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Intrinsic character of minimal hypersurfaces in flat spaces.

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Introduction.

A minimal variety in a Riemannian space is defined as a variety which realizes an extremal of the volume integral, and is characterized, from a stand-point of the differential geometry in the small, by the property that the mean curvature vanishes. Although many properties of such spaces immersed in an enveloping space are known, it seems to me that their intrinsic properties have not been yet discussed.

In this paper we investigate the *intrinsic properties of minimal hypersurfaces in flat spaces.* At the beginning of Section 3, the tensors $S_{p)ij}$ are defined in terms of the curvature tensor and they play an important rôle throughout the paper. The first two sections are devoted to explain how the tensors are derived. In Section 3, by means of these tensors, the coefficients of the second fundamental form are written in terms of the curvature tensor, and then, from the Gauss equation, we obtain the identities which are satisfied by the components of the curvature tensor of a minimal hypersurface.

In Section 4, the classification theorem of minimal hypersurfaces is obtained with the aid of the tensors $S_{p)ij}$, and then we get the imbedding theorem of a Riemannian space as a minimal hypersurface in a flat space.

There exists a special class of minimal hypersurfaces, which will be called to be of type M^{∞} and for which we can not determined the coefficients of the second fundamental form by the general method used in Section 3. Any minimal surface of ordinary space belongs to this class. In the final two sections, we shall treat the Einstein spaces, conformally flat spaces, and 3-dimensional spaces as simple examples of such an exceptional case.

§ 1. Minimal hypersurfaces of type M^1.

We consider a minimal hypersurface V^n in a flat space and denote by H_{ij} the second fundamental tensor of V^n. Then H_{ij} satisfies the so-called Gauss equation:

$$R_{hijk}=e(H_{hj}H_{ik}-H_{hk}H_{ij}), \qquad (e=\pm 1), \tag{1.1}$$

where R_{hijk} is the curvature tensor of V^n. It is well known that the necessary and sufficient condition for V^n to be a minimal hypersurface is that the mean curvature vanishes, that is,

$$g^{ij}H_{ij}=0, \tag{A$_0$}$$

where g^{ij} is the fundamental tensor of V^n.

Transvecting (1.1) by g^{hk} and making use of (A_0), we have

$$R_{ij}=-eH_{ia}H_{jb}g^{ab}, \tag{B$_1$}$$

where R_{ij} is the Ricci tensor of V^n. It can be easily verified by direct substitution from (B_1) that

$$H_{ij}H_{ak}R^a{}_l=H_{ij}H_{al}R^a{}_k.$$

Interchanging j and l, and subtracting the resulting equation from the above, we have as a consequence of (1.1)

$$H_{ij}H_{ak}R^a{}_l-H_{il}H_{ak}R^a{}_j=eR_{ailj}R^a{}_k. \tag{1.2}$$

Transvection of (1.2) by g^{kl} yields, in virture of (B_1),

$$R^{ab}H_{ab}H_{ij}=S_{ij}\equiv e(R_{iajb}R^{ab}-R_{ia}R^a{}_j). \tag{C$_1$}$$

Moreover, transvection of (C_1) by R^{ij} leads at once to

$$(R^{ab}H_{ab})^2=S_{ij}R^{ij}\equiv S. \tag{D$_1$}$$

In the following we shall assume that the quantity S does not vanish and say that such a V^n is of type M^1. Since we can put $S=\sigma^{-2}$, it follows from (D_1) that $R^{ab}H_{ab}=\sigma^{-1}$, and hence we can rewrite (C_1) as

$$H_{ij}=\sigma S_{ij}. \tag{E$_1$}$$

As a result we can deduce the expression of H_{ij} in terms of the curvature tensor of V^n. By means of the defining expression of S_{ij} we can immediately show that H_{ij} as above determined satisfies the characteristic property (A_0).

Substituting in (1.1) from (E_1), we obtain

$$(F_1) \qquad R_{hijk} = \sigma^2 (S_{hj} S_{ik} - S_{hk} S_{ij}) .$$

Consequently the curvature tensor of V^n, which is of type M^1 and is imbedded in a flat space as a minimal hypersurface, must satisfy the equation (F_1).

§ 2. Minimal hypersurfaces of type M^2.

If the quantity S of V^n as defined by (D_1) vanishes, then it follows from (C_1) and (D_1) that the tensor S_{ij} must vanish and H_{ij} satisfies the equation

$$(A_1) \qquad R^{ab} H_{ab} = 0 .$$

In this case, when we transvect (1.1) by R^{hj} and make use of (A_1) and $S_{ij}=0$, the equation

$$(B_2) \qquad R_{ia} R^a{}_j = -e H_{ia} H_{jb} R^{ab}$$

is deduced. Transvecting (1.2) by R^{kl}, we have by virtue of (B_2)

$$(C_2) \qquad R^{ab}_{2)} H_{ab} H_{ij} = S_{2)ij} ,$$

where by definition

$$R_{2)ij} = R_{ia} R^a{}_j, \quad S_{2)ij} = e(R_{iajb} R^{ab}_{2)} - R_{ia} R^a_{2)j}) .$$

Moreover transvection of (C_2) by $R^{ij}_{2)}$ yields

$$(D_2) \qquad (R^{ab}_{2)} H_{ab})^2 = S_{2)ij} R^{ij}_{2)} \equiv S_2 .$$

In the rest of this section we restrict our considerations to a V^n for which $S_{ij}=0$ and $S_2 \neq 0$, and we say that such a V^n is of type M^2. In this case, since we can put $S_2 = (\sigma_2)^{-2}$, it follows from (D_2) that $R^{ab}_{2)} H_{ab} = (\sigma_2)^{-1}$,and hence equations (C_2) are written as

$$(E_2) \qquad H_{ij} = \sigma_2 S_{2)ij} .$$

Thus H_{ij} is expressible in terms of the curvature tensor of V^n. It is not difficult to verify that H_{ij} as just defined satisfies the equations (A_0) and (A_1). Inserting the expression (E_2) of H_{ij} in (1.1), we have

$$(F_2) \qquad R_{hijk} = (\sigma_2)^2 (S_{2)hj} S_{2)ik} - S_{2)hk} S_{2)ij}) ,$$

this having the meaning similar to (F_1).

§ 3. Algebraic properties of minimal hypersurfaces in a flat space.

In this section we shall generalize the treatments of minimal hypersurfaces as described in the preceeding sections. For this purpose it is convenient to introduce the tensors $R_{p)ij}$ and $S_{p)ij}$ as follows:

$$R_{p)ij}=R_{ia}R_{p-1)}{}^{a}{}_{j}, \qquad R_{1)ij}=R_{ij},$$
$$S_{p)ij}=e(R_{iajb}R_{p)}{}^{ab}-R_{p+1)ij}), \qquad S_{1)ij}=S_{ij},$$
$$S_{p}=S_{p)ij}R_{p)}{}^{ij}, \qquad S_{1}=S,$$

where the index p takes the values $1, 2, 3, \cdots$. It is obvious that $R_{p)ij}$ and consequently $S_{p)ij}$ are both symmetric tensors. Also we can easily derive equations

(3.1) $$R_{p)ij}R_{q)}{}^{ij}=R_{r)ij}R_{s)}{}^{ij}, \qquad (p+q=r+s),$$

(3.2) $$g^{ij}S_{p)ij}=0,$$

(3.3) $$S_{p)ij}R_{q)}{}^{ij}=S_{q)ij}R_{p)}{}^{ij}.$$

In consequence of (B_1) the tensors $R_{p)ij}$ are expressed in terms of H_{ij} as

(3.4) $$R_{p)ij}=(-e)^{p}H_{i}^{a_1}H_{a_1}^{a_2}\cdots H_{a_{2p-2}}^{a_{2p-1}}H_{a_{2p-1}j}.$$

Substituting from (1.1) and (3.4) in the defining expression of $S_{p)ij}$, we have

(3.5) $$S_{p)ij}=(-e)^{p}e\,H_{2p+1}H_{ij},$$

where by definition

$$H_{2p+1}=H_{a_{2p}}^{a_0}H_{a_0}^{a_1}\cdots H_{a_{2p-2}}^{a_{2p-1}}H_{a_{2p-1}}^{a_{2p}}, \qquad H_1=H_a^a.$$

The discussions of the first and second sections lead us to the following definition.

DEFINITION. *If a Riemannian space V^n is such that the equations*

$$S_{1)ij}=\cdots=S_{r-1)ij}=0, \qquad S_r\neq 0$$

are satisfied, then V^n is said to be of type M^r, regardless V^n is a minimal hypersurface in a flat space or not.

In the following we shall treat a minimal hypersurface V^n of type M^r. From (3.5) it follows that

(3.6) $$S_r=(-1)^{r}eH_{2r+1}H_{ij}R_{r)}{}^{ij}.$$

Hence, for V^n of type M^r, we have in consequence of (A_0), (3.5) and (3.6),

$$H_1 = H_3 = \cdots = H_{2r-1} = 0\,, \qquad H_{2r+1} \neq 0\,, \tag{3.7}$$

$$R_{r)}{}^{ij} H_{ij} \neq 0\,, \tag{3.8}$$

From (3.3) and (3.5) it follows that

$$S_{r)ij} R_{p)}{}^{ij} = (-1)^r e\, H_{2r+1} R_{p)}{}^{ij} H_{ij} = S_{p)ij} R_{r)}{}^{ij}\,,$$

and hence, for V^n of type M^r, we have by virtue of (3.7)

$$R_{p)}{}^{ij} H_{ij} = 0\,, \qquad (p = 1, \cdots, r-1)\,. \tag{A_p}$$

Now we shall find the similar expression of H_{ij} of V^n with (E_1) and (E_2). The equation (3.4) gives

$$R_{r)ij} = -e\, H_{ia} H_{jb} R_{r-1)}{}^{ab} \tag{B_r}$$

Transvecting (1.2) by $R_{r-1)}{}^{kl}$ and making use of (B_r), we obtain

$$H_{ij} H_{ak} R^a{}_l R_{r-1)}{}^{kl} = e\, R_{ailj} R^a{}_k R_{r-1)}{}^{kl} - e\, R^a{}_j R_{r-1)ai} = S_{r)ij}\,,$$

from which it follows that

$$H_{ij} R_{r)}{}^{ab} H_{ab} = S_{r)ij}\,. \tag{C_r}$$

Furthermore transvection by $R_{r)}{}^{ij}$ gives

$$(R_{r)}{}^{ab} H_{ab})^2 = S_r\,. \tag{D_r}$$

Since, by hypotheses $S_r \neq 0$, we can put $S_r = (\sigma_r)^{-2}$, then (C_r) is written in the form

$$H_{ij} = \sigma_r S_{r)ij}\,. \tag{E_r}$$

As a result we have the expression of H_{ij} in terms of the curvature tensor of V^n. From (3.2) and (3.3) it is clear that H_{ij} as defined satisfies (A_p) $(p = 0, 1, \cdots, r-1)$. Substituting in (1.1) from (E_r), we get

$$R_{hijk} = (\sigma_r)^2 (S_{r)hj} S_{r)ik} - S_{r)hk} S_{r)ij})\,. \tag{F_r}$$

These facts permit us to state

THEOREM 1. *If a V^n is of type M^r and a minimal hypersurface of a flat space, then the equation* (F_p) *is satisfied and the coefficients H_{ij} of the second fundamental form of V^n are expressed as* (E_r) *in terms of the curvature tensor of V^n.*

It is to be remarked that there exists such a Riemannian space that the tensors $S_{p)ij}$ vanish for all indices p. As an example, we have any Einstein space and hence any 2-dimensional Riemannian space, as easily verified by direct calculation with the aid of the equation $R_{ij}=(R/n)g_{ij}$. We shall say that such a space is *of type* M^{∞}, for which the above theorem can not be applied.

§ 4. Classification of minimal hypersurfaces in a flat space.

The principal normal curvatures ρ_a of a hypersurface V^n are defined as the roots of the determinantial equation

$$|\rho g_{ij}-H_{ij}|=0.$$

Let $\lambda_{a)}{}^i$ be the orthogonal ennuple determined by the equation

$$(\rho_a g_{ij}-H_{ij})\lambda_{a)}{}^j=0$$

Then the tensors g_{ij} and H_{ij} are respectively written in the following forms:

$$g_{ij}=\sum_{a=1}^{n} e_a \lambda_{a)i}\lambda_{a)j}, \qquad H_{ij}=\sum_{a=1}^{n} e_a \rho_a \lambda_{a)i}\lambda_{a)j}, \qquad (e_a=\pm 1),$$

from which we have

$$H_p=H^{a_1}_{a_2}H^{a_2}_{a_3}\cdots H^{a_p}_{a_1}=\sum_{a=1}^{n}(\rho_a)^p.$$

Therefore (3.7) are expressed in terms of ρ_a as

$$\text{(4.1)} \qquad \begin{aligned} &\sum_{a=1}^{n}(\rho_a)^p=0, \qquad (p=1,3,\cdots,2r-1), \\ &\sum_{a=1}^{n}(\rho_a)^{2r+1}=0. \end{aligned}$$

We put $P_\alpha=\sum_{a=1}^{n}(\rho_a)^\alpha$ and denote by p_α the elementary symmetric function of degree $\alpha\,(\alpha=1,\cdots,n)$ with respect to ρ_a. Following the theory of the symmetric polynomials, these polynomials P_α can be written in terms of p_α by means of the Newton formula [1] as follows:

$$\text{(4.2)} \qquad P_\alpha+\sum_{\beta=1}^{\alpha-1}(-1)^\beta p_\beta P_{\alpha-\beta}+(-1)^\alpha \alpha p_\alpha=0, \qquad (\alpha=1,2,\cdots),$$

where by definition

$$\text{(4.3)} \qquad p_\alpha=0, \qquad (\alpha>n).$$

Making use of (4.2) and applying the mathematical inductions, we can readily prove the following

LEMMA. *If* $P_1=P_3=\cdots=P_{2p-1}=0$, *then* $p_1=p_3=\cdots=p_{2p-1}=0$, *and* $P_{2p+1}=(2p+1)\,p_{2p+1}$.

For a minimal hypersurface V^n of type M^r, it follows from (4.1) and the lemma that

$$p_1=p_3=\cdots=p_{2r-1}=0\,, \qquad p_{2r+1}\neq 0\,.$$

But, it follows from (4.3) that $p_{2r+1}=0$ $(2r+1>n)$, so that we have

THEOREM 2 (CLASSIFICATION THEOREM). *The only type numbers* M^r *of minimal hypersurfaces of an* $(n+1)$*-dimensional flat space are equal to* $M^1,\cdots,M^p$ $(2p+1\leqq n)$ *or* M^∞. *For a* V^n *of type* M^r (r = *finite or infinite*), *the equations*

$$p_1=p_3=\cdots=p_{2r-1}=0\,, \qquad p_{2r+1}\neq 0$$

are satisfied, where p_α $(\alpha\leqq n)$ *are the elementary symmetric functions of degree* α *with respect to the principal normal curvatures of* V^n *and by definition* $p_\alpha=0$ $(\alpha>n)$.

§ 5. Imbedding of Riemannian spaces in flat spaces as minimal hypersurfaces.

From Theorem 1 it follows that a necessary condition for a Riemannian n-space V^n of type M^r to be imbedded in a flat space as a minimal hypersurface is that the equation (F_r) be satisfied, and then the solution (H_{ij}) of the system of the equations (A_0) and (1.1) is uniquely determined by (E_r).

It is well known that a V^n can be imbedded in a flat $(n+1)$-space, if and only if there exist H_{ij} which satisfies the Gauss and Codazzi equations. The equation (F_r) is equivalent to the Gauss equation. On the other hand, substitution from (E_r) in the Codazzi equation, that is,

$$H_{ij,k}-H_{ik,j}=0 \tag{5.1}$$

yields

$$S_{r)ij}\frac{\partial \log \sigma_r}{\partial x^k}-S_{r)ik}\frac{\partial \log \sigma_r}{\partial x^j}+S_{r)ij,k}-S_{r)ik,j}=0\,. \tag{5.2}$$

T. Y. Thomas [2] proved that if the matrix (H_{ij}) is of rank $\geqq 4$, then the Codazzi equation is a consequence of the Gauss equation. This theorem and the above results enable us to establish the

THEOREM 3 (IMBEDDING THEOREM). *Let V^n be a Riemannian n-space of type M^r and the rank of the matrix* $(S_{r)ij})$ *be more than* 3. *The necessary and sufficient condition that V^n be imbedded in a flat space as a minimal hypersurface is that the equation* (F_r) *be satisfied. On the other hand, if the matrix is of rank* 2 *or* 3, *then the further condition* (5.2) *must be added.*

If the matrix $(S_{r)ij})$ is of rank 1, then it follows from (F_r) that the curvature tensor vanishes, so that we have a contradiction to the assumption $S_r \neq 0$. This fact permits us to state

THEOREM 4. *There exists no minimal hypersurface of type M^r in a flat space for which the matrix* $(S_{r)ij})$ *is of rank* 1.

§ 6. Einstein spaces as minimal hypersurfaces in flat spaces.

As already remarked at the end of Section 3, any Einstein space is of type M^∞, and so Theorem 3 can not be applied to the space. However, C. B. Allendoerfer gave a necessary and sufficient condition that an Einstein space of $n\,(\geqq 4)$-dimensions having non-vanishing scalar curvature R may be imbedded in a flat $(n+1)$-space [3]. Then he deduced the equation

$$H_{hi}H_{jk}=\frac{eR}{n(n-2)}g_{hi}g_{jk} + \frac{en}{2R(n-2)}(R_{a)}{}^{b}{}_{hj}R_{b)}{}^{a}{}_{ik}-2R_{h)}{}^{a}{}_{ib}R_{j)}{}^{b}{}_{ka}), \tag{6.1}$$

from which H_{ij} are determined. When an Einstein V^n is a minimal hypersurface in a flat space, then we have from (6.1) by transvection with g^{hi}

$$R^{abc}{}_{i}R_{abcj}=\frac{n-1}{n^2}eR^2 g_{ij}\,. \tag{6.2}$$

Therefore we obtain

THEOREM 5. *Let $V^n\,(n\geqq 4)$ be an Einstein space whose scalar curvature R does not vanish. If V^n is imbedded in a flat space as a minimal hypersurface, then the equation* (6.2) *is satisfied.*

§ 7. Minimal hypersurfaces of 3-dimensions, and those which are conformal to a flat space.

We consider a conformally flat space V^n $(n\geqq 4)$. The curvature tensor is expressed as

$$R_{hijk}=g_{hj}\,l_{ik}-g_{hk}\,l_{ij}+g_{ik}\,l_{hj}-g_{ij}\,l_{hk}\,, \tag{7.1}$$

where we have put

$$l_{ij}=\frac{1}{n-2}\left(R_{ij}-\frac{R}{2(n-1)}\,g_{ij}\right).$$

It is well known that any 3-dimensional space satisfies (7.1). We now first treat the spaces for which (7.1) is satisfied, and which are not necessarily minimal hypersurfaces.

From (7.1) and the defining expression of S_{ij} it follows that

$$S_{ij}=\frac{en}{n-2}\,(\alpha\,g_{ij}+\beta\,R_{ij}-R_{2)ij})\,, \tag{7.2}$$

where by definition

$$\alpha=\frac{1}{n}\left(R_{2)}{}^{a}{}_{a}-\frac{R^2}{n-1}\right), \qquad \beta=\frac{R}{n-1}\,.$$

If S_{ij} vanishes, then (7.2) gives

$$R_{2)ij}=\alpha\,g_{ij}+\beta\,R_{ij}\,. \tag{7.3}$$

We shall generally prove the following equations:

$$R_{p)ij}=\alpha_{p-1}\,g_{ij}+\beta_{p-1}\,R_{ij}\,, \qquad (p\geqq 2)\,. \tag{7.4}$$

Indeed, if we suppose that (7.4) holds good, then we obtain from the definition of $R_{p+1)ij}$

$$R_{p+1)ij}=\alpha_{p-1}\,R_{ij}+\beta_{p-1}\,R_{2)ij}\,.$$

Substitution from (7.3) gives

$$R_{p+1)ij}=\alpha_{p}\,g_{ij}+\beta_{p}\,R_{ij}\,,$$

where we have put

$$\alpha_p=\alpha\,\beta_{p-1}\,, \qquad \beta_p=\alpha_{p-1}+\beta\,\beta_{p-1}\,. \tag{7.5}$$

Hence (7.4) has been established and we now have the relation (7.5). As a consequence of (7.4) and (7.5), we have $S_{p)ij}=0$ $(p=2,\cdots)$, provided $S_{ij}=0$. (We have already seen this fact for a V^3 at the end of Section 4.)

When we denote the Ricci principal directions by $\lambda_{a)i}$, the tensors g_{ij} and R_{ij} are expressible as follows:

$$g_{ij}=\sum_{a=1}^{n} e_a \lambda_{a)i} \lambda_{a)j}, \qquad R_{ij}=\sum_{a=1}^{n} e_a \tau_a \lambda_{a)i} \lambda_{a)j}, \qquad (e_a=\pm 1),$$

where τ_a are the mean curvatures of V^n for the direction $\lambda_{a)}{}^i$. Inserting these in (7.3), it can be seen that τ_a must satisfy

$$\tau_i^2=\frac{1}{n}\left\{\sum_{a=1}^{n}\tau_a^2-\frac{1}{n-1}\left(\sum_{a=1}^{n}\tau_a\right)^2\right\}+\frac{\tau_i}{n-1}\sum_{a=1}^{n}\tau_a, \tag{7.6}$$

from which it is readily concluded that there exist only two following cases:

(1) All of the mean curvatures τ_i are equal. In this case V^n is clearly an Einstein space, so that, according to a theorem due to J. A. Schouten and D. J. Struik [4], V^n is of constant curvature.

(2) $\tau_1=\cdots=\tau_r=\tau$, $\tau_{r+1}=\cdots=\tau_n=\tau'$, $\tau\neq\tau'$, $(1\leqq r<n)$, and τ and τ' satisfy the relation

$$(n-r-1)\,\tau+(r-1)\,\tau'=0\,.$$

Gathering the above results we have

THEOREM 6. *If a V^n is of 3-dimensions or conformally flat* $(n\geqq 4)$, *and such that the tensor S_{ij} vanishes, then all of the tensors $S_{p)ij}$ vanish, and the equation* (7.3) *holds good. Such a V^n is of constant curvature or such that the mean curvaturses τ_i are related as given by* (2) *above.*

We return to the consideration of a V^3 or a conformally flat V^n $(n\geqq 4)$, which is not of constant curvature (the case (2)) and may be imbedded in a flat space as a minimal hypersurface. It was shown by the present author [5] that the tensor

$$K_{ij}=nl_{ij}-lg_{ij}, \qquad (l=l^i{}_i),$$

has the property that the determinant $|K_{ij}|$ does not vanish and

$$K_{ij}H_{hk}-K_{hk}H_{ij}=0\,.$$

From these facts it follows easily that there exists a scalar $\pi\neq 0$, satisfying the equation

$$H_{ij}=\pi K_{ij}\,. \tag{7.7}$$

Substitution in (1.1) from (7.7) gives

$$R_{hijk}=\pi^2\,(K_{hj}K_{ik}-K_{hk}K_{ij})\,, \tag{7.8}$$

from which we obtain a necessary condition for a V^n under consideration to be a minimal hypersurface as follows:

$$\begin{vmatrix} R_{abcd} & K_{ac}K_{bd}-K_{ad}K_{bc} \\ R_{hijk} & K_{hj}K_{ik}-K_{hk}K_{ij} \end{vmatrix}=0\,. \tag{7.9}$$

Conversely, if (7.9) is satisfied, then there exist non-trivial p and q such that

$$p\,R_{hijk}=q\,(K_{hj}K_{ik}-K_{hk}K_{ij})\,.$$

It is easily verified by means of $|K_{ij}|\neq 0$ that p does not vanish, so that we can obtain the quantity π satisfying (7.8) and further define H_{ij} by (7.7), which is the solution of the system of equations (1.1) and (A_0). Consequently, making use of a theorem due to T. Y. Thomas mentioned in Section 5, we are led to conclusion that

THEOREM 7. *Let V^n $(n\geqq 4)$ be conformal to a flat space and not of constant curvature. The necessary and sufficient condition that V^n be imbedded in a flat space as a minimal hypersurface is that the determinant $|K_{ij}|\neq 0$ and (7.9) hold good.*

On the other hand, for a V^3, the Codazzi equation should be taken into account. Substituting in (5.1) from (7.7), we get

$$K_{ij}\frac{\partial \log \pi}{\partial x^k}-K_{ik}\frac{\partial \log \pi}{\partial x^j}+K_{ij,k}-K_{ik,j}=0\,. \tag{7.10}$$

For a V^3 conformal to a flat space, (7.10) is written in a simpler form. Since

$$K_{ij,k}-K_{ik,j}=\left(R_{ij,k}-R_{ik,j}+\frac{1}{4}R_{,j}\,g_{ik}-\frac{1}{4}R_{,k}\,g_{ij}\right)$$
$$+l_{,j}\,g_{ik}-l_{,k}\,g_{ij}\,,$$

and the term in parentheses vanishes for such a V^3, (7.10) is now reducible to the following:

$$K_{ij}\frac{\partial \log \pi}{\partial x^k}-K_{ik}\frac{\partial \log \pi}{\partial x^j}=g_{ij}\frac{\partial l}{\partial x^k}-g_{ik}\frac{\partial l}{\partial x^j}\,. \tag{7.11}$$

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References

[1] M. Bôcher: Introduction to higher algebra. Macmillan, New York, 1907, p. 244.
[2] T. Y. Thomas: *Riemann spaces of class one and their algebraic characterization,* Acta Math., 67 (1936), pp. 169–211, § 6.
[3] C. B. Allendoerfer: *Einstein spaces of class one,* Bull. Amer. Math. Soc., 43 (1937), pp. 265–270.
[4] J. A. Schouten and D. J. Struik: *On some properties of general manifolds relating to Einstein's theory of gravitation,* Amer. Jour. Math., 43 (1921), pp. 213–216.
[5] M. Matsumoto: *Conformally flat Riemann spaces of class one,* Jour. Math. Soc. Japan, 3 (1951), pp. 306–309. See (1.4) and Lemma.