Journal of the Mathematical Society of Japan Vol. 9, No. 1, January, 1957

On Riemannian spaces admitting groups of conformal transformations.

By Hitosi HIRAMATU

(Received Nov. 7, 1955)
(Revised Jun. 15, 1956)

On Riemannian spaces admitting groups of conformal transformations, the following theorem was obtained independently by S. Sasaki [3][1], A. H. Taub [4] and K. Yano [6]:

THEOREM 1. *The maximum order of groups of conformal transformations in N-dimensional Riemannian spaces for* $N\geqq 3$ *is* $\frac{1}{2}(N+1)(N+2)$ *and if a Riemannian space admits a group of conformal transformations of the maximum order then the space is conformally flat.*

The author believes that it might not be useless to study the structure of a Riemannian space admitting a group of conformal transformations of order less than the maximum order. In this connection, Y. Mutō [2] recently obtained the following interesting

THEOREM 2. *If an N-dimensional Riemannian space for* $N>4$ *admits a group of conformal transformations of order r such that*

$$r>\frac{1}{2}(N+1)(N+2)-2N+6$$

then the space is conformally flat.

The main purpose of the paper is to prove that in an N-dimensional Riemannian space there exists no group of conformal transformations of order r such that

$$\frac{1}{2}N(N+1)+2<r<\frac{1}{2}(N+1)(N+2)$$

and that an N-dimensional Riemannian space admitting a group of conformal transformations of order larger than $\frac{1}{2}(N-1)(N-2)+2$ is

1) See the Bibliography at the end of the paper.
2) Personal communication.

conformally flat, under some restrictions for dimension N. The sections 1–4 are devoted to the preliminaries and the main theorems will appear in the last section 5. Throughout the paper, we concern essentially with local properties.

§ 1. We consider an N-dimensional Riemannian space R_N with positive definite metric $ds^2=g_{jk}dx^j dx^k$, referred to a coordinate system (x^i) $(a, b, c,\cdots, i, j, k,\cdots=1, 2,\cdots, N)$, and assume that R_N admits a group G_r of conformal transformations

$$T_a:\ x'^i=f^i(x\,;a)\equiv f^i(x^1,\cdots, x^N; a^1,\cdots, a^r)$$

depending on r essential parameters $a^\alpha(\alpha=1, 2,\cdots, r)$. Then we have

$$g_{jk}(x')=h^2(x\,;a)g_{ab}(x)\,\frac{\partial x^a}{\partial x'^j}\frac{\partial x^b}{\partial x'^k}\,, \tag{1.1}$$

where $\left\|\dfrac{\partial x^i}{\partial x'^j}\right\|$ is the inverse of the matrix $\left\|\dfrac{\partial x'^i}{\partial x^j}\right\|$ and $h(x\,;a)$ a positive valued function of x^i and a^α. We notice that $h(x\,;a)$ is a scalar in R_N. If we denote $h(x\,;a)$ by $\alpha(P, T_a)$ symbolically, then we have

$$\alpha(P, T_bT_a)=\alpha(T_aP, T_b)\alpha(P, T_a)\,. \tag{1.2}$$

We take an arbitrary point P_0 with coordinates x_0^i in R_N and denote the group of stability at P_0 by $G_{l_0}(P_0)$, where l_0 represents the order. To each T_a of $G_{l_0}(P_0)$ corresponds a linear transformation T_a^* defined by

$$y'^i=f^i_j(x_0\,;a)y^j,\ f^i_j(x_0\,;a)\equiv\frac{\partial f^i(x_0\,;a)}{\partial x_0^j}\,,$$

where y^i are coordinates of a point in the tangent Euclidean space $E_N(P_0)$ at P_0, and the correspondence φ is a homomorphism of $G_{l_0}(P_0)$ into the linear group consisting of all the T_a^*. We denote the kernel of φ by $K_{s_0}(P_0)$, where s_0 is the order. Since the linear group is of order l_0-s_0, we put $p_0=l_0-s_0$ and denote it by $L^*_{p_0}(P_0)$.

We shall say that a transformation T_a of $G_{l_0}(P_0)$ is *isometric* or *homothetic* at P_0 according as $\alpha(P_0, T_a)$ is equal to 1 or not and that $G_{l_0}(P_0)$ is *isometric* or *homothetic* at P_0 according as all the transformations are isometric at P_0 or not.

First, in the case in which $G_{l_0}(P_0)$ is isometric at P_0, $L^*_{p_0}(P_0)$ is a rotation group. Next, we consider the case in which $G_{l_0}(P_0)$ is homothetic at P_0. We see from (1.2) that a correspondence α defined by

$T_a \to \alpha(P_0, T_a)$ for T_a of $G_{l_0}(P_0)$ is a homomorphism of $G_{l_0}(P_0)$ into the multiplicative group A of real positive numbers and the kernel of α is the maximal subgroup of $G_{l_0}(P_0)$, which is isometric at P_0. Since A is of order one the kernel must be of order l_0-1. We denote it by $M_{l_0-1}(P_0)$. The image of $M_{l_0-1}(P_0)$ by φ is the maximal subgroup of rotations of $L^*_{p_0}(P_0)$ and is of order $p_0-1=l_0-s_0-1$. We denote it by $R^*_{p_0-1}(P_0)$.

In each of cases: the case in which $G_{l_0}(P_0)$ is isometric at P_0 or the case in which $G_{l_0}(P_0)$ is homothetic at P_0, we denote by $M(P_0)$ the totality of all the transformations of $G_{l_0}(P_0)$, which are isometric at P_0 and by $R^*(P_0)$ the totality of all the rotations of $L^*_{p_0}(P_0)$. In the former case we have $M(P_0)=G_{l_0}(P_0)$ and $R^*(P_0)=L^*_{p_0}(P_0)$ and in the latter case $M(P_0)=M_{l_0-1}(P_0)$ and $R^*(P_0)=R^*_{p_0-1}(P_0)$.

§ 2. Hereafter we assume that R_N is of dimension $N \geqq 3$. If we denote by δ the covariant differential, then a conformal circle is defined as a curve represented by a set of solutions of the system of differential equations

$$(2.1) \qquad \frac{\delta^3 x^i}{ds^3} = H^i\left(x, \frac{dx}{ds}, \frac{\delta^2 x}{ds^2}\right) \equiv -\left[g_{ab}(x)\frac{\delta^2 x^a}{ds^2}\frac{\delta^2 x^b}{ds^2} - \Pi_a{}^0{}_b(x)\frac{dx^a}{ds}\frac{dx^b}{ds}\right]\frac{dx^i}{ds} - g^{ia}(x)\Pi_a{}^0{}_b(x)\frac{dx^b}{ds}$$

with the arc length s as variable, where $\Pi_a{}^0{}_b$ are the components of a tensor defined by

$$\Pi_j{}^0{}_k = -\frac{R_{jk}}{N-2} + \frac{Rg_{jk}}{2(N-1)(N-2)} \quad (R_{jk} \equiv R^a{}_{jka},\ R \equiv R^a{}_a),$$

$R^i{}_{jkl}$ being the components of the curvature tensor of R_N [5].

A projective parameter t on a conformal circle with the equations $x^i = x^i(s)$ is uniquely determined by

$$(2.2) \qquad \{t, s\} \equiv \frac{\dfrac{d^3 t}{ds^3}}{\dfrac{dt}{ds}} - \frac{3}{2}\left(\frac{\dfrac{d^2 t}{ds^2}}{\dfrac{dt}{ds}}\right)^2 = \frac{1}{2} g_{ab}(x)\frac{\delta^2 x^a}{ds^2}\frac{\delta^2 x^b}{ds^2} - \Pi_a{}^0{}_b(x)\frac{dx^a}{ds}\frac{dx^b}{ds}$$

up to linear fractional transformations [5].

We shall denote briefly by

$$\frac{\delta^3 x^i}{dt^3} = F^i\left(t,\, x,\, \frac{dx}{dt},\, \frac{\delta^2 x}{dt^2}\right) \tag{2.3}$$

the system of differential equations of conformal circles which can be obtained from (2.1) by a parameter transformation of the arc length s to a projective parameter t. Let P_0 be any given point in R_N and p^i and q^i be any given vectors at the point. Then (2.3) has a unique set of solutions which have the initial conditions: $x^i = x^i_0$, $\left(\frac{dx^i}{dt}\right)_{t=0} = p^i$ and $\left(\frac{\delta^2 x^i}{dt^2}\right)_{t=0} = q^i$ for $t=0$. We express the dependence of the solutions on their initial conditions by writing

$$x^i = x^i(t\,;\, x_0, p, q)\,. \tag{2.4}$$

Hereafter we shall not consider the solutions of (2.3) such that p^i is a zero-vector and we shall say that the conformal circle C represented by (2.4) has the tangent vector p^i at P_0. Then to any two different values in a small open interval $|t| < \mu$ correspond two different points on C. We call the *side* of C with respect to P_0 each of two sets: a set of all the points on C corresponding to all the values of $0 < t < \mu$ and a set of all the points corresponding to $-\mu < t < 0$ and say that one of these two sides is the opposite side of the rest.

THEOREM 2.1. *When $N \geqq 3$, a necessary and sufficient condition that solutions* (2.4) *and*

$$x^i = x^i(t\,;\, x_0, p', q') \tag{2.5}$$

represent the same conformal circle is that we have a relation of the form

$$p'^i = a p^i, \quad q'^i = b p^i + a^2 q^i \quad (a \neq 0)\,. \tag{2.6}$$

For, if (2.4) and (2.5) represent the same conformal circle, then there exists a suitable fractional function $\sigma(t) \equiv \frac{at}{ct+1}$ $(a \neq 0)$ such that the functions $x^i(t\,;\, x_0, p', q')$ and $x^i(\sigma(t)\,;\, x_0, p, q)$ coincide as functions of the argument t and from this fact we have

$$p'^i = a p^i, \quad q'^i = -2ac p^i + a^2 q^i\,.$$

Conversely, if we have the relation of the form (2.6), then the functions $x^i(t\,;\, x_0, p', q')$ and $x^i(\sigma(t)\,;\, x_0, p, q)$ coincide as functions of the argument t and consequently (2.4) and (2.5) represent the same conformal circle, where $\sigma(t)\equiv\dfrac{at}{-\dfrac{b}{2a}t+1}$.

We remark the following. When (2.4) and (2.5) represent the same conformal circle C, for any value t in a small open interval containing zero, the points with coordinates $x^i(t\,;x_0,p,q)$ and $x^i(t\,;x_0,p',q')$ lie on the same side of C with respect to P_0 or lie respectively on the opposite sides according as a in (2.6) is positive or negative.

From the fact that a conformal circle is defined by a system of ordinary differential equations of the third order, we have the following

THEOREM 2.2. *When* $N\geqq 3$, (I) *for any given two different points* P_0 *and* P_1 *which are close to each other and any given vector* p^i *at* P_0 *there exists one and only one conformal circle passing through these points and having* p^i *as the tangent vector at* P_0, (II) *for any given three different points* P_0, P_1 *and* P_2 *which are close to each other there exists one and only one conformal circle passing through these points.*

Referring the calculations done by K. Yano [5] we have, after some calculations,

THEOREM 2.3. *When* $N\geqq 3$, *any conformal circle* C *is transformed by any* T_a *of* G_r *into a conformal circle* C' *and a projective parameter on* C *is at the same time a projective parameter on* C'.

Therefore if C is represented by (2.4) then the functions

$$x'^i=f^i(x(t\,;x_0,p,q)\,;a) \tag{2.7}$$

representing C' are also solutions of (2.3). From (2.4) and (2.7), we have

$$\left\{\begin{aligned}\frac{dx'^i}{dt}&=\frac{\partial f^i(x\,;a)}{\partial x^a}\frac{dx^a}{dt}\\ \frac{\delta^2x'^i}{dt^2}&=\frac{\partial f^i(x\,;a)}{\partial x^a}\left[\frac{\delta^2x^a}{dt^2}+\frac{2}{h(x\,;a)}\frac{\partial h(x\,;a)}{\partial x^b}\frac{dx^b}{dt}\frac{dx^a}{dt}\right.\\ &\qquad\left.-\frac{1}{h(x\,;a)}g_{bc}(x)\frac{dx^b}{dt}\frac{dx^c}{dt}\frac{\partial h(x\,;a)}{\partial x^d}g^{ad}(x)\right]\end{aligned}\right. \tag{2.8}$$

which shows the relation between the vectors at the corresponding points on C and C'.

We remark the following. The property that any two different points lie on the same side of C with respect to P_0 or lie respectively on the opposite sides is invariant under any transformation of G_r.

§ 3. The discussions in this section hold no matter whether $G_{l_0}(P_0)$ is isometric or homothetic at a point P_0. We shall often use the fact that the rotation corresponding to any transformation of $K_{s_0}(P_0)$ is the identity.

To each T_a of $K_{s_0}(P_0)$ corresponds a vector $\psi(P_0, T_a)$ at P_0 with components $\frac{\partial h(x_0\,;a)}{\partial x_0^a} g^{ia}(x_0)$, x_0^i being the coordinates of P_0. If we denote this correspondence by $\dot{\psi}$, then we have the following

THEOREM 3.1. *If $N \geqq 3$, then $K_{s_0}(P_0)$ is isomorphic, under the ψ, to an s_0-dimensional linear space $B_{s_0}(P_0)$ consisting of all the $\psi(P_0, T_a)$ and consequently $K_{s_0}(P_0)$ must be of order $0 \leqq s_0 \leqq N$.*

For, take a transformation T_b of $K_{s_0}(P_0)$. Then we have, from (1.2),

$$\left[\frac{\partial}{\partial x^j}\{h(x'\,;b)h(x\,;a)\}\right]_{x^i=x_0^i} = \frac{\partial h(x_0\,;b)}{\partial x_0^j} + \frac{\partial h(x_0\,;a)}{\partial x_0^j} \quad (x'^i = f^i(x\,;a)),$$

from which

$$\psi(P_0, T_bT_a) = \psi(P_0, T_b) + \psi(P_0, T_a).$$

We consider any T_a of $K_{s_0}(P_0)$ such that $\psi(P_0, T_a)$ is a zero-vector. Let P ($\neq P_0$) be an arbitrary point which is near to P_0 and C an arbitrary conformal circle passing through P_0 and P. If C is represented by (2.4) then C is transformed by T_a into a conformal circle represented by (2.7) which is a set of solutions of (2.3). By using (2.8), we can see that (2.4) and (2.7) have the same initial conditions and coincide well as functions with the argument t. This shows that T_a leaves C point-wisely invariant and consequently leaves P invariant. Since P was arbitrary T_a must be the identity, and we have the theorem.

THEOREM 3.2. *When $N \geqq 3$, if $R^*(P_0)$ is of order $\frac{1}{2}N(N-1)$ then $K_{s_0}(P_0)$ must be of order $s_0 = 0$ or $s_0 = N$.*

We take an arbitrary vector at P_0 and denote it by v^i. If we assume that $K_{s_0}(P_0)$ is of order $s_0 \geqq 1$, then we can take a transformation T_c of $K_{s_0}(P_0)$ such that the vector $\psi(P_0, T_c)$ has the same length as that of v^i. Since $R^*(P_0)$ is of order $\frac{1}{2}N(N-1)$, there exists in $M(P_0)$ a transformation T_b such that the rotation T_b^* carries $\psi(P_0, T_c)$ to v^i, that is,

$$v^i = \frac{\partial f^i(x_0\,;b)}{\partial x_0^j}\frac{\partial h(x_0\,;c)}{\partial x_0^k}g^{jk}(x_0)\,.$$

Putting $T_a = T_b T_c T_b^{-1}$ or $T_a T_b = T_b T_c$, we have, from (1.2),

$$\left[\frac{\partial}{\partial x^k}\{h(x'\,;a)h(x\,;b)\}\right]_{x^i=x_0^i} = \frac{\partial h(x_0\,;a)}{\partial x_0^j}\frac{\partial f^j(x_0\,;b)}{\partial x_0^k} + \frac{\partial h(x_0\,;b)}{\partial x_0^k}$$
$$(x'^i = f^i(x\,;b))$$

and

$$\left[\frac{\partial}{\partial x^k}\{h(x'\,;b)h(x\,;c)\}\right]_{x^i=x_0^i} = \frac{\partial h(x_0\,;b)}{\partial x_0^k} + \frac{\partial h(x_0\,;c)}{\partial x_0^k}$$
$$(x'^i = f^i(x\,;c))\,,$$

from which

$$\frac{\partial h(x_0\,;a)}{\partial x_0^j}\frac{\partial f^j(x_0\,;b)}{\partial x_0^k} = \frac{\partial h(x_0\,;c)}{\partial x_0^k}\,.$$

Multiplying this relation by $\frac{\partial f^i(x_0\,;b)}{\partial x_0^l}g^{kl}(x_0)$, summing with respect to the index k and using the relation

$$g^{ij}(x_0) = g^{kl}(x_0)\frac{\partial f^i(x_0\,;b)}{\partial x_0^k}\frac{\partial f^j(x_0\,;b)}{\partial x_0^l}$$

which can be obtained from (1.1), we have $\frac{\partial h(x_0\,;a)}{\partial x_0^j}g^{ij}(x_0) = v^i$. Since $K_{s_0}(P_0)$ is a normal subgroup of $M(P_0)$, T_a is contained in $K_{s_0}(P_0)$. Thus we see that, for any vector v^i at P_0, there exists a transformation T_a of $K_{s_0}(P_0)$ such that $\psi(P_0, T_a) = v^i$. Hence $B_{s_0}(P_0)$ must coincide with the tangent space at P_0 and $K_{s_0}(P_0)$ is of order $s_0 = N$ by virtue of Theorem 3.1.

From now, we shall prove some lemmas useful to prove the next Theorem 3.3.

LEMMA 1. *When $N \geq 3$, a necessary and sufficient condition that a transformation T_a of $K_{s_0}(P_0)$ leave invariant a conformal circle passing through P_0 is that the vector $\psi(P_0, T_a)$ be proportional to the tangent vector of the conformal circle at the point.*

For, take a conformal circle C passing through P_0 and assume that C is represented by (2.4). By T_a of $K_{s_0}(P_0)$ C is transformed into a conformal circle represented by (2.7) which is a set of solutions of (2.3), and we have, from (2.8),

$$\begin{cases} \left(\dfrac{dx'^i}{dt}\right)_{t=0}=p^i \\ \left(\dfrac{\delta^2 x'^i}{dt^2}\right)_{t=0}=2\dfrac{\partial h(x_0\,;\,a)}{\partial x_0^a}p^a p^i-g_{ab}(x_0)p^a p^b\dfrac{\partial h(x_0\,;\,a)}{\partial x_0^c}g^{ic}(x_0)+q^i\,. \end{cases}$$

By using the above relation, Theorem 2.1 and the fact that $g_{ab}(x_0)p^a p^b$ is different from zero, we obtain the lemma.

LEMMA 2. *When $N \geqq 3$ and $K_{s_0}(P_0)$ is of order $s_0 \geqq 1$, if we take an arbitrary conformal circle such that the tangent vector at P_0 is contained in $B_{s_0}(P_0)$ and two arbitrary point P and P' on the same side of the conformal circle with respect to P_0, then there exists in $K_{s_0}(P_0)$ a transformation which carries P to P'.*

We take a conformal circle C such that the tangent vector at P_0 is contained in $B_{s_0}(P_0)$. By Theorem 3.1 and Lemma 1, the totality of all the transformations of $K_{s_0}(P_0)$ leaving C invariant forms a subgroup of order one of $K_{s_0}(P_0)$. Any group of transformations of order one is isomorphic to a group of translations in a one-dimensional Euclidean space. As was stated in § 2, each of the two sides of C with respect to P_0 is respectively homeomorphic to some small open interval. Therefore, we see that the above stated subgroup acts transitively on each of the sides.

LEMMA 3. *When $N \geqq 3$ and $K_{s_0}(P_0)$ is of order $s_0=N$, if we take two arbitrary conformal circles C and C' which are tangent at P_0 then there exists in $K_{s_0}(P_0)$ a transformation which carries C to C'.*

By using Theorem 2.1, we can assume without loss of generality that C and C' are respectively represented by (2.4) and $x^i=x^i(t\,;\,x_0, p, q')$ and that p^i is a unit vector. We shall examine whether $K_{s_0}(P_0)$ contains a transformation which carries C to C' or not. To do this, if we consider a set of equations

$$q^i+2\frac{\partial h(x_0\,;\,a)}{\partial x_0^j}p^j p^i-\frac{\partial h(x_0\,;\,a)}{\partial x_0^j}g^{ij}(x_0)=q'^i$$

with unknown vector $\dfrac{\partial h(x_0\,;\,a)}{\partial x_0^j}g^{ij}(x_0)$ and solve these equations, then we have

$$\frac{\partial h(x_0\,;a)}{\partial x_0^j}\, g^{ij}(x_0)=q^i-q'^i-2(q^a-q'^a)p_a p^i \quad (p_a=g_{ab}(x_0)p^b)\,.$$

Since $K_{s_0}(P_0)$ is of order $s_0=N$, from Theorem 3.1, for the vector $q^i-q'^i-2(q^a-q'^a)p_a p^i$ at P_0 there exists a transformation T_a of $K_{s_0}(P_0)$ such that

$$\psi(P_0,\, T_a)=\frac{\partial h(x_0\,;a)}{\partial x_0^j}\, g^{ij}(x_0)=q^i-q'^i-2(q^a-q'^a)p_a p^i\,.$$

By using (2.8), we can see that for this transformation the solutions $f^i(x(t\,;x_0,p;q)\,;a)$ and $x^i(t\,;x_0,p,q')$ of (2.3) have the same initial conditions and coincide well as functions with the argument t. This means that T_a carries C to C'.

LEMMA 4. *When $N\geqq 3$ and $R^*(P_0)$ and $K_{s_0}(P_0)$ are of orders $\frac{1}{2}N(N-1)$ and $s_0=N$ respectively, if we take two arbitrary conformal circles C and C' passing through P_0, then $M(P_0)$ contains a transformation which carries C to C', especially contains a transformation which carries C to its opposite conformal circle, in other words, carries any point on one of two sides of C with respect to P_0 to a point on the opposite side.*

From Theorem 2.1, we can assume without loss of generality that C and C' have the unit tangent vectors p^i and p'^i at P_0 respectively. Since $R^*(P_0)$ is of order $\frac{1}{2}N(N-1)$, there exists in $M(P_0)$ a transformation T_a such that the rotation T_a^* carries p^i to p'^i. Consequently T_a transforms C into a conformal circle C_1 having p'^i as the tangent vector at P_0. Since C_1 and C' are tangent at P_0 and $K_{s_0}(P_0)$ is of order $s_0=N, K_{s_0}(P_0)$ contains a transformation T_b which carries C_1 to C' by virtue of Lemma 3. Hence the product T_bT_a transforms C to C'.

THEOREM 3.3. *When $N\geqq 3$ and $R^*(P_0)$ and $K_{s_0}(P_0)$ are of orders $\frac{1}{2}N(N-1)$ and $s_0=N$ respectively, if we take two arbitrary different points $P(\neq P_0)$ and $P'(\neq P_0)$ which are close to P_0, then there exists in $M(P_0)$ a transformation which carries P to P'.*

From Theorem 2.2, there exists one and only one conformal circle C_1 passing through the points P_0, P and P'. First we consider the case in which P and P' lie on the same side of C_1 with respect to P_0. From Lemma 2, $K_{s_0}(P_0)$ contains a transformation which carries P to P'. Next we consider the case in which P and P' lie respectively on the opposite side. From Lemma 4, $M(P_0)$ contains a transformation T_a which carries C_1 to the opposite conformal circle. Therefore, if P_1 is the transformed point of P by T_a, then P_1 and P' lie on the same side of C_1. Since a suitable transformation T_b of $K_{s_0}(P_0)$ carries P_1 to P', the product T_bT_a carries P to P'.

COROLLARY. *When* $N\geqq 3$, *if we assume that* $R^*(P_0)$ *and* $K_{s_0}(P_0)$ *are of orders* $\frac{1}{2}N(N-1)$ *and* $s_0=N$ *respectively at each point of the space, then* G_r *is transitive.*

If there are given arbitrary different points P and P', then we can take suitable odd points $P_1, P_2, \cdots, P_{2m+1}$ in such a way that any two neighboring points of a series of points: $P, P_1, P_2, \cdots, P_{2m+1}$ and P' are sufficiently near. Since Theorem 3.3 holds at each point of the space, a suitable transformation belonging to $M(P_1)$ carries P to P_2, a suitable transformation belonging to $M(P_3)$ carries P_2 to $P_4, \cdots$ and a suitable transformation belonging to $M(P_{2m+1})$ carries P_{2m} to P'. Thus the product of these transformations carries P to P'.

§ 4. Hereafter the Greek indices take the following values:

$$\begin{cases} \alpha, \beta, \gamma=1,\cdots, r\,;\ \delta=1,\cdots, l_0\,;\ \theta=1,\cdots, s_0\,; \\ \lambda=s_0+1,\cdots, l_0\,;\ \pi, \omega=l_0+1,\cdots, r\,. \end{cases}$$

If we put $\xi^i_\alpha \equiv \dfrac{\partial f^i(x\,;a_0)}{\partial a^\alpha_0}$, then we have

$$\xi^a_\alpha \frac{\partial \xi^i_\beta}{\partial x^a} - \xi^a_\beta \frac{\partial \xi^i_\alpha}{\partial x^a} = C_{\alpha\beta}{}^\gamma \xi^i_\gamma\,, \tag{4.1}$$

where a^α_0 are the values of parameters of the identity of G_r and $C_{\alpha\beta}{}^\gamma$ are the constants of structure of the group. Let L_α be the operator giving Lie derivative [6] with respect to the vector ξ^i_α, then we have, by using (1.1),

$$L_\alpha g_{jk} \equiv \xi_{\alpha j;\,k} + \xi_{\alpha k;\,j} = 2\phi_\alpha g_{jk} \quad (\xi_{\alpha j} = g_{ij}\xi^i_\alpha)\,,$$

where the semi-colon denotes covariant differentation and ϕ_α is a scalar defined by $\phi_\alpha \equiv \dfrac{\partial h(x\,;a_0)}{\partial a^\alpha_0}$. We have

$$[L_\alpha, L_\beta]g_{jk} \equiv (L_\alpha L_\beta - L_\beta L_\alpha)g_{jk} = 2(L_\alpha\phi_\beta - L_\beta\phi_\alpha)g_{jk}$$

and, on the other hand [6],

$$[L_\alpha, L_\beta]g_{jk} = C_{\alpha\beta}{}^\gamma L_\gamma g_{jk} = 2C_{\alpha\beta}{}^\gamma \phi_\gamma g_{jk}$$

and consequently

$$\xi^a_\alpha \frac{\partial\phi_\beta}{\partial x^a} - \xi^a_\beta \frac{\partial\phi_\alpha}{\partial x^a} = C_{\alpha\beta}{}^\gamma\phi_\gamma . \tag{4.2}$$

If we define the so-called Weyl's conformal curvature tensor $C^i{}_{jkl}$ and the tensor $C^0{}_{jkl}$ by

$$C^i{}_{jkl} \equiv R^i{}_{jkl} + \Pi_j{}^0{}_k\delta^i_l - \Pi_j{}^0{}_l\delta^i_k + g_{jk}g^{ia}\Pi_a{}^0{}_l - g_{jl}g^{ia}\Pi_a{}^0{}_k$$

and

$$C^0{}_{jkl} \equiv \Pi_j{}^0{}_{k;l} - \Pi_j{}^0{}_{l;k}$$

respectively, then we have the identities:

$$\begin{cases} C^a{}_{jka} = 0 \quad C^i{}_{jkl} + C^i{}_{klj} + C^i{}_{ljk} = 0 \\ C_{ijkl} = -C_{ijlk} = C_{klij} (= -C_{jikl}) \end{cases} \tag{4.3}$$

and

$$C^0{}_{jkl} = -C^0{}_{jlk}, \quad C^0{}_{jkl} + C^0{}_{klj} + C^0{}_{ljk} = 0, \quad g^{jk}C^0{}_{jkl} = 0 . \tag{4.4}$$

We have

$$L_\alpha C^i{}_{jkl} = C^i{}_{jkl;a}\xi^a_\alpha - \xi^i_{\alpha f}C^f{}_{jkl} + \xi^f_{\alpha j}C^i{}_{fkl} + \xi^f_{\alpha k}C^i{}_{jfl} + \xi^f_{\alpha l}C^i{}_{jkf} = 0$$

and

$$L_\alpha C^0{}_{jkl} = C^0{}_{jkl;a}\xi^a_\alpha + \xi^f_{\alpha j}C^0{}_{fkl} + \xi^f_{\alpha k}C^0{}_{jfl} + \xi^f_{\alpha l}C^0{}_{jkf} = -\phi_{\alpha i}C^i{}_{jkl} ,$$

from which, by using $\xi_{\alpha jk} = \phi_\alpha g_{jk} + \frac{1}{2}(\xi_{\alpha jk} - \xi_{\alpha kj})$,

$$C_{ijkl;a}\xi^a_\alpha + 2\phi_\alpha C_{ijkl} + \frac{1}{2}(\xi_{\alpha bc} - \xi_{\alpha cb})E^{bc}{}_{ijkl} = 0 \tag{4.5}$$

and

$$C^0{}_{jkl;a}\xi^a_\alpha + 3\phi_\alpha C^0{}_{jkl} + \frac{1}{2}(\xi_{\alpha bc} - \xi_{\alpha cb})F^{bc}{}_{jkl} + \phi_{\alpha d}C^d{}_{jkl} = 0 , \tag{4.6}$$

where

$$E^{bc}{}_{ijkl} \equiv g^{bf}(\delta^c_i C_{fjkl} + \delta^c_j C_{ifkl} + \delta^c_k C_{ijfl} + \delta^c_l C_{ijkf}),$$
$$F^{bc}{}_{jkl} \equiv g^{jf}(\delta^c_j C^0{}_{fkl} + \delta^c_k C^0{}_{jfl} + \delta^c_l C^0{}_{jkf})$$

and

$$\xi^i_{\alpha j} \equiv \xi^i_{\alpha;j}, \quad \xi_{\alpha jk} \equiv \xi_{\alpha j;k}, \quad \phi_{\alpha j} \equiv \phi_{\alpha;j} = \frac{\partial \phi_\alpha}{\partial x^j}.$$

Now, when $G_{l_0}(P_0)$ is isometric at P_0, we can take parameters of G_r such that when and only when a transformation T_α is contained in $G_{l_0}(P_0)$ the T_a has values of parameters of the form (a^δ, a^π_0) and when and only when a transformation T_a is contained in $K_{s_0}(P_0)$ the T_a has values of parameters of the form $(a^\theta, a^\lambda_0, a^\pi_0)$, where a^α_0 are the values of parameters of the identity transformation in such parameters. From now, we shall adopt such parameters. From the relations $x^i_0 = f^i(x_0; a^\delta, a^\pi_0)$, $h(x_0; a^\delta, a^\pi_0) = 1$ and $f^i_j(x_0; a^\theta, a^\lambda_0, a^\pi_0) = \delta^i_j$, we have $\xi^i_\delta(x_0) = 0$, $\phi_\delta(x_0) = 0$ and $\dfrac{\partial \xi^i_\theta(x_0)}{\partial x^j_0} = 0$ respectively, and consequently $\xi_{\delta jk}(x_0) = \dfrac{\partial \xi_{\delta j}(x_0)}{\partial x^k_0}$.

THEOREM 4.1. *When $N \geqq 3$, if $G_{l_0}(P_0)$ is isometric at P_0 then the order s_0 of $K_{s_0}(P_0)$ satisfies the relations $s_0 \leqq N - r + l_0$ or $p_0 = l_0 - s_0 \geqq r - N$.*

In fact, from the relations (4.1), $\xi^i_\theta(x_0) = 0$ and $\dfrac{\partial \xi^i_\theta(x_0)}{\partial x^j_0} = 0$ and the fact that the matrix $\|\xi^i_\pi(x_0)\|$ is of rank $r - l_0 (\leqq N)$, we have $C_{\theta\pi}{}^\omega = 0$. Hence, from the relations (4.2), $\xi^i_\theta(x_0) = 0$ and $\phi_\delta(x_0) = 0$, we have

$$\phi_{\theta i}(x_0)\xi^i_\pi(x_0) = 0.$$

On the other hand, since, as was stated in Theorem 3.1, $K_{s_0}(P_0)$ is isomorphic to $B_{s_0}(P_0)$, it follows that the matrix $\|\phi_{\theta j}(x_0)\|$ is of rank s_0. Hence we can obtain the relations in the theorem.

Since $l_0 = r - N$ holds if G_r is transitive, we have

COROLLARY. *In Theorem 4.1, if we moreover assume that G_r is transitive, then $s_0 = 0$.*

THEOREM 4.2. *When $G_{l_0}(P_0)$ is isometric at P_0, if $N \geqq 3$ and $L^*_{p_0}(P_0)$ is of order $\dfrac{1}{2}N(N-1)$ or if $N > 4, \neq 8$ and $L^*_{p_0}(P_0)$ is of order $\dfrac{1}{2}(N-1)(N-2)$, then R_N is conformally flat at P_0, that is, $C^i{}_{jkl} = C^0{}_{jkl} = 0$ at P_0.*

We introduce in R_N a coordinate system such that $g_{jk}(x_0)=\delta_{jk}$. In such a coordinate system we see that $p_0(=l_0-s_0)$ $\xi_{\lambda jk}(x_0)$ satisfying the relations

$$\xi_{\lambda jk}(x_0)+\xi_{\lambda kj}(x_0)=0$$

derived from the fact that $||f^i_j(x_0\,;a^\theta_0, a^\lambda, a^\pi_0)||$ is an orthogonal matrix form a basis of the Lie ring of the orthogonal group of order p_0 corresponding to the rotation group $L^*_{p_0}(P_0)$. Hence the matrix $||\xi_{\lambda jk}(x_0)||$ $(j<k)$ of $\frac{1}{2}N(N-1)$ columns and p_0 rows, λ indicating the rows and (j, k) the columns, must be of rank p_0.

From the relations (4.5), (4.6), $\xi^i_\lambda(x_0)=0$ and $\phi_\lambda(x_0)=0$, we have

$$\sum_{b<c}\xi_{\lambda bc}(x_0)[E^{bc}{}_{ijkl}(x_0)-E^{cb}{}_{ijkl}(x_0)]=0 \tag{4.7}$$

and

$$\sum_{b<c}\xi_{\lambda bc}(x_0)[F^{bc}{}_{jkl}(x_0)-F^{cb}{}_{jkl}(x_0)]+\phi_{\lambda d}(x_0)C^d{}_{jkl}(x_0)=0 \tag{4.8}$$

with $g_{jk}(x_0)=\delta_{jk}$. Since $L^*_p(P_0)$ is of the maximum order, that is, of order $\frac{1}{2}N(N-1)$, we have, from (4.7),

$$E^{bc}{}_{ijkl}(x_0)-E^{cb}{}_{ijkl}(x_0)=0\ ,$$

from which, by using (4.3), $C^i{}_{jkl}(x_0)=0$. Consequently, we have, from (4.8),

$$F^{bc}{}_{jkl}(x_0)-F^{cb}{}_{jkl}(x_0)=0$$

from which, by using (4.4), $C^0{}_{jkl}(x_0)=0$. Thus the first half of the Theorem is proved.

Next we shall prove the latter half of the theorem. Since $L^*_{p_0}(P_0)$ is of order $\frac{1}{2}(N-1)(N-2)$ and $N\neq 4, 8, L^*_{p_0}(P_0)$ fixes one and only one direction by virtue of the theorem due to D. Montgomery and H. Samelson [2]. Consequently there exists a coordinate system of R_N in which $g_{jk}(x_0)=\delta_{jk}$ and moreover the first vector of the natural frame of reference at P_0 is in the direction. In such a coordinate system, we have $f^1_j(x_0\,;a^\theta_0, a^\lambda, a^\pi_0)=\delta^1_j$ from which $\xi_{\lambda 1c}(x_0)\,(=\xi_{\lambda b1}(x_0))=0$, and consequently the matrix $||\xi_{\lambda pq}(x_0)||$ $(p<q)$ $(p, q, r, s, t, u=2, 3,\cdots, N)$ is of rank $\frac{1}{2}(N-1)(N-2)$.

We have, from (4.7),

$$E^{pq}{}_{ijkl}(x_0)-E^{qp}{}_{ijkl}(x_0)=0\ ,$$

from which, by using (4.3),

$$C_{1stu}(x_0)=C_{r11u}(x_0)=C_{rstu}(x_0)=0\ .$$

$C_{ijkl}(x_0)=0$ follows from (4.3) and the above relations. We get, from (4.8),

$$F^{pq}{}_{jkl}(x_0)-F^{qp}{}_{jkl}(x_0)=0\ .$$

By using (4.4) and the assumption that N is not equal to 3, we have

$$C^0{}_{s1u}(x_0)=C^0{}_{11u}(x_0)=C^0{}_{stu}(x_0)=0\ .$$

$C^0{}_{jkl}(x_0)=0$ follows from (4.4) and the above relations. Thus we have the theorem.

Here we mention the theorem which was obtained by S. Ishihara and M. Obata [1]: *When* $N\geqq 3$, *if* $G_{l_0}(P_0)$ *is homothetic at* P_0 *then* R_N *is conformally flat at* P_0.

§ 5. We first prove the following theorem:

THEOREM 5.1. *In* R_N *for* $N\geqq 3, \neq 4$, *there exists no group of conformal transformations of order* r *such that*

$$\frac{1}{2}N(N+1)+2<r<\frac{1}{2}(N+1)(N+2)\ .$$

Assume that G_r is of order $r>\frac{1}{2}N(N+1)+2$. When $G_{l_0}(P_0)$ is isometric at P_0,

$$\text{order } p_0 \text{ of } R^*(P_0)=l_0-s_0\geqq r-N-s_0>\frac{1}{2}(N-1)(N-2)\ .$$

According to D. Montgomery and H. Samelson [2], in an N-dimensional Euclidean space for $N\neq 4$ there exists no proper subgroup of order greater than $\frac{1}{2}(N-1)(N-2)$, and we must have $l_0-s_0=\frac{1}{2}N(N-1)$. Therefore we have

$$r\leqq l_0+N=\frac{1}{2}N(N+1)+s_0\ ,$$

from which $2<s_0\leqq N$. Hence we have $s_0=N$ from Theorem 3.2. When $G_{l_0}(P_0)$ is homothetic at P_0,

order p_0-1 of $R^*(P_0)=l_0-s_0-1\geqq r-N-s_0-1>\frac{1}{2}(N-1)(N-2)$.

Since $N\neq 4$, we have $l_0-s_0-1=\frac{1}{2}N(N-1)$. Therefore we have

$$r\leqq l_0+N=\frac{1}{2}N(N+1)+s_0+1\,,$$

from which $1<s_0\leqq N$. Hence we have $s_0=N$. Thus, from corollary to Theorem 3.3, G_r is transitive. If we assume that $G_{l_0}(P_0)$ is isometric at P_0, then, from corollary to Theorem 4.1, $K_{s_0}(P_0)$ must be of order $s_0=0$. But, as was already stated, $s_0=N$ holds. This is a contradiction. Therefore $G_{l_0}(P_0)$ must be homothetic at P_0 and we have $r=\frac{1}{2}(N+1)(N+2)$.

THEOREM 5.2. *If R_N for $N\geqq 3, \neq 4$ admits G_r of order r such that*

$$\frac{1}{2}N(N-1)+1<r\leqq\frac{1}{2}(N+1)(N+2)\,,$$

then the R_N is conformally flat.

When $G_{l_0}(P_0)$ is isometric at P_0, the order p_0 of $L^*_{p_0}(P_0)$ satisfies

$$p_0=l_0-s_0\geqq r-N>\frac{1}{2}(N-1)(N-2)$$

by virtue of Theorem 4.1. Therefore, from the assumption that $N\neq 4$, we have $p_0=\frac{1}{2}N(N-1)$. Hence, from Theorem 4.2, R_N is conformally flat at P_0. When $G_{l_0}(P_0)$ is homothetic at P_0, from a theorem of S. Ishihara and M. Obata, R_N is conformally flat at P_0. The point P_0 being arbitrary, we have the theorem.

THEOREM 5.3. *If R_N for $N>4, \neq 8$ admits G_r of order $r=\frac{1}{2}N(N-1)+1$, then the R_N is conformally flat.*

When $G_{l_0}(P_0)$ is isometric at P_0, we have, from Theorem 4.1,

$$p_0=l_0-s_0\geqq r-N=\frac{1}{2}(N-1)(N-2)\,.$$

Since $N\neq 4$, we have $p_0=\frac{1}{2}(N-1)(N-2)$ or $p_0=\frac{1}{2}N(N-1)$. Therefore, from the assumption that $N>4, \neq 8, R_N$ is conformally flat at P_0 by virtue of Theorem 4.2. When $G_{l_0}(P_0)$ is homothetic at P_0, R_N is conformally flat at P_0. The point P_0 being arbitrary, we have the theorem.

THEOREM 5.4. *Except for finite number of N's, if R_N for $N \geqq 3$ admits G_r of order r such that*

$$\frac{1}{2}(N-1)(N-2)+2<r<\frac{1}{2}N(N-1)+1,$$

then the R_N is conformally flat.

When $G_{l_0}(P_0)$ is isometric at P_0, we have, from Theorem 4.1,

$$p_0=l_0-s_0 \geqq r-N>\frac{1}{2}(N-2)(N-3).$$

According to D. Montgomery and H. Samelson [2], except for finite number of N's in an N-dimensional Euclidean space the rotation group has no subgroup of order t such that

$$\frac{1}{2}(N-1)(N-2)<t<\frac{1}{2}N(N-1),$$

$$\frac{1}{2}(N-2)(N-3)<t<\frac{1}{2}(N-1)(N-2),$$

the exceptional values of N's depending on the special types of the Killing-Cartan's classification of simple groups. Therefore, we must have $p_0=\frac{1}{2}(N-1)(N-2)$ or $p_0=\frac{1}{2}N(N-1)$, and R_N is, from Theorem 4.2, conformally flat at P_0. When $G_{l_0}(P_0)$ is homothetic at P_0, R_N is conformally flat at P_0. The point P_0 being arbitrary, we have the theorem.

We remark that the above theorems are true for a Riemannian space admitting a group of motions or a group of homothetic transformations which are special groups of conformal transformations.

The author wishes to express here his gratitude to Professor Kentaro Yano for his many valuable suggestions.

Mathematical Institute,
Kumamoto University.

Bibliography

[1] S. Ishihara and M. Obata, On the group of conformal transformations of a Riemannian manifold, Proc., Japan Acad., 31 (1955), pp. 426–429.
[2] D. Montgomery and H. Samelson, Transformation groups of spheres, Ann. of Math., 44 (1943), pp. 454–470.
[3] S. Sasaki, Geometry of the conformal connexion, Sci. Rep. Tôhoku Imp. Univ., Ser. 1, 29 (1940), pp. 219–267.
[4] A. H. Taub, A characterization of conformally flat spaces, Bull. Amer. Math. Soc., 55 (1949), pp. 85–89.
[5] K. Yano, Sur la théorie des espaces à connexion conforme, Jour. Fac. Sci. Imp. Univ. of Tokyo, Sect. 1, 4 (1939), pp. 1–59.
[6] ———, Groups of transformations in generalized spaces, Tokyo, 1949.