Phi_q)=0=\Omega\,.$$

These results can be unified as follows:

$$-d\Pi=\Omega\,,$$

$$\Pi=\frac{(-1)^m}{2^m\pi^{\frac{m-1}{2}}}\sum_{\lambda=0}^{\left[\frac{m-1}{2}\right]}\frac{(-1)^\lambda}{\lambda!\,\Gamma\left(\frac{m+1}{2}-\lambda\right)}\Phi_\lambda\,. \tag{9}$$

2. Index of singularity of a vector field. Let $\mathfrak{X}$ be a unit vector field over M with singularities at $p_1,\cdots,p_s$, i.e. a cross section of T_0 on $M-\{p_1,\cdots,p_s\}$, Let $\{\tilde U, G, \varphi\}\in\mathfrak{F}$, $\tilde p\in\tilde U$ be such that $\varphi(\tilde U)\ni p_i$, $\varphi(\tilde p)=p_i$ and let $\mathfrak{X}_{\tilde U}$ be the corresponding unit vector field over $\tilde U$. Then $\mathfrak{X}_{\tilde U}$ has a singularity at $\tilde p$. Let $I_{\tilde p}(\mathfrak{X}_{\tilde U})$ be the index of singularity of $\mathfrak{X}_{\tilde U}$ at $\tilde p$ in the usual sense and put

$$(10) \qquad I_{p_i}(\mathfrak{X}) = \frac{1}{N_{G_{\tilde{p}}}} I_{\tilde{p}}(\mathfrak{X}_{\tilde{U}}) ,$$

$N_{G_{\tilde{p}}}$ denoting the order of the isotropy subgroup $G_{\tilde{p}}$ of G at $\tilde{p}$. It is clear that this definition of $I_{p_i}(\mathfrak{X})$ does not depend on the choice of $\tilde{U}$ and $\tilde{p}$ and is uniquely determined by $\mathfrak{X}$ and p_i. We call $I_{p_i}(\mathfrak{X})$ the *index of singularity* of $\mathfrak{X}$ at p_i. (It should be noted that $I_{p_i}(\mathfrak{X})$ is not necessarily an integer.)

Now let S_i be a sufficiently small geodesic sphere around p_i and consider it to be oriented by the induced orientation with respect to the outer normal vector field on it. Then we have the following "integral formula of Kronecker":

$$(11) \qquad \frac{1}{O_{m-1}} \int_{S_i} (\omega_{1m} \cdots \omega_{m-1\,m}) \circ \mathfrak{X} = I_{p_i}(\mathfrak{X}) .$$

For, $\{\tilde{U}, G, \varphi\}$, $\tilde{p}$ being as above, we have by (3), (10) and by the integral formula (11) in the ordinary case

$$\text{the left side of (11)} = \frac{1}{N_{G_{\tilde{p}}}} \frac{1}{O_{m-1}} \int_{\tilde{S}_i} (\omega_{1m\tilde{U}} \cdots \omega_{m-1\,m\tilde{U}}) \circ \mathfrak{X}_{\tilde{U}}$$

$$= \frac{1}{N_{G_{\tilde{p}}}} I_{\tilde{p}}(\mathfrak{X}_{\tilde{U}}) = I_{p_i}(\mathfrak{X}) ,$$

$\tilde{S}_i$ being the corresponding geodesic sphere around $\tilde{p}$.

3. Gauss-Bonnet formula for a compact *V*-manifold. Let M be a compact, orientable, Riemannian V-manifold of dimension m. We shall consider the integral $\int_M \Omega$. For that purpose, let $\mathfrak{X}$ be an arbitrary unit vector field on M with singularities at $p_1, \cdots, p_s$ [7] and K_i be geodesic balls around p_i with sufficiently small radius ρ. (We take ρ so small that $K_i \cap K_j = \phi$ for $i \neq j$.) Then we have

$$\int_M \Omega = \lim_{\rho \to 0} \int_{M - \bigcup_i K_i} \Omega ,$$

and by (9) and 'Stokes formula'

7) The existence of such a vector field can be proved easily.

$$\int_{M-\cup K_i} \Omega = -\int_{M-\cup K_i} d(\Pi \circ \mathfrak{X})$$

$$= \sum_i \int_{S_i} \Pi \circ \mathfrak{X}$$

S_i being the boundary of K_i oriented as above. Now since Φ_k is of $2k$-th degree with respect to du^i as is seen from (4), (5), (8), we have

$$\int_{S_i} \Phi_k \circ \mathfrak{X} = O(\rho^{2k})$$

and so

$$\lim_{\rho \to 0} \int_{S_i} \Phi_k \circ \mathfrak{X} = 0 \qquad \left(1 \leqq k \leqq \left[\frac{m-1}{2}\right]\right).$$

On the other hand, we have by (11)

$$\int_{S_i} \Phi_0 \circ \mathfrak{X} = (m-1)! \int_{S_i} (\omega_{1\,m} \cdots \omega_{m-1\,m}) \circ \mathfrak{X}$$

$$= (m-1)!\, O_{m-1}\, I_{p_i}(\mathfrak{X}).$$

From these we get the following fundamental formula of Chern

$$\int_M \Omega = (-1)^m \sum_{i=1}^{s} I_{p_i}(\mathfrak{X}). \tag{12}$$

The left side of this formula being independent of the choice of $\mathfrak{X}$ and the right side being independent of the Riemannian metric g, we see that this number is determined only by the V-manifold structure of M. We shall denote this number by $\chi_V(M)$.

Let us now assume that M has a triangulation $M = \bigcup |s_k|$ ($|s_k|$ denoting the carrier of the singular simplex s_k) such that all the irreducible components of S (subvariety of all singular points) become subcomplexes of $\{s_k\}$.[8] Then considering a canonical vector field $\mathfrak{X}_0$ attached to this triangulation (the vector field of Stiefel-Whitney) with singularities at each barycenters p_k of s_k, we have

$$\chi_V(M) = \sum_k I_{p_k}(\mathfrak{X}_0) = \sum_k (-1)^{\dim s_k} \frac{1}{N_{s_k}},$$

the summation being taken over all simplexes of this triangulation

8) It is very plausible that every C^∞-V-manifolds allow such a triangulation, though no proof is yet obtained.

and N_{s_k} denoting the order of the isotropy group of s_k (i. e. that of the isotropy group of an inner point p of $|s_k|$, which is, by our assumption, independent of the choice of p). Thus, $\chi_V(M)$ being an analogue of the Euler characteristic, we call it the ***Euler characteristic of M as a V-manifold.*** It should be noted that generally $\chi_V(M)$ does not coincide with the ordinary Euler characteristic $\chi(M)$ and that $\chi_V(M)$ is not necessarily an integer.

We have thus obtained the following results:

THEOREM 2. *Let M be an orientable, compact, Riemannian V-manifold of even dimension m. Then we have*

$$\frac{2}{O_m}\int_M K\,dw = \chi_V(M) \tag{13}$$

Let us remark that in the preceding considerations the assumption of the orientability of M is inessential. Namely considering dw as an n-form on M of the 2nd kind (in the sense of de Rham). Theorem 2 holds also for non-orientable M. We remark also that a (C^∞-) V-manifold has always a (C^∞-) Riemannian metric just as an ordinary C^∞-manifold has one. We have therefore

THEOREM 3. *Let M be a compact V-manifold. Then for any (unit) vector field $\mathfrak{X}$ with singularities at $p_1, \cdots, p_s$, we have*

$$\sum I_{p_i}(\mathfrak{X}) = \chi_V(M)$$

THEOREM 4. *If M is a compact V-manifold of odd dimension, we have*

$$\chi_V(M) = 0 .$$

Theorem 2, 3 generalize the theorems of Allendoerfer-Weil [1] and of Hopf [4], respectively, to the case of V-manifold. Theorem 4 is trivial (by the duality of Poincaré) in the case of ordinary manifold.

4. The case of V-manifold with boundaries. We define the notion of V-manifold with boundaries as follows. Let $(M, \mathfrak{F})$ be a V-manifold. We assume that for each $\{\tilde{U}, G, \varphi\} \in \mathfrak{F}$ there is given a closed subset $\tilde{U}_0$ of $\tilde{U}$ such that $\tilde{U}_0$ is the closure of its interior and that the boundary $\mathrm{b}\tilde{U}_0$ of $\tilde{U}_0$ is a part of a finite union of C^∞-subvarieties of dimension $m-1$ of $\tilde{U}$. We assume furthermore that $\bigcup \varphi(\mathrm{b}\tilde{U}_0)$ becomes a part of a V-subvariety B of dimension

$m-1$ of M. Then we call $M_0=\bigcup \varphi(\tilde{U}_0)$ a *V-manifold with boundaries.* The boundary $\mathrm{b}M_0$ of M_0 is clearly a part of B.

Now let M_0 be an orientable compact Riemannian V-manifold with boundaries. Then, considering (unit) vector fields $\mathfrak{X}$ on M_0 with finite number of singularities in the interior of M_0 coinciding with the inner (unit) normal vector field $\mathfrak{N}$ on $\mathrm{b}M_0$, we can prove just as in **3** the following

THEOREM 5. *Let M_0 be an orientable, compact Riemannian V-manifold with boundaries and let $\mathfrak{N}$ be the inner unit normal vector field on* $\mathrm{b}M_0$. *Then we have*

$$\int_{M_0}\Omega=\chi'_V(M_0)-\int_{\mathrm{b}M_0}\Pi\circ\mathfrak{N} \tag{14}$$

$\mathrm{b}M_0$ *being oriented by the induced orientation with respect to the outer normal vector field on* $\mathrm{b}M_0$.

$\chi'_V(M_0)$ is a rational number depending only on the V-manifold structure of M_0 and may be called the *inner Euler characteristic of M_0 as a V-manifold with boundaries.* In case M_0 has a triangulation $M=\bigcup|s_k|$ such that all irreducible components of S and of $\mathrm{b}M_0$ become subcomplexes of $\{s_k\}$, $\chi'_V(M_0)$ is given as follows

$$\chi'_V(M_0)=\sum_k(-1)^{\dim s_k}\frac{1}{N_{s_k}},$$

the summation being taken over all simplexes of this triangulation which are in the interior of M_0.

§ 4. Application to the Siegel space $\mathfrak{V}_n$.

1. Some results of Siegel. Let $\mathfrak{H}_n$ be the generalized upper half-plane of degree n, i.e. the space of all complex symmetric matrices $Z=X+iY$ of degree n with imaginary parts $Y>0$, and let $\boldsymbol{M}_n$ be Siegel's modular group, i.e. the group of all symplectic transformations $\sigma=\begin{pmatrix}A & B\\ C & D\end{pmatrix}$ of degree 2n with integral coefficients, operating on $\mathfrak{H}_n$ in the following form

$$\sigma(Z)=(AZ+B)(CZ+D)^{-1}.$$

Then, $\boldsymbol{M}_n$ being a properly discontinuous group of analytic automorphisms of $\mathfrak{H}_n$, the quotient space $\mathfrak{V}_n=\boldsymbol{M}_n\backslash\mathfrak{H}_n$ becomes a complex analytic (hence orientable) V-manifold with respect to the family $\mathfrak{F}=\{\{\tilde{U}, G, \varphi\}\}$, where $\tilde{U}$ is any connected open neighbourhood of any $\tilde{p}\in\mathfrak{H}_n$ such that $\sigma(\tilde{U})=\tilde{U}$ for σ with $\sigma(\tilde{p})=\tilde{p}$ and $\sigma(\tilde{U})\cap\tilde{U}=\phi$ otherwise and φ the restriction of the canonical map $\mathfrak{H}_n\to\boldsymbol{M}_n\backslash\mathfrak{H}_n$ to $\tilde{U}$. $\mathfrak{V}_n$ is of real dimension $m=n(n+1)=2p$.

An invariant Riemannian metric of $\mathfrak{H}_n$ (the symplectic metric) is given by

$$ds^2=Tr(Y^{-1}dZY^{-1}dZ)\,.$$

Concerning this metric Siegel [8] obtained the following results:

A) The volume element is given by

$$dw=2^{p-n}dv=2^{p-n}\prod_{i\leqq j}dx_{ij}dY_{ij}\,,$$

where $X=(x_{ij})$, $Y^{-1}=(Y_{ij})$. Calculating the Gaussian curvature K, we have

$$\Omega=\frac{2}{O_m}K\,dw=(-1)^p\frac{a_n}{2^{n^2}p\,!\,\pi^p}dv\,,$$

a_n denoting a certain combinatorial number depending only on n. $\left(\frac{a_n}{p\,!}\text{ is an integer.}\right)$ (See Siegel [9], III, Theorem 5)

B) The volume of $\mathfrak{V}_n=\boldsymbol{M}_n\backslash\mathfrak{H}_n$ is finite and is given as follows.

$$\begin{aligned}\int_{\mathfrak{V}_n}dv&=2\prod_{k=1}^{n}\{(k-1)\,!\,\pi^{-k}\,\zeta(2k)\}\\&=2^{n^2+1}\pi^p\prod_{k=1}^{n}\left\{\frac{(k-1)\,!}{(2k)\,!}B_{2k}\right\}.\end{aligned}$$

B_{2k} denoting the absolute value of the k-th Bernoulli number. (See Siegel [9], VIII, Theorem 11)

Combining A), B), we obtain

$$\int_{\mathfrak{V}_n}\Omega=(-1)^p2\frac{a_n}{p\,!}\prod_{k=1}^{n}\left\{\frac{(k-1)\,!}{(2k)\,!}B_{2k}\right\}. \tag{15}$$

We shall prove in the next paragraph that $\int_{\mathfrak{V}_n}\Omega$ expresses the Euler characteristic χ_n of $\mathfrak{V}_n$ as an open V-manifold (in the sense specified below). For some small values of n we have the following numerical table.

n	1	2	3	4	
$\frac{a_n}{p!}$	1	6	90	?	
χ_n	$-\frac{1}{6}$	$-\frac{1}{720}$	$\frac{1}{725760}$	$\frac{a_4}{10!}$	$\frac{1}{13168189440000}$

2. Gauss-Bonnet formula for $\mathfrak{B}_n$. Since $\mathfrak{B}_n = \boldsymbol{M}_n \backslash \mathfrak{H}_n$ is not compact, we can not apply directly the formula (13) to $\mathfrak{B}_n$. But we can procede as follows.

Let $\mathfrak{H}(\eta)$ be the set of all $Z \in \mathfrak{H}_n$ such that $|Y| \geqq \eta^n$ and put $K_\infty(\eta) = \boldsymbol{M}_n \backslash \boldsymbol{M}_n\, \mathfrak{H}(\eta)$. We shall first prove the following

LEMMA 3. *$\mathfrak{B}_n - K_\infty(\eta)$ is a compact V-manifold with boundaries.*

PROOF. Let $\boldsymbol{F}_n$ be Siegel's fundamental region of $\boldsymbol{M}_n$ in $\mathfrak{H}_n$. Then by Siegel's reduction theory [8] we have for any $Z \in \boldsymbol{F}_n$, $\sigma \in \boldsymbol{M}_n$, $|\mathfrak{I}\sigma(Z)| \leqq |\mathfrak{I}Z|$, $\mathfrak{I}Z$ denoting the imaginary part of Z. Thus $|\mathfrak{I}\sigma(Z)| \geqq \eta^n$ implies $|\mathfrak{I}Z| \geqq \eta^n$. Hence $\mathfrak{H}(\eta) \subset \boldsymbol{M}_n(\boldsymbol{F}_n \cap \mathfrak{H}(\eta))$. From this follows easily that $\mathfrak{B}_n - K_\infty(\eta)$ becomes a V-manifold with boundaries. It follows also from Siegel's reduction theory that if $Z = X + iY \in \boldsymbol{F}_n$ then $-\frac{1}{2} \leqq x_{ij} \leqq \frac{1}{2}$ and Y is reduced in the sense of Minkowski so that, in particular, we have $y_{11} \leqq y_{22} \leqq \cdots \leqq y_{nn}$, $-\frac{y_{ii}}{2} \leqq y_{ij} \leqq \frac{y_{ii}}{2}$ and $\prod_i y_{ii} \leqq \frac{1}{c_n} |Y|$ with some $c_n > 0$ depending only on n. It holds also $y_{11} \geqq \frac{\sqrt{3}}{2}$ for $Z \in \boldsymbol{F}_n$. Hence the set $\{Z;\, Z \in \boldsymbol{F}_n,\ |Y| \leqq \eta^n\}$ is compact. It follows that $\mathfrak{B}_n - K_\infty(\eta)$ is compact, q. e. d.

Now, applying theorem 5 to $\mathfrak{B}_n - K_\infty(\eta)$, we have

$$\int_{\mathfrak{B}_n - K_\infty(\eta)} \Omega = \chi'_V(\mathfrak{B}_n - K_\infty(\eta)) - \int_{\mathrm{b}K_\infty(\eta)} \Pi \circ \mathfrak{N}_\eta\,, \tag{16}$$

$\mathfrak{N}_\eta$ denoting the outer unit normal vector field on $\mathrm{b}\mathfrak{H}(\eta)$. We shall next prove that

$$\lim_{\eta \to \infty} \int_{\mathrm{b}K_\infty(\eta)} \Pi \circ \mathfrak{N}_\eta = 0\,.$$

For that purpose, we shall give some lemmas.

LEMMA 4. *We have*

$$\Pi \circ \mathfrak{N}_\eta = c_\eta dw_\eta^* ,$$

dw_η^* *denoting the volume element (with respect to the symplectic metric) on* $\mathrm{b}\mathfrak{H}(\eta)=\{Z;\ |Y|=\eta^n\}$ *and* c_η *a constant depending only on* η.

PROOF. The group $\mathfrak{G}$ of all symplectic transformations of the form

$$\sigma(Z)=AZ\,{}^tA+B, \qquad |A|=1$$

leaves $\mathrm{b}\mathfrak{H}(\eta)$ invariant and operates on it transitively. Since Π is clearly a symplectic invariant, $\Pi\circ\mathfrak{N}_\eta$ is a $\mathfrak{G}$-invariant $(m-1)$-form on $\mathrm{b}\mathfrak{H}(\eta)$ and so is also dw_η^*. From these follows the lemma immediately.

LEMMA 5. *The constant* c_η *in the preceding lemma is of the form*

$$c_\eta = c_1 \eta^{-(m-1)}$$

PROOF. Let us compare $\Pi\circ\mathfrak{N}_\eta$ and dw_η^* with $\Pi\circ\mathfrak{N}_1$ and dw_1^*, respectively. We denote by τ_η the symplectic transformation $Z\to\eta Z$ of $\mathfrak{H}_n$, which transforms $\mathrm{b}\mathfrak{H}(1)$ onto $\mathrm{b}\mathfrak{H}(\eta)$. Then, since Π is a symplectic invariant, we have $(\Pi\circ\mathfrak{N}_\eta)\circ\tau_\eta=\Pi\circ\mathfrak{N}_1$. On the other hand, we have clearly $(dw_\eta^*)\circ\tau_\eta=\eta^{m-1}dw_1^*$. These prove the lemma.

LEMMA 6.

$$\int_{\mathrm{b}K_\infty(\eta)} dw^* = O(\eta^{\frac{m}{2}-1})$$

PROOF. Since we have

$$dw = 2^{\frac{p-n}{2}} \prod_{i\leq j} dx_{ij}\cdot dw' ,$$

dw' denoting the volume element in the space of positive definite symmetric matrices Y with respect to the invariant metric $ds^2 = Tr(Y^{-1}dY)^2$, it follows easily that

$$dw_\eta^* = 2^{\frac{p-n}{2}} \prod_{i\leq j} dx_{ij}\cdot dw_\eta'^*$$

$dw_\eta'^*$ denoting the volume element in the hyperplane $\{Y;\ |Y|=\eta^n\}$ with respect to the above metric. Now from the proof of Lemma 3 we can see that $\mathrm{b}K_\infty(\eta)$ is obtained from $\boldsymbol{F}_n\cap\mathrm{b}\mathfrak{H}(\eta)=\{Z;\ Z\in\boldsymbol{F}_n,\ |Y|=\eta^n\}$ by identifying the equivalent points on the boundary.

Hence we have by Siegel's reduction theory

$$\int_{\mathfrak{b}K_\infty(\eta)} dw^* = \int_{\boldsymbol{F}_n \cap \mathfrak{b}\mathfrak{H}(\eta)} dw^*$$

$$\leqq 2^{\frac{p-n}{2}} \prod_{i \leqq j} \int_{-\frac{1}{2}}^{\frac{1}{2}} dx_{ij} \cdot \int_{\substack{|Y|=\eta^n \\ Y:\text{ reduced}}} dw'^*$$

$$= (2^{\frac{p-n}{2}} \int_{\substack{|Y|=1 \\ Y:\text{ reduced}}} dw_1'^*) \eta^{\frac{m}{2}-1}, \qquad \text{q. e. d.}$$

From Lemmas 4, 5, 6, we get

$$\int_{\mathfrak{b}K_\infty(\eta)} \Pi \circ \mathfrak{N}_\eta = O(\eta^{\frac{m}{2}}),$$

which proves

$$\lim_{\eta \to \infty} \int_{\mathfrak{b}K_\infty(\eta)} \Pi \circ \mathfrak{N}_\eta = 0.$$

On the other hand, since the subvariety S of all singular points of $\mathfrak{V}_n$ is a union of a finite number of irreducible subvarieties (as is seen from the fact that $\boldsymbol{F}_n$ is bounded by a finite number of algebraic surfaces[9]), the denominator of $\chi_V' \ (\mathfrak{V}_n - K_\infty(\eta))$ is bounded when $\eta \to \infty$. These facts together with (16) prove the following

THEOREM 6. *$\chi_V' \ (\mathfrak{V}_n - K_\infty(\eta))$ becomes constant for sufficiently large η. Denoting this number by χ_n, we have*

$$\frac{2}{O_m} \int_{\mathfrak{V}_n} K \, dw = \chi_n. \tag{17}$$

3. An application.

We shall now apply the above result to the study of the least common multiple N_n of the orders of all isotropy subgroups of $\boldsymbol{M}_n / \{\pm E_{2n}\}$.

First, it is clear that $N_n \chi_n$ is an integer. This gives a lower estimation of N_n. An upper estimation of N_n is given by a method of Minkowski [6] as follows.

We denote by $\boldsymbol{M}_n$ the 'homogeneous' Siegel's modular group, i. e. $\boldsymbol{M}_n = Sp(2n, \boldsymbol{Z})$, and put

9) This follows from Siegel [9], VI, Theorem 8.

$$\boldsymbol{M}_n(l)=\{\sigma\,;\ \sigma\in\boldsymbol{M}_n,\ \sigma\equiv E_{2n}\ (\mathrm{mod}\ l)\}\,,$$

l being any prime number. Then

$\boldsymbol{M}_n/\boldsymbol{M}_n(l)\cong Sp\,(2n,\,l)$ (the symplectic group of degree $2n$ over $GF(l)$).

We shall first prove the following

LEMMA 7. *If $l\neq 2$, $\boldsymbol{M}_n(l)$ contains no finite subgroup. If $l=2$, any finite subgroup of $\boldsymbol{M}_n(2)$ is conjugate in $\boldsymbol{M}_n$ with a subgroup consisting of elements of the form*

$$\sigma=\begin{pmatrix}A & 0\\ 0 & A\end{pmatrix},\qquad A=\begin{pmatrix}\pm1 & & 0\\ & \ddots & \\ 0 & & \pm1\end{pmatrix}.$$

PROOF. The first half of this lemma is well-known (and is easy to prove).10) Also it is easy to see that $\boldsymbol{M}_n(4)$ contains no finite subgroup. It follows that if G is a finite subgroup of $\boldsymbol{M}_n(2)$, we have $\sigma^2=1$ for all $\sigma\in G$ (hence, in particular, G is abelian). Therefore to prove the lemma it will be enough to show that any $\sigma\in\boldsymbol{M}_n(2)$ with $\sigma^2=1$ is conjugate in $\boldsymbol{M}_n$ with an element of the form described above. Let $\sigma\in\boldsymbol{M}_n(2)$, $\sigma^2=1$ and $\mathfrak{w}$ be an eigen vector of σ (with the eigen value ±1). We can take $\mathfrak{w}$ so as to be a primitive integral vector. Then there exists a $\tau\in\boldsymbol{M}_n$ with $\mathfrak{w}$ as the first column vector.11) Then $\tau^{-1}\sigma\tau$ having $\begin{pmatrix}\pm1\\ 0\\ \vdots\\ 0\end{pmatrix}$ as the first column vector, we have by the conditions of symplectic matrices

$$\tau^{-1}\sigma\tau=\begin{pmatrix}\pm1 & {}^t\mathfrak{a} & b & {}^t\mathfrak{b}_1\\ 0 & A^* & \mathfrak{b}_2 & B^*\\ 0 & 0 & \pm1 & 0\\ 0 & C^* & \mathfrak{d} & D^*\end{pmatrix},$$

where $\sigma^*=\begin{pmatrix}A^* & B^*\\ C^* & D^*\end{pmatrix}\in\boldsymbol{M}_{n-1}(2)$, $\sigma^{*2}=1$. Hence, using the induction on n, we can find $\tau_1\in\boldsymbol{M}_n$ such that

10) Cf. Minkowski [5], [6], Siegel [9], VII, Theorem 10.
11) See Siegel [9], VIII, Lemma 15.

$$\tau_1^{-1}\sigma\tau_1=\begin{pmatrix} A & B \\ 0 & {}^tA^{-1}\end{pmatrix}, \qquad A=\begin{pmatrix} \pm 1 & & * \\ & \ddots & \\ 0 & & \pm 1\end{pmatrix}.$$

Then from $\sigma^2=1$ and $\sigma\in \boldsymbol{M}_n(2)$, we have (in a suitable choice of τ_1)

$$A=\begin{pmatrix} E_r & 2P \\ 0 & -E_{n-r}\end{pmatrix}, \qquad B=\begin{pmatrix} -2(Q_1{}^tP+PQ_2) & 2Q_1 \\ 2Q_2 & 0\end{pmatrix}$$

so that putting

$$\tau_2=\begin{pmatrix} S & T \\ 0 & {}^tS^{-1}\end{pmatrix}, \qquad S=\begin{pmatrix} E_r & P \\ 0 & E_{n-r}\end{pmatrix}, \qquad T=\begin{pmatrix} 0 & Q_2 \\ -Q_2 & 0\end{pmatrix},$$

we obtain

$$\tau_2\tau_1^{-1}\sigma\tau_1\tau_2^{-1}=\begin{pmatrix} A_1 & 0 \\ 0 & A_1\end{pmatrix}, \qquad A=\begin{pmatrix} E_r & 0 \\ 0 & -E_{n-r}\end{pmatrix}, \qquad \text{q. e. d.}$$

It follows that, denoting by G any finite subgroup of $\boldsymbol{M}_n$, the restriction of the canonical homomorphism $\boldsymbol{M}_n\to Sp(2n, l)$ to G is an isomorphism, if $l\neq 2$, and has a kernel of the order dividing 2^n, if $l=2$. Hence, denoting by $\overline{N}_n$ the half of the greatest common divisor of the orders of $Sp(2n, l)$ $(l\neq 2)$ and of the 2^n times order of $Sp(2n, 2)$, we have $N_n\mid\overline{N}_n$.

Now since

$$\text{the order of } Sp(2n, l)=l^{n^2}(l^{2n}-1)(l^{2n-2}-1)\cdots(l^2-1),$$

we have

$$\overline{N}_n=\frac{1}{2}\, G.\,C.\,M.\ \{l^{n^2}(l^{2n}-1)\cdots(l^2-1)\ (l\neq 2),\ 2^{n^2+n}(2^{2n}-1)\cdots(2^2-1)\}\,.$$

This number can be calculated easily by means of the following lemma of Minkowski.

LEMMA 8.[12] *The greatest common divisor of* $(l^{2n}-1)(l^{2n-2}-1)\cdots(l^2-1)$, *$l$ being all prime numbers, is equal to*

$$\overline{2n}\rvert=\prod_{l:\,\text{prime}} l^{\sum_{i=0}^{\infty}\left[\frac{2n}{l^i(l-1)}\right]}, \tag{18}$$

l running over all prime numbers and [ ] *denoting the symbol of Gauss.*

It is also known that

12) See Minkowski [6].

$$\overline{2n}| = 2^{2n} b_1 b_2 \cdots b_n ,$$

$$b_k = \prod_{l-1|2k} l^{1+\mathrm{ord}_l k} = \text{the denominator of } \frac{B_{2k}}{k} ,$$

l running over all prime numbers such that $l-1|2k$ and $\mathrm{ord}_l k$ denoting the exponent of the highest power of l dividing k.

Now we have

$$\sum_{i=0}^{\infty} \left[\frac{2n}{l^i(l-1)} \right] \leqq n^2$$

for $l \neq 2$ and also

$$\sum_{i=0}^{\infty} \left[\frac{2n}{2^i} \right] \leqq n^2 + n$$

except for the cases $n=1, 2$, for which we have

$$\sum_{i=0}^{\infty} \left[\frac{2n}{2^i} \right] = n^2 + n + 1 .$$

Hence we have

$$(19) \quad \bar{N}_n = \begin{cases} \dfrac{1}{4} \overline{2n}| = 2^{2n-2} \displaystyle\prod_{k=1}^{n} \left(\text{denominator of } \frac{B_{2k}}{k} \right) & (n=1, 2) \\ \dfrac{1}{2} \overline{2n}| = 2^{2n-1} \displaystyle\prod_{k=1}^{n} \left(\text{denominator of } \frac{B_{2k}}{k} \right) & (n \geqq 3) \end{cases}$$

We have thus proved the following

THEOREM 7. *Let N_n be the least common multiple of the orders of the isotropy subgroups of $\boldsymbol{M}_n/\{\pm E_{2n}\}$. Then $N_n \chi_n$ is an integer and N_n is a divisor of $\bar{N}_n$, χ_n, $\bar{N}_n$ being numbers given by* (15), (19), *respectively.*

It follows, in particular, that $\bar{N}_n \chi_n$ is an integer. From (15), (19), we have

$$\bar{N}_n \chi_n = \pm 2^n \frac{a_n}{p!} \prod_{k=1}^{n} \left\{ \frac{(k-1)!}{(2k-1)!} \left(\text{numerator of } \frac{B_{2k}}{k} \right) \right\} \qquad (n \geqq 3) .$$

(We must replace 2^n by 2^{n-1} for $n=1, 2$.) This formula seems to involve a new relationship between the numerators of the Bernoulli numbers. For some small values of n we have the following table.

n	1	2	3	4
$\overline{N}_n$	6	1440	1451520	696729600
$\overline{N}_n\chi_n$	-1	-2	2	$\frac{a_4}{10!}\ \frac{1}{18900}$

From this and the other tables we obtain finally

$$N_1=6,\ N_2=720 \text{ or } 1440,\ N_3=725760 \text{ or } 725760\times 2.$$

REMARK. The same method can be applied, for instance, to the case of $\mathfrak{B}_n'=\Gamma_n\backslash P_n^1$, P_n^1 being the space of all positive definite symmetric matrices of degree n with the determinant 1 and Γ_n the unimodular group (with the determinant ± 1) operating on P_n^1. But, it is proved that the Gaussian curvature of $P_n^1=SL(n, \boldsymbol{R})/SO(n)$ is $=0$ for $n>2$.[13] In fact, in virtue of the invariance of the algebraic relations, it is sufficient to prove this in replacing $SL(n, \boldsymbol{R})$ by any other real form of $SL(n, \boldsymbol{C})$, say by $SU(n)$, and for this latter case the Gauss-Bonnet theorem (for compact Riemannian manifold) can be applied. On the other hand the Euler characteristic of $SU(n)/SO(n)$ can be calculated and proved to be $=0$ for $n>2$. Hence application of the Gauss-Bonnet theorem to $\mathfrak{B}_n'$ gives us nothing new. On the other hand, the least common multiple of the orders of the isotropy subgroups of $\Gamma_n/\{\pm E_n\}$ has been already calculated by Minkowski [6] with the result that it is equal to $\frac{1}{2}\overline{n|}$.

Added in Proof. The author was informed recently that Hirzebruch has calculated a_n in (15) explicity, obtaining the result that

$$\frac{a_n}{p!}=\prod_{k=1}^{n}\frac{(2k-1)!}{(k-1)!}.$$

It follows that

$$\chi_n=\frac{(-1)^p}{2^{n-1}}\prod_{k=1}^{n}\frac{B_{2k}}{k}$$

13) The author owes this remark to Professor A. Weil, to whom he wishes to express here his hearty thanks. The fact that $\chi(SU(n)/SO(n))=0$ $(n>2)$ follows also from a result of H. Hopf and H. Samelson, Ein Satz über die Wirkungsräume geschlossener Liescher Gruppen, Comm. Math. Helv., 13 (1940/41) pp. 240-251.

and that the fact $\overline{N}_n\chi_n$ is an integer is trivial (hence containing no relationship between the Bernoulli numbers). He remarks also that the values for $n=3$ in our tables are false. The true values are as follows:

$$\frac{a_3}{6!}=360,\ \chi_3=\frac{1}{181440},\ \overline{N}_3\chi_3=8,\ N_3=181440\times 2^{\nu} \qquad (0\leqq\nu\leqq 3).$$

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