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On the theory of ordinal numbers

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In a former paper [3], the author formalized the theory of ordinal numbers in a logical system G^1LC introduced in [4], and constructed in that theory the set theory of Fraenkel-von Neumann. The system G^1LC is a large system containing the concept of 'arbitrary predicates'. Such a logical system is convenient, on the one hand, in application. It allows us to form the theory of ordinal numbers, for example, on a quite simple system of axioms. On the other hand however, the nature of G^1LC itself is not yet clarified. One does not know whether an analogue of Gentzen's theorem [1]: "Every provable sequence in the system is provable without cut" ("Fundamental conjecture") does or does not hold in G^1LC.

In this paper, we introduce a new logical system *FLC*, obtained by a slight modification of *GLC*. It will not contain the concept of "arbitrary predicates" but will contain that of "arbitrary functions". We shall prove that the "fundamental conjecture" is valid in *FLC*.

The theory of ordinal numbers will be then reformalized in *FLC*, and the set theory of Fraenkel-von Neumann will be constructed in it. The author has in view to show, in a forth-coming paper, that, conversely, the theory of ordinal numbers as formalized in this paper can be constructed in the Fraenkel-von Neumann set theory.

The logical system *FLC* is, as said above, a modification of *GLC*. The formulas in *FLC* consist of variables, functions of various types, predicates and logical symbols described below. It is to be noticed that we have a wider domain of types than in *GLC*.

Chapter I. The system of logic *FLC*.

§ 1. Type symbols.

The type symbol is defined recursively as follows.

1.1.1. 0 is a type symbol

1.1.2. If $\alpha_1,\cdots,\alpha_n$ are type symbols, then $(\alpha_1,\cdots,\alpha_n)$ is a type symbol. $(n=1,2,3,\cdots)$

The *height* of a type symbol is defined recursively as follows.

1.2.1. The height of the type symbol 0 is zero.

1.2.2. The height of the type symbol $(\alpha_1,\cdots,\alpha_n)$ is $h+1$, where h is the maximal number of the heights of $\alpha_1,\cdots,\alpha_n$.

§ 2. Variables Functions etc.

2.1. Variables, which are also called function of type 0 or variable of type 0.

2.1.1. Free variables $a, b, c,\cdots$

2.1.2. Bound variables $x, y, z,\cdots$

2.1.3. Special variables $0, \omega,\cdots$

2.2. Functions of type α (the height of $\alpha>0$), which are also called variable of type α.

2.2.1. Free functions of type α $f\alpha, g\alpha, h\alpha,\cdots$

2.2.2. Bound functions of type α $p\alpha, q\alpha, r\alpha,\cdots$

2.2.3. Special functions $*_1'$, $\max(*_1, *_2),\cdots$

2.3. Predicates $*_1=*_2$, $*_1<*_2$

2.4. Logical symbols $\neg, \wedge, \vee, \forall, \exists$

If no confusion is likely to occur, the notation may be abbreviated by the following conventions; we may write f, p for $f\alpha_1, p\alpha_2$ respectively.

§ 3. Terms and functionals.

Terms and functionals are defined recursively as follows.

3.1. Every free or special variable is a term.

3.2. Let $f_1,\cdots,f_n$ be free functions of type $\alpha_1,\cdots,\alpha_n$ respectively, where $f_i\neq f_j$, for $i\neq j$. Moreover, let $p_1,\cdots,p_n$ be bound functions of types $\alpha_1,\cdots,\alpha_n$ respectively such that $p_1,\cdots,p_n$ are not contained in a term T and $p_i\neq p_j$ for $i\neq j$. If we substitute p_i for f_i in T for all $i(1\leqq i\leqq n)$ at all places where f_i are, and add $\{p_1,\cdots, p_n\}$ in front of the so constructed figure, then we obtain a functional of type $(\alpha_1,\cdots,\alpha_n)$.

3.3. Let f be a free or a special function of type $(\alpha_1,\cdots,\alpha_n)$ and F_i be a functional of type $\alpha_i(i=1,\cdots,n)$. Then $f(F_1,\cdots,F_n)$ is a term.

Term is called functional of type 0.

§ 4. Formula.

Formula is defined recursively as follows.

4.1. If T_1 and T_2 are terms, then $T_1 = T_2$ and $T_1 < T_2$ are formulas.

4.2. If A_1 and A_2 are formulas, then $\neg A_1$, $A_1 \wedge A_2$, $A_1 \vee A_2$ are formulas.

4.3. Let p be a bound function of the same type with a free function f and be not contained in a formula A. If we substitute p for f in A at all places where f are, and add $\forall p$ or $\exists p$ in front of the so constructed figure, then we obtain a formula.

Indication, homologousness and substitution are defined in the same way as in [4].

§ 5. Inference-figures.

Inference-figures on structure of sequences, cut, inferences on $\neg$, on $\wedge$, and on $\vee$ are given with the same schemata as in [4].

$\forall$ left $\dfrac{A(F), \Gamma \rightarrow \Delta}{\forall p A(p), \Gamma \rightarrow \Delta}$

(F is an arbitrary functional of the same type with p)

$\forall$ right $\dfrac{\Gamma \rightarrow \Delta, A(f)}{\Gamma \rightarrow \Delta, \forall p A(p)}$

(There is no f in the lower sequence)

$\exists$ left $\dfrac{A(f), \Gamma \rightarrow \Delta}{\exists p A(p), \Gamma \rightarrow \Delta}$

(There is no f in the lower sequence)

$\exists$ right $\dfrac{\Gamma \rightarrow \Delta, A(F)}{\Gamma \rightarrow \Delta, \exists p A(p)}$

(F is an arbitrary functional of the same type with p)

§ 6. Proof-figure.

Proof-figure is defined in the same way as in [4].

By the same method as in [1], we have easily the following theorem.

THEOREM 1. If $\mathfrak{S}$ is a provable sequence, then $\mathfrak{S}$ is provable without cut.

Chapter II. The theory of ordinal numbers.

§ 1. Axioms of the theory of ordinal numbers.

First we shall list the axioms of the theory of ordinal numbers.

I. 1. $\forall x(x=x)$

2. $0<\omega$

3. $\forall x\forall y(x<y\vee x=y\vee y<x)$

4. $\forall x\forall y\neg(x=y\wedge x<y)$

5. $\forall x\forall y\neg(x<y\wedge y<x)$

6. $\forall x\forall y\forall z(x<y\wedge y<z\rightarrow x<z)$

7. $\forall x(0<x\vee x=0)$

8. $\forall x\forall y(x<y\rightarrow x'=y\vee x'<y)$

where $*'$ is a special function of type (0)

9. $\forall x(x<x')$

10. $\forall x\forall y(x'=y'\rightarrow x=y)$

11. $\forall x(x<\omega\rightarrow x'<\omega)$

12. $\forall x\forall y(x\leqq y\leftrightarrow \max(x,y)=y)$

where $x\leqq y$ means $x<y\vee x=y$ as usual, and max is a special function of type (0,0).

13. $\forall x\forall y(y\leqq x\leftrightarrow \max(x,y)=x)$

14. $N(0)=0'$

where N is a special function of type (0).

15. $\forall x(x>0\rightarrow N(x)=0)$

We use sometimes $a>b$ for $b<a$ and $a\geqq b$ for $b\leqq a$ as usual.

16. $\forall x\forall y(y<x\leftrightarrow \mathrm{Iq}(y,x)=0)$

where Iq is a special function of type (0,0).

17. $\forall x\forall y(x=y\rightarrow \mathrm{Eq}(x,y)=0)$

where Eq is a special function of type (0,0).

18. $\forall x(\delta(x')=x)$

where δ is a special function of type (0).

19. $\forall x\forall y(x<y\rightarrow \delta(x)\leqq\delta(y))$

20. $\forall x(x<\omega\rightarrow(\delta(x))'=x\vee x=0)$

21. $\forall x(j(g^1(x),g^2(x))=x)$

where j is a special function of type (0,0) and g^1 and g^2 are special functions of type (0).

22. $\forall x\forall y(g^1(j(x,y))=x)$

23. $\forall x\forall y(g^2(j(x,y))=y)$

2.4. $\forall x \forall y \forall u \forall v(j(x, y) < j(u, v) \leftrightarrow \max(x, y) < \max(u, v)$
$\vee (\max(x, y) = \max(u, v) \wedge (y < v \vee (y = v \wedge x < u))))$

II. 1. Equality axiom
$\forall p \forall x \forall y((x = y \rightarrow p(x) = p(y))$

2. Axiom of mininum
a) $\forall p \forall x(p(x) = 0 \rightarrow p(\mathrm{Min}(p)) = 0 \wedge x \geqq \mathrm{Min}(p))$
b) $\forall p(p(\mathrm{Min}(p)) = 0 \vee \mathrm{Min}(p) = 0)$

where Min is a special function of type ((0)).

3. Axiom of upper bound
$\forall p \forall x \forall y(y < x \rightarrow p(y) < \mathrm{up}(p, x))$

where up is a special function of type ((0), 0).

4. Axiom of contraction
$\forall p \forall x \forall y((y < x \rightarrow \mathrm{Con}(p, x, y) = p(y)) \wedge (y \geqq x \rightarrow \mathrm{Con}(p, x, y) = 0))$

where Con is a special function of type ((0), 0,0).

5. Axiom of gap
$\forall p \forall x \forall y(\mathrm{gap}(p, x) = p(y) \rightarrow y \geqq x)$

where gap is a special function of type ((0), 0).

6. Axiom of sum
$\forall p \forall q \forall r \forall x((p(x) = 0 \rightarrow S(p, q, r, x) = q(x))$
$\wedge (p(x) > 0 \rightarrow S(p, q, r, x) = r(x)))$

where S is a special function of type ((0), (0), (0), 0)

7. Axiom of recursive function
$\forall p \forall x(\mathrm{Rec}(p, x) = p(\{y\}\mathrm{Con}(\{z\}\mathrm{Rec}(p, z), x, y), x))$

where Rec is a special function of type (((0), 0), 0).

8. Axiom of cardinal
$\forall p \forall x(\mathrm{gap}(p, x) < \chi(x))$

The system of all these axioms I. 1—24 and II. 1—8, that is, the juxtaposition of these axioms by the agency of commata, will be denoted by Γ_0.

In this paper, we shall use the following abbreviated wording. When $\Gamma_0, \Gamma \rightarrow \Delta$ is provable, we shall say briefly that $\Gamma \rightarrow \Delta$ is provable, or that we have $\Gamma \rightarrow \Delta$; if, moreover, $\Gamma_0 \rightarrow A$ is provable, then we say that A is provable or that we have A.

Propositions, of which we shall omit easy proofs are, marked by *.

Next we shall define several elementary functions and state their elementary properties.

$\mathrm{Min}(f, a)$ is defined by $\mathrm{Min}(\{x\}\mathrm{Eq}(f(x), a))$.

*We have

$$f(T)=a\to f(\mathrm{Min}(f, a))=a\wedge T\geqq \mathrm{Min}(f, a)$$

and

$$\forall x\neg(f(x)=a)\to \mathrm{Min}(f, a)=0 .$$

$E(f)$ is defined by $f(\mathrm{Min}(f))$.

*We have

$$E(f)=0\leftrightarrow \exists x(f(x)=0) .$$

$E(f, a)$ is defined by $E(\{x\}\mathrm{Eq}(f(x), a))$.

*We have

$$E(f, a)=0\leftrightarrow \exists x(f(x)=a) .$$

§ 2. Primitive formula.

A formula A is called a *primitive formula* (abbreviated by *pf*), if A satisfies the following condition. If $\forall p$ or $\exists p$ is contained in A, then p is of the type 0, that is, $\forall p$ or $\exists p$ is of the form $\forall x$ or $\exists x$ respectively.

Now, we prove the following theorem.

THEOREM 2. *Let $F(f,\cdots, g, a,\cdots, b)$ be a primitive formula, where $F(p,\cdots, q, x,\cdots, y)$ contains no free function nor free variable. Then there exists such a term $T(f,\cdots, g, a,\cdots, b)$ that $T(p,\cdots, q, x,\cdots, y)$ has no free function nor free variable and*

$$\forall p\cdots\forall q\forall x\cdots\forall y(T(p,\cdots, q, x,\cdots, y)=0 \\ \leftrightarrow F(p,\cdots, q, x,\cdots, y))$$

is provable.

PROOF. We prove this by the induction on the number n of logical symbols in $F(f,\cdots, g, a,\cdots, b)$.

1) The case, when $n=0$. In this case $F(f,\cdots, g, a,\cdots, b)$ is of the form

$$U(f,\cdots, g, a,\cdots, b)=V(f,\cdots, g, a,\cdots, b)$$

or

$$U(f,\cdots, g, a,\cdots, b)<V(f,\cdots, g, a,\cdots, b) .$$

Accordingly, we set as $T(f,\cdots, g, a,\cdots, b)$.

$$\mathrm{Eq}(U(f,\cdots, g, a,\cdots, b), V(f,\cdots, g, a,\cdots, b))$$

or

$$\mathrm{Iq}(U(f,\cdots, g, a,\cdots, b), V(f,\cdots, g, a,\cdots, b))$$

Then the proposition is clear.

2) The case, when $n>0$.

a) The case, when $F(f,\cdots,g,a,\cdots,b)$ is of the form $\neg G(f,\cdots,g,a,\cdots,b)$. By the hypothesis of the induction, there exists such a term $U(f,\cdots,g,a,\cdots,b)$ that $U(p,\cdots,q,x,\cdots,y)$ contains no free function nor free variable and $\forall p\cdots\forall q\forall x\cdots\forall y(U(p,\cdots,q,x,\cdots,y)=0\mapsto G(p,\cdots,q,x,\cdots,y))$ is provable. We set $N(U(f,\cdots,g,a,\cdots,b))$ as $T(f,\cdots,g,a,\cdots,b)$. Then the proposition is clear.

b) The case, when $F(f,\cdots,g,a,\cdots,b)$ is of the form $G(f,\cdots,g,a,\cdots,b)\wedge H(f,\cdots,g,a,\cdots,b)$. By the hypothesis of the induction, there exist such terms $U(f,\cdots,g,a,\cdots,b)$ and $V(f,\cdots,g,a,\cdots,b)$ that $U(p,\cdots,q,x,\cdots,y)$ and $V(p,\cdots,q,x,\cdots y)$ contain no free function nor free variable and

$$\forall p\cdots\forall q\forall x\cdots\forall y(U(p,\cdots,q,x,\cdots,y)=0 \\ \mapsto G(p,\cdots,q,x,\cdots,y))$$

and

$$\forall p\cdots\forall q\forall x\cdots\forall y(V(p,\cdots,q,x,\cdots,y)=0 \\ \mapsto H(p,\cdots,q,x,\cdots,y))$$

are provable. We set

$$\max(U(f,\cdots,g,a,\cdots,b),\ V(f,\cdots,g,a,\cdots,b))$$

as $T(f,\cdots,g,a,\cdots,b)$. Then the proposition is clear.

c). The case, when $F(f,\cdots,g,a,\cdots,b)$ is of the form $\exists zG(z,f,\cdots,g,a,\cdots,b)$. By the hypothesis of the induction, there exists such a term $U(c,f,\cdots,g,a,\cdots,b)$ that $U(z,p,\cdots,q,x,\cdots,y)$ contains no free function nor free variable and

$$\forall z\forall p\cdots\forall q\forall x\cdots\forall y(U(z,p,\cdots,q,x,\cdots,y)=0 \\ \mapsto G(z,p,\cdots,q,x,\cdots,y))\,.$$

We set $$E(\{z\}U(z,f,\cdots,g,a,\cdots,b))$$

as $$T(f,\cdots,g,a,\cdots b)\,.$$

Then the proposition is clear.

By theorem 2 the following propositions are easily proved.

LEMMA 1. *Let $A(a)$ be pf. Then the following formula is provable.*

$$\forall x\forall y(x=y\vdash(A(x)\mapsto A(y)))\,.$$

LEMMA 2. (Transfinite induction). *Let $A(a)$ be pf. Then the fol-*

lowing sequences are provable.

$$\forall x(x<T \vdash (\forall y(y<x \vdash A(y)) \vdash A(x))) \to \forall x(x<T \vdash A(x)).$$

and $$\forall x(\forall y(y<x \vdash A(y)) \vdash A(x)) \to \forall x A(x).$$

LEMMA 3. (Mathematical induction). *Let* $A(a)$ *be pf. Then the following sequence is provable.*

$$A(0),\ \forall x(A(x) \vdash A(x')),\ T<\omega \to A(T).$$

LEMMA 4. *Let* $A(a)$ *be pf, and* $U(a)$ *and* $V(a)$ *be terms. Then there exists such a term* $T(a)$ *that the following formula is provable.*

$$\forall x((A(x) \vdash T(x)=U(x)) \wedge (\neg A(x) \vdash T(x)=V(x))).$$

This proposition is clearly generalized as follows.

Let $A_1(a),\cdots, A_n(a)$ *be pf's and* $\forall x(A_1(x) \vee \cdots \vee A_n(x))$ *and* $\forall x \neg (A_i(x) \wedge A_j(x))$, *for* $i \neq j$, *be provable. Moreover let* $T_1(a),\cdots, T_n(a)$ *be terms. Then there exists such a term* $T(a)$ *that the following formula is provable.*

$$\forall x((A_1(x) \vdash T(x)=T_1(x)) \wedge \cdots \wedge (A_n(x) \vdash T(x)=T_n(x))).$$

LEMMA 5. *Let* $T(\{z\}\mathrm{Con}(\{u\}f(u), a, z), a)$ *be a term, where* $T(\{z\}\mathrm{Con}(\{u\}p(u), x, z), x)$ *contains neither* f *nor* a. *Then there exists such a term* $U(a)$ *that* $U(x)$ *has no* a *and the following formula is provable.*

$$\forall x(U(x)=T(\{z\}\mathrm{Con}(\{v\}U(u), x, z), x)).$$

LEMMA 6. *Let* $A_1(b),\cdots, A_n(b)$ *be pf's and* $\forall x(A_1(x) \vee \cdots \vee A_n(x))$ *and* $\forall x \neg (A_i(x) \wedge A_j(x))$, *for* $i \neq j$, *be provable. Moreover, let* $T_1(\{z\}\mathrm{Con}(\{u\}f(u), a, z), a),\cdots, T_n(\{z\}\mathrm{Con}(\{u\}f(u), a, z), a)$ *be terms, where* $T_i(\{z\}\mathrm{Con}(\{u\}p(u), x, z), x)$ *contains neither* f *nor* a *for each* i $(1 \leqq i \leqq n)$. *Then there exists such a term* $T(a)$ *that the following formula is provable.*

$$\forall x((A_1(x) \vdash T(x)=T_1(\{z\}\mathrm{Con}(\{u\}T(u), x, z), x)) \wedge \cdots \wedge$$
$$(A_n(x) \vdash T(x)=T_n(\{z\}\mathrm{Con}(\{u\}T(u), x, z), x))).$$

In this case, we say that the function $T(a)$ *is defined by the formula:*

$$\forall x((A_1(x) \vdash T(x)=T_1(\{z\}\mathrm{Con}(T, x, z), x)) \wedge \cdots \wedge$$
$$(A_n(x) \vdash T(x)=T_n(\{z\}\mathrm{Con}(T, x, z), x))).$$

LEMMA 7. *Let* $A_1(b),\cdots, A_n(b)$ *be pf's and* $\forall x(A_1(x) \vee \cdots \vee A_n(x))$ *and* $\forall x \neg (A_i(x) \wedge A_j(x))$, *for* $i \neq j$, *be provable. Moreover, let* $B_1(\{z\}\mathrm{Con}(\{u\}f(u),$

$a, z), a), \cdots, B_n(\{z\}\mathrm{Con}(\{u\}f(u), a, z), a)$ be pf's, where $B_i(\{z\}\mathrm{Con}(\{u\}p(u), x, z), x)$ contains neither f nor a for each $i\ (1 \leqq i \leqq n)$. Then there exists such a term $T(x)$ that the following formula is provable.

$$\forall x((A_1(x) \vdash (T(x)=0 \vdash\!\dashv B_1(\{z\}\mathrm{Con}(\{u\}T(u), x, z), x))) \wedge \cdots \wedge (A_n(x) \vdash (T(x)=0 \vdash\!\dashv B_n(\{z\}\mathrm{Con}(\{u\}T(u), x, z), x)))) .$$

Let $A(a)$ be pf. Then there exists such a term $T(a)$ that

$$\forall x((T(x)=0 \vdash\!\dashv A(x)) \wedge T(x) \leqq 1) .$$

We define $\mathrm{Min}(x)A(x)$ *by* $\mathrm{Min}(\{x\}T(x))$. *Clearly the following sequences are provable.*

$$A(U) \rightarrow A(\mathrm{Min}(x)A(x)) ,$$

$$\rightarrow A(\mathrm{Min}(x)A(x)), \mathrm{Min}(x)A(x)=0 ,$$

$$\rightarrow \forall x(A(x) \vdash x \geqq \mathrm{Min}(y)A(y)) .$$

§ 3. Construction of several functions.

$\sup(f, a)$ is defined by $\mathrm{Min}(x)\forall y(y<a \vdash f(y)<x)$.
By the axiom of upper bound, we have easily

$$\forall p\forall x\forall y(y<x \vdash p(y)<\sup(p, x))$$

and

$$\forall p\forall x\forall u(\forall y(y<x \vdash p(y)<u) \vdash \sup(p, x) \leqq u) .$$

$\mathrm{mg}(f, a)$ is defined by $\mathrm{Min}(x)\forall y(x=f(y) \vdash y \geqq a)$.
By the axiom of gap, we have easily

$$\forall p\forall x\forall y(\mathrm{mg}(p, x)=p(y) \vdash y \geqq x)$$

and

$$\forall p\forall x\forall u(\forall y(u=p(y) \vdash y \geqq x) \vdash u \geqq \mathrm{mg}(p, x)) .$$

It is remarkable that $\mathrm{Min}(x)\forall p(\mathrm{mg}(p, a)<x)$ cannot be defined.

$a+b$ is defined by the following formula.

$$a+0=a \wedge \forall x(x>0 \vdash a+x=\sup(\{z\}\mathrm{Con}(\{u\}(a+u), x, z), x))$$

*We have

$$\forall x\forall y(x \leqq x+y)$$

$$\forall x\forall y\forall z(z<y \vdash (x+z)' \leqq x+y)$$

$$\forall x\forall y\forall u(\forall z(z<y \vdash (x+z)' \leqq u) \wedge x \leqq u \vdash x+y \leqq u)$$

$$\forall x\forall y(x+y'=(x+y)')$$

$$\forall x(x+0=x)$$

$$\forall x(0+x=x)$$

$\forall x \forall y(x<\omega \wedge y<\omega \vdash x+y=y+x)$

$\forall x \forall y(x<\omega \wedge y<\omega \vdash x+y<\omega)$

$\forall x \forall y \forall z(y<z \vdash\dashv x+y<x+z)$

$\forall x \forall y \forall z((x+y)+z=x+(y+z))$.

*We have furthermore

$\forall x \forall y(x<\omega \wedge y<\omega \vdash j(x, y)<\omega)$

$j(\omega, 0)=\omega$

$j(0, \omega)=\omega \cdot 2$

$j(\omega, \omega)=\omega \cdot 3$

where $a \cdot 2$ or $a \cdot 3$ is the abbreviation of $a+a$ or $a+a+a$.

$\delta_0(a)$ is defined by the following formula.

$(a<\omega \vdash \delta_0(a)=\delta(a)) \wedge(a \geqq \omega \vdash \delta_0(a)=a)$.

We define $g^{11}(a), g^{12}(a), g^{21}(a)$ and $g^{22}(a)$ by $g^1(g^1(a)), g^1(g^2(a)), g^2(g^1(a))$ and $g^2(g^2(a))$ respectively.

$\delta_1(a)$ is defined by the following formula.

$(g^1(a)=0 \vdash \delta_1(a)=g^2(a)) \wedge(g^1(a)>0 \vdash \delta_1(a)=\delta_0(g^1(a)))$

$\delta_0(a, b)$ is defined by $\delta_1(j(a, b))$

*We have

$\delta_0(0, b)=b$

and $\quad a>0 \rightarrow \delta_0(a, b)=\delta_0(a)$

$S(a, b), S_1(a), T(a, b), T_1(a), T_2(a), T_3(a)$ and $T_4(a)$

are defined by the following:

$S(a, b)$ is $j(\delta_0(g^1(a), \delta(b)), \delta_0(g^2(a), \delta(b)))$

$S_1(a)$ is $j(g^1(a), \delta(g^2(a)))$

$T(a, b)$ is $j(j(g^1(a), g^1(b)), j(g^2(a), g^2(b)))$

$T_1(a)$ is $j(g^{11}(a), g^{12}(a))$

$T_2(a)$ is $j(g^{21}(a), g^{22}(a)')$

$T_3(a)$ is $j(g^{11}(a), g^{12}(a)')$

$T_4(a)$ is $j(g^{21}(a), g^{22}(a))$

$K(a)$ is defined by the following formula

$(g^1(a) \geqq g^2(a) \vdash K(a)=0)$

$$\wedge (g^1(a) < g^2(a) \wedge g^{12}(a) = g^2(a) \vdash K(a) = g^1(a))$$
$$\wedge (g^1(a) < g^2(a) \wedge g^{12}(a) < g^2(a) \wedge \delta(g^2(a)) < g^2(a)$$
$$\vdash K(a) = S(\mathrm{Con}(K, a, S_1(a)), g^2(a)))$$
$$\wedge (g^1(a) < g^2(a) \wedge g^{12}(a) < g^2(a) \wedge \delta(g^2(a)) = g^2(a) \wedge g^{12}(a) \leqq g^{22}(a)$$
$$\vdash K(a) = T(\mathrm{Con}(K, a, T_1(a)), (\mathrm{Con}(K, a, T_2(a)))))$$
$$\wedge (g^1(a) < g^2(a) \wedge g^{12}(a) < g^2(a) \wedge \delta(g^2(a)) = g^2(a) \wedge g^{12}(a) > g^{22}(a)$$
$$\vdash K(a) = T(\mathrm{Con}(K, a, T_3(a)), \mathrm{Con}(K, a, T_4(a))))$$

$K(a, b)$ is defined to be $K(j(a, b))$

*We have

$$a \geqq b \rightarrow K(a, b) = 0$$
$$a < b, g^1(b) = b \rightarrow K(a, b) = a$$
$$a < b, g^1(b) < b, \delta(b) < b \rightarrow K(a, b) = S(K(a, \delta(b)), b)$$
$$a < b, g^1(b) < b, \delta(b) = b, g^1(b) \leqq g^2(b)$$
$$\rightarrow K(a, b) = T(K(g^1(a), g^1(b)), K(g^2(a), g^2(b)'))$$
$$a < b, g^1(b) < b, \delta(b) = b, g^1(b) > g^2(b)$$
$$\rightarrow K(a, b) = T(K(g^1(a), g^1(b)'), K(g^2(a), g^2(b)))$$
$$a < \omega, b < \omega \rightarrow \exists z(z < \omega \wedge K(z, \omega) = j(a, b))$$
$$a < \omega \rightarrow g^1(K(a, \omega)) < \omega \wedge g^2(K(a, \omega)) < \omega$$

By the transfinite induction on a, we have easily

$$\forall x \forall y \exists z(x < a \wedge y < a \wedge \omega \leqq a \vdash z < a \wedge K(z, a) = j(x, y))$$

and $$\forall x(x < a \wedge \omega \leqq a \vdash g^1(K(x, a)) < a \wedge g^2(K(x, a)) < a).$$

$G(a, b\,; c)$ is defined by $\mathrm{Min}(z)\,(K(z, c) = j(a, b))$.

*We have

$$\omega \leqq a \rightarrow \forall x \forall y(x < a \wedge y < a \vdash G(x, y\,; a) < a \wedge K(G(x, y\,; a), a) = j(x, y)).$$

Now we define successively $0' = 1$, $1' = 2$, $2' = 3$, $3' = 4$, $4' = 5$, $5' = 6$, $6' = 7$, $7' = 8$, $8' = 9$, $9' = 10$, $10' = 11$, $11' = 12$.

$J(a, b)$ is defined by the following formula.

$$J(0, b) = 0$$
$$\wedge (\delta(a) < a \vdash J(a, b) = \mathrm{Con}(\{u\}J(u, b), a, \delta(a)) + b)$$
$$\wedge (0 < a \wedge \delta(a) = a \vdash J(a, b) = \sup(\{z\}\mathrm{Con}(\{u\}J(u, b), a, z), a)).$$

*We have

$$b>0\rightarrow\forall x\forall y(x<y\vdash\dashv J(x,b)<J(y,b))$$
$$\forall\exists xy\exists z(J(y,b)+z=x\wedge z<b).$$

$j(c,a,b)$ is defined by $J(j(a,b);9)+c$.

*We have

$$\forall u\forall v\forall x\forall y\forall z\forall w(u<9\wedge v<9\vdash(j(u,x,y)<j(v,z,w)$$
$$\vdash\dashv j(x,y)<j(z,w)\vee(j(x,y)=j(z,w)\wedge u<v)))$$
$$\forall u\forall v\forall x\forall y\forall z\forall w(u<9\wedge v<9\wedge j(u,x,y)=j(v,z,w)$$
$$\vdash u=v\wedge x=z\wedge y=w)$$

$g_0(a)$ is defined by $\mathrm{Min}(z)\exists x\exists y(a=j(z,x,y)\wedge z<9)$.
$g_1(a)$ is defined by $\mathrm{Min}(z)\exists x\exists y(a=j(x,z,y)\wedge x<9)$.
$g_2(a)$ is defined by $\mathrm{Min}(z)\exists x\exists y(a=j(x,y.z)\wedge x<9)$.

*We have

$$\forall x(g_0(x)<9)$$
$$\forall x(j(g_0(x),g_1(x),g_2(x))=x)$$
$$\forall x(x\geqq g_1(x))$$
$$\forall x(x\geqq g_2(x))\wedge\forall x(x>0\vdash x>g_2(x)).$$

$\tilde{j}(c,a,b)$ is defined by $J(j(a,b),12)+c$.

*We have

$$\forall u\forall v\forall x\forall y\forall z\forall w(u<12\wedge v<12\vdash(\tilde{j}(u,x,y)<\tilde{j}(v,z,w)$$
$$\vdash\dashv j(x,y)<j(z,w)\vee(j(x,y)=j(z,w)\wedge u<v)))$$
$$\forall u\forall v\forall x\forall y\forall z\forall w(u<12\wedge v<12\wedge\tilde{j}(u,x,y)=\tilde{j}(v,z,w)$$
$$\vdash u=v\wedge x=z\wedge y=w).$$

In the same way as above, we have three functions $\tilde{g}_0$, $\tilde{g}_1$, $\tilde{g}_2$ satisfying

$$\forall x(\tilde{g}_0(x)<12)$$
$$\forall x(\tilde{j}(\tilde{g}_0(x),\tilde{g}_1(x),\tilde{g}_2(x))=x)$$
$$\forall x(x\geqq\tilde{g}_1(x))$$
$$\forall x(x\geqq\tilde{g}_2(x))\wedge\forall x(x>0\vdash x>\tilde{g}_2(x)).$$

In the same way as above, we have a function $\tilde{K}(a,b)$ satisfying

$$\forall u\forall x\forall y\exists z(u<12\wedge x<a\wedge y<a\wedge\omega\leqq a$$

$$\vdash z < a \wedge \tilde{K}(z, a) = \tilde{j}(u, x, y))$$

and $$\forall x(x < a \wedge \omega \leqq a \vdash g_1(\tilde{K}(x, a)) < a \wedge g_2(\tilde{K}(x, a)) < a).$$

$a-b$ is defined by $\mathrm{Min}(x)\,(b+x=a)$.

*We have

$$b \leqq a \rightarrow b + (a-b) = a$$

$$b \leqq a, b \leqq c \rightarrow a \leqq c \vdash\!\dashv (a-b) \leqq (c-b).$$

Chapter III. Construction of set theory.

§ 1. The model Δ.

Let $<(f, b, c)$ be $b > c \wedge f(j(b, c)) = 0$

$=(f, b, c)$ be $(b \leqq c \wedge f(j(b, c)) = 0) \vee (b \geqq c \wedge f(j(c, b)) = 0)$

$\preccurlyeq(f, b, c)$ be $\exists x(=(f, x, c) \wedge <(f, b, x))$

$=(f : b\,;\{c\,; d\})$ be $\forall x(x < b \vdash (\preccurlyeq(f, b, x) \vdash\!\dashv =(f, x, c) \vee =(f, x, d)))$
$\wedge \exists x(x < b \wedge =(f, x, c)) \wedge \exists x(x < b \wedge =(f, x, d))$

$\preccurlyeq(f : b\,;\{c\,; d\})$ be $\exists x(x < b \wedge \preccurlyeq(f, b, x) \wedge =(f : x\,; \{c\,; d\}))$

$=(f : b\,; <c\,; d>)$ be $\exists x \exists y(x < b \wedge y < b \wedge =(f : b\,; \{x\,; y\})$
$\wedge =(f : x\,; \{c\,; c\}) \wedge =(f : y\,; \{c\,; d\}))$

$\preccurlyeq(f : b\,; <c\,; d>)$ be
$\exists x(x < b \wedge \preccurlyeq(f, b, x) \wedge =(f : x\,; <c\,; d>))$

$=(f : b\,; <c\,; d\,; e>)$ be
$\exists x(x < b \wedge =(f : b\,; <c\,; x>) \wedge =(f : x\,; <d\,; e>))$

$\preccurlyeq(f : b\,; <c\,; d\,; e>)$ be
$\exists x(x < b \wedge \preccurlyeq(f, b, x) \wedge =(f : x\,; <c\,; d\,; e>))$.

Moreover let

$H_1(f, a)$ be $=(f, g^2(a), g_1(g^1(a))) \vee =(f, g^2(a), g_2(g^1(a)))$

$H_2(f, a)$ be $\preccurlyeq(f, g_1(g^1(a)), g^2(a)) \wedge \exists x \exists y(x < g^2(a) \wedge y < g^2(a)$
$\wedge \preccurlyeq(f, y, x) \wedge =(f : g^2(a)\,; <x\,; y>))$

$H_3(f, a)$ be $\preccurlyeq(f, g_1(g^1(a)), g^2(a)) \wedge \neg \preccurlyeq(f, g_2(g^1(a)), g^2(a))$

$H_4(f, a)$ be $\preccurlyeq(f, g_1(g^1(a)), g^2(a)) \wedge \exists x \exists y(x < g^2(a) \wedge y < g^2(a)$
$\wedge =(f : g^2(a)\,; <x\,; y>) \wedge \preccurlyeq(f, g_2(g^1(a)), y))$

$H_5(f, a)$ be $\exists x(x < g_1(g^1(a)) \wedge \preccurlyeq(f : g_1(g^1(a))\,; <x\,; g^2(a)>))$

$H_6(f,a)$ be $\exists x \exists y(x<g_1(g^1(a)) \wedge y<g_1(g^1(a))$
$\wedge \leqslant(f: g_1(g^1(a)) ; <x ; y>) \wedge =(f: g^2(a) ; <y ; x>))$

$H_7(f,a)$ be $\exists x \exists y \exists z(x<g_1(g^1(a)) \wedge y<g_1(g^1(a)) \wedge z<g_1(g^1(a))$
$\wedge \leqslant(f: g_1(g^1(a)) ; <x ; y ; z>) \wedge =(f: g^2(a) ; <y ; z ; x>))$

$H_8(f,a)$ be $\exists x \exists y \exists z(x<g_1(g^1(a)) \wedge y<g_1(g^1(a)) \wedge z<g_1(g^1(a))$
$\wedge \leqslant(f: g_1(g^1(a)) ; <x ; y ; z>) \wedge =(f: g^2(a) ; <x ; z ; y>))$

$H_9(f,a)$ be $\forall x(x<g^2(a) \vdash (\leqslant(f, g^1(a), x) \vdash\!\dashv \leqslant(f, g^2(a), x)))$

Then there exists such a $fn(a)$ that the following sequences are provable.

$$g^1(a)>g^2(a), g_0(g^1(a))=0 \rightarrow fn(a)=0$$
$$g^1(a)>g^2(a), g_0(g^1(a))=1 \rightarrow fn(a)=0 \vdash\!\dashv H_1(\{z\}\mathrm{Con}(fn, a, z), a)$$
$$g^1(a)>g^2(a), g_0(g^1(a))=2 \rightarrow fn(a)=0 \vdash\!\dashv H_2(\{z\}\mathrm{Con}(fn, a, z), a)$$
$$g^1(a)>g^2(a), g_0(g^1(a))=3 \rightarrow fn(a)=0 \vdash\!\dashv H_3(\{z\}\mathrm{Con}(fn, a, z), a)$$
$$g^1(a)>g^2(a), g_0(g^1(a))=4 \rightarrow fn(a)=0 \vdash\!\dashv H_4(\{z\}\mathrm{Con}(fn, a, z), a)$$
$$g^1(a)>g^2(a), g_0(g^1(a))=5 \rightarrow fn(a)=0 \vdash\!\dashv H_5(\{z\}\mathrm{Con}(fn, a, z), a)$$
$$g^1(a)>g^2(a), g_0(g^1(a))=6 \rightarrow fn(a)=0 \vdash\!\dashv H_6(\{z\}\mathrm{Con}(fn, a, z), a)$$
$$g^1(a)>g^2(a), g_0(g^1(a))=7 \rightarrow fn(a)=0 \vdash\!\dashv H_7(\{z\}\mathrm{Con}(fn, a, z), a)$$
$$g^1(a)>g^2(a), g_0(g^1(a))=8 \rightarrow fn(a)=0 \vdash\!\dashv H_8(\{z\}\mathrm{Con}(fn, a, z), a)$$
$$g^1(a) \leqq g^2(a) \qquad \rightarrow fn(a)=0 \vdash\!\dashv H_9(\{z\}\mathrm{Con}(fn, a, z), a).$$

Let $<(b,c)$, $=(b,c)$, $\leqslant(b,c)$, $=(b;\{c;d\})$, $\leqslant(b;\{c;d\})$, $=(b;<c;d>)$, $\leqslant(b;<c;d>)$, $=(b;<c;d;e>)$, $\leqslant(b;<c;d;e>)$ be $<(fn,b,c)$, $=(fn,b,c)$, $\leqslant(fn,b,c)$, $=(fn:b;\{c;d\})$, $\leqslant(fn:b;\{c;d\})$, $=(fn:b;<c;d>)$, $\leqslant(fn:b;<c;d>)$, $=(fn:b;<c;d;e>)$, $\leqslant(fn:b;<c;d;e>)$ respectively. We write $c \in b$ (or $b \ni c$), $b \equiv c$, $\{b,c\}$, $<c,d>$, $<c,d,e>$, $\mathrm{Od}(a)$, $C(a)$, $b \dot{-} c$, $b \cdot c$, $a \subseteq b$ and $a \subset b$ for $\leqslant(b,c)$, $=(b,c)$, $j(1,b,c)$, $\{\{c,c\}, \{c,d\}\}$, $<c, <d,e>>$, $\mathrm{Min}(z)(z \equiv a)$, $\mathrm{Min}(z)(z \in a)$, $j(3,b,c)$, $b \dot{-} (b \dot{-} c)$, $\forall x(x \in a \vdash x \in b)$ and $a \subseteq b \wedge \neg(a \equiv b)$ respectively. Then, in the same way as in pp. 209—214 of [3], we have

$$g_0(b)=0 \rightarrow a \in b \vdash\!\dashv \exists x(x \equiv a \wedge x<b)$$
$$\rightarrow \{b,c\} \ni d \vdash\!\dashv b \equiv d \vee c \equiv d$$
$$a \equiv b, c \equiv d \rightarrow \{a,c\} \equiv \{b,d\}$$
$$\rightarrow a \equiv c \wedge b \equiv d \vdash\!\dashv <a,b> \equiv <c,d>$$
$$\rightarrow a \equiv b \wedge c \equiv d \wedge e \equiv f \vdash\!\dashv <a,c,e> \equiv <b,d,f>$$

$g_0(b)=2\rightarrow b\ni c\vdash\!\dashv c\in g_1(b)\wedge\exists x\exists y(x\in y\wedge c\equiv<x,y>)$

$\rightarrow a\in(b\dot{-}c)\vdash\!\dashv a\in b\wedge\neg(a\in c)$

$\rightarrow a\in(b\cdot c)\vdash\!\dashv a\in b\wedge a\in c$

$g_0(b)=4\rightarrow b\ni c\vdash\!\dashv g_1(b)\ni c\wedge\exists x\exists y(c\equiv<x,y>\wedge g_2(b)\ni y)$

$g_0(b)=5\rightarrow b\ni c\vdash\!\dashv\exists x(g_1(b)\ni<x,c>)$

$g_0(b)=6\rightarrow b\ni c\vdash\!\dashv\exists x\exists y(g_1(b)\ni<x,y>\wedge c\equiv<y,x>)$

$g_0(b)=7\rightarrow b\ni c\vdash\!\dashv\exists x\exists y\exists z(g_1(b)\ni<x,y,z>\wedge c\equiv<y,z,x>)$

$g_0(b)=8\rightarrow b\ni c\vdash\!\dashv\exists x\exists y\exists z(g_1(b)\ni<x,y,z>\wedge c\equiv<x,z,y>)$

$\rightarrow a\equiv\mathrm{Od}(a)$

$a\equiv b\rightarrow b\geqq\mathrm{Od}(a)$

$a\in b\rightarrow\mathrm{Od}(a)<\mathrm{Od}(b)$

$\exists x(x\in a)\rightarrow C(a)\in a\wedge\forall x(x\in a\vdash x\geqq C(a))$

$\rightarrow\neg(a\in a)$

$\rightarrow 0\in\omega\wedge\forall x(x\in\omega\vdash\exists y(y\in\omega\wedge x\subset y))$

$\rightarrow\forall x\neg(x\in 0)$

$\rightarrow\forall y\forall z(z\in y\wedge y\in a\vdash z\in j(0,a,a))\,.$

We write $a\in f$ for $f(a)=0$. Moreover we denote with $\mathrm{cl}(f)$ the formula $\forall x\exists y\forall z(z\in f\wedge z\in x\vdash\!\dashv z\in y)$.

First, we see from lemma 2 that the following sequences are provable.

$\rightarrow\exists p\forall x(p(x)=0\vdash\!\dashv x\in a)$

$\rightarrow\exists p\forall x(p(x)=0\vdash\!\dashv f(x)=0\wedge\neg g(x)=0)$

$\rightarrow\exists p\forall x(p(x)=0\vdash\!\dashv f(x)=0\wedge g(x)=0)$

$\rightarrow\exists p\forall x(p(x)=0\vdash\!\dashv f(x)=0\vee g(x)=0)$

$\rightarrow\exists p\forall x(p(x)=0\vdash\!\dashv\exists y\exists z(x\equiv<y,z>\wedge f(x)=0))$

$\rightarrow\exists p\forall x(p(x)=0\vdash\!\dashv\exists yf(<y,x>)=0)$

$\rightarrow\exists p\forall u(p(u)=0\vdash\!\dashv\exists x\exists y(f(<x,y>)=0\wedge u\equiv<y,x>))$

$\rightarrow\exists p\forall u(p(u)=0\vdash\!\dashv\exists x\exists y\exists z(f(<x,y,z>)=0\wedge u\equiv<y,z,x>))$

$\rightarrow\exists p\forall u(p(u)=0\vdash\!\dashv\exists x\exists y\exists z(f(<x,y,z>)=0\wedge u\equiv<x,z,y>))$

$\rightarrow\exists p\forall u(p(u)=0\vdash\!\dashv\exists x\exists y(u\equiv<y,x>\wedge f(y)=0))$

$\rightarrow\exists p\forall u(p(u)=0\vdash\!\dashv\exists y\exists z(u\equiv<y,z>\wedge f(y)=0\wedge g(z)=0))$

$\rightarrow\exists p\forall x(p(x)=0\vdash\!\dashv\exists y(f(<x,y>)=0))$

$\rightarrow\exists p\forall x(p(x)=0\vdash\!\dashv\exists y(f(<x,y>)=0\wedge g(y)=0))$

$$\rightarrow \exists p \forall x(p(x)=0 \vdash\!\dashv \exists y(f(<x, y>)=0 \wedge y \in a))$$
$$\rightarrow \exists p \forall x(p(x)=0 \vdash\!\dashv f(<x, a>)=0) .$$

Hence by the same calculation as in the chapter V p. 40 in [2], we have

$$\forall x \neg(x \in f) \rightarrow \mathrm{cl}(f)$$
$$\forall x(x \in f) \rightarrow \mathrm{cl}(f)$$
$$\mathrm{cl}(f), a \in f, a \equiv b \rightarrow b \in f$$
$$\rightarrow \forall x \exists p(\mathrm{cl}(p) \wedge \forall y(y \in p \vdash\!\dashv y \in x))$$
$$\forall x(x \in f \vdash\!\dashv \exists y \exists z(x \equiv <y, z> \wedge y \in z)) \rightarrow \mathrm{cl}(f)$$
$$\rightarrow \forall p \forall q \exists r(\mathrm{cl}(p) \wedge \mathrm{cl}(q) \vdash \mathrm{cl}(r) \wedge \forall x(x \in r \vdash\!\dashv x \in p \wedge \neg x \in q))$$
$$\rightarrow \forall p \forall q \exists r(\mathrm{cl}(p) \wedge \mathrm{cl}(q) \vdash \mathrm{cl}(r) \wedge \forall x(x \in r \vdash\!\dashv x \in p \vee x \in q))$$
$$\mathrm{cl}(f) \rightarrow \exists p(\mathrm{cl}(p) \wedge \forall x(x \in p \vdash\!\dashv \exists y(<y, x> \in f)))$$
$$\mathrm{cl}(f) \rightarrow \exists p(\mathrm{cl}(p) \wedge \forall u(u \in p \vdash\!\dashv \exists x \exists y(<x, y> \in f \wedge u \equiv <y, x>)))$$
$$\mathrm{cl}(f) \rightarrow \exists p(\mathrm{cl}(p) \wedge \forall u(u \in p \vdash\!\dashv \exists x \exists y \exists z(<x, y, z> \in f \wedge u \equiv <y, z, x>)))$$
$$\mathrm{cl}(f) \rightarrow \exists p(\mathrm{cl}(p) \wedge \forall u(u \in p \vdash\!\dashv \exists x \exists y \exists z(<x, y, z> \in f \wedge u \equiv <x, z, y>)))$$
$$\mathrm{cl}(f) \rightarrow \exists p(\mathrm{cl}(p) \wedge \forall u(u \in p \vdash\!\dashv \exists x \exists y(u \equiv <y, x> \wedge y \in f)))$$
$$\mathrm{cl}(f), \mathrm{cl}(g) \rightarrow \exists p(\mathrm{cl}(p) \wedge \forall x(x \in p \vdash\!\dashv \exists y \exists z(x \equiv <y, z> \wedge y \in f \wedge z \in g)))$$
$$\mathrm{cl}(f) \rightarrow \exists p(\mathrm{cl}(p) \wedge \forall x(x \in p \vdash\!\dashv \exists y(<x, y> \in f)))$$
$$\mathrm{cl}(f), \mathrm{cl}(g) \rightarrow \exists p(\mathrm{cl}(p) \wedge \forall x(x \in p \vdash\!\dashv \exists y(<x, y> \in f \wedge y \in g)))$$
$$\mathrm{cl}(f) \rightarrow \exists p(\mathrm{cl}(p) \wedge \forall x(x \in p \vdash\!\dashv \exists y(<x, y> \in f \wedge y \in a)))$$
$$\mathrm{cl}(f) \rightarrow \exists p(\mathrm{cl}(p) \wedge \forall x(x \in p \vdash\!\dashv <x, a> \in f)) .$$

As in p. 215 of [3], we have

$$\mathrm{cl}(f), \forall x \forall y \forall z(<x, z> \in f \wedge <y, z> \in f \vdash x \equiv y)$$
$$\rightarrow \exists x \forall y(y \in x \vdash\!\dashv \exists z(z \in a \wedge <y, z> \in f)) .$$

Therefore, in order to construct a model of set theory, we have only to prove

$$\forall x \exists y \forall z(z \subseteq x \vdash z \in y)$$

which is proved in § 2.

§ 2. Proof of $\forall x \exists y \forall z(z \subseteq x \vdash z \in y)$.

To prove $\forall x \exists y \forall z(z \subseteq x \vdash z \in y)$ we have only to prove

$$\omega \leqq b, c \subseteq b, d = j(0, j(0, x(b), 0), 0) \rightarrow c \in d .$$

Therefore we assume in this section that $\omega \leqq b, c \subseteq b,$ and $d = j(0, j(0, \chi(b), 0), 0)$ hold.

First we define clos(f) as an abbreviation of

$$\forall u \forall x \forall y(u < 9 \wedge f(x) = 0 \wedge f(y) = 0 \vdash f(j(u, x, y)) = 0)$$
$$\wedge \forall x(f(x) = 0 \vdash f(C(x)) = 0 \wedge f(g_1(x)) = 0 \wedge f(g_2(x)) = 0)$$
$$\wedge \forall x(x < \omega \vdash f(x) = 0) .$$

In the same way as in the proof of 12.3 in the chapter VIII in [2] (pp. 54—61) we have first

(1) $$\forall x \forall y(x < a_0 \wedge y < a_0 \vdash (x < y \vdash\!\dashv G(x) < G(y))) ,$$
$$\forall x(x < a_0 \vdash f(G(x)) = 0) ,$$
$$\forall x \exists y(f(x) = 0 \vdash x = G(y) \wedge y < a_0) ,$$
$$\text{clos}(f), e < a_0, c < a_0 \rightarrow c \supseteq e \vdash\!\dashv G(c) \supseteq G(e) .$$

Now, we assume the following propositions (2) and (3), and prove $c \in d$.

(2) $$\exists x \forall y(f(y) = 0 \vdash y < x)$$
$$\rightarrow \exists p \exists u(\forall x \forall y(x < u \wedge y < u \vdash (x < y \vdash\!\dashv p(x) < p(p)))$$
$$\wedge \forall x(x < u \vdash f(p(x)) = 0)$$
$$\wedge \forall x \exists y(f(x) = 0 \vdash y < u \wedge x = p(y)))$$

(3) $$\omega \leqq b \rightarrow \exists p \exists q(\forall x(x < b \vdash p(x) = 0) \wedge \text{clos}(p) \wedge p(c) = 0$$
$$\wedge \forall x \exists y(p(x) = 0 \vdash x = q(y) \wedge y < b)) .$$

From 3) and the assumption in the beginning of this section, we may assume that there exist f and g satisfying

(4.1) $$\text{clos}(f)$$

(4.2) $$\forall x(x < b \vdash f(x) = 0) \wedge f(c) = 0$$

(4.3) $$\forall x \exists y(f(x) = 0 \vdash x = g(y) \wedge y < b)$$

Then by (4.3) and the axiom of upper bound we have an ordinal number a satisfying

(5) $$\forall x(f(x) = 0 \vdash x < a)$$

Therefore we see from 2) that there exist a_0 and G satisfying

(6.1) $$\forall x \forall y(x < a_0 \wedge y < a_0 \vdash (x < y \vdash\!\dashv G(x) < G(y)))$$

(6.2) $$\forall x(x < a_0 \vdash f(G(x)) = 0)$$

(6.3) $$\forall x \exists y(f(x) = 0 \vdash y < a_0 \wedge x = G(y)) .$$

From (6.1), (6.2), (6.3), (5) and (4.3) follows

(7) $$a_0 < \chi(b).$$

Let $\hat{c}$ be an ordinal number satisfying $\hat{c} < a_0$ and $G(\hat{c}) = c$. Then we have from 1)

$$\forall x(x < b \vdash (\hat{c} \ni x \mapsto c \ni x))$$

and so $c \equiv \hat{c} \cdot b.$

Since $\hat{c} \cdot b < d$, we have $\hat{c} \cdot b \in d$, whence $c \in d$ follows.

§ 3. Proof of 2) of § 2.

If $\forall x(f(x) > 0)$ holds, then the proposition is clear. Therefore we may assume that $\exists x(f(x) = 0)$ and $\forall x(f(x) = 0 \vdash x < a)$. Then G is defined by the following formula

$$G(0) = \mathrm{Min}(f)$$

$$\wedge \forall x(x > 0 \vdash G(x) = \mathrm{Min}(z)(f(z) = 0 \wedge \forall y(\neg \mathrm{Con}(G, x, y) = z))).$$

And G^{-1} is defined by the formula

$$\forall x(G^{-1}(x) = \mathrm{Min}(z)(G(z) = x)).$$

And b is defined by $\sup(G^{-1}, a)$. Then we see clearly that the following formulas hold.

$$\forall x \forall y(x < b \wedge y < b \vdash (x < y \mapsto G(x) < G(y)))$$

$$\forall x(x < b \vdash f(G(x)) = 0)$$

and $$\forall x \exists y(f(x) = 0 \vdash y < b \wedge x = G(y)).$$

Therefore the proposition is proved.

§ 4. Proof of 3) of § 2.

In this section we shall prove

$$\omega \leqq b \rightarrow \exists p \exists q(\forall x(x < b \vdash p(x) = 0) \wedge p(c) = 0 \wedge \mathrm{clos}(p)$$
$$\wedge \forall x \exists y(p(x) = 0 \vdash x = q(y) \wedge y < b)).$$

To the end, we define several functions.

$A_0(a, b)$ is defined by the following formula.

$$\forall x((g^1(K(x,b))=0\vdash A_0(x,b)=g^2(K(x,b)))$$
$$\wedge(g^1(K(x,b))=1\vdash A_0(x,b)=b+\tilde{K}(g^2(K(x,b)),b))$$
$$\wedge(g^1(K(x,b))>1\vdash A_0(x,b)=0)).$$

*We have

$$\omega\leqq b\rightarrow\forall x\exists y(x<b+\tilde{j}(0,b,0)\vdash y<b\wedge A_0(y,b)=x)$$

$B_0(a,b)$ is defined by the following formula

$$B_0(0,b)=0\wedge\forall x((x>0\wedge x\geqq\omega\vdash B_0(x,b)=0)$$
$$\wedge(x>0\wedge x<\omega\vdash B_0(x,b)=B_0(\delta(x),b)+b)).$$

Since $B_0(\delta(x),b)$ is equal to $\mathrm{Con}(\{y\}B_0(y,b),x,\delta(x))$, this definition is legitimate. Hereafter we use similar abbreviated definitions.

$B_0(b)$ is defined by $\sup(\{x\}B_0(x,b),\omega)$

*We have

$$\forall x(x<B_0(b)\vdash\exists y\exists z(y<\omega\wedge z<b\wedge B_0(y,b)+z=x))$$
$$n_1<\omega,a_1<b,n_2<\omega,a_2<b\rightarrow B_0(n_1,b)+a_1<B_0(n_2,b)+a_2$$
$$\mapsto(n_1<n_2)\vee(n_1=n_2\wedge a_1<a_2).$$

$C_0(a,b)$ is defined by $\mathrm{Min}(x)(B_0(x',b)>a)$.

*We have

$$\forall x(x<B_0(b)\vdash C_0(x,b)<\omega\wedge\exists y(y<b\wedge B_0(C_0(x,b),b)+y=x)).$$

We can define easily a function $A_1(n,a,b)$ satisfying

$$A_1(0,a,b)=a\wedge\forall x((0<x\wedge x<\omega\vdash A_1(x,a,b)$$
$$=A_0(A_1(\delta(x),a,b),\sup((y)A_1(\delta(x),y,b),b)))$$
$$\wedge(\omega\leqq x\vdash A_1(x,a,b)=0)).$$

In fact, if we rewrite this formula in using $\tilde{A}_1$ instead of A_1, such that $\tilde{A}_1(j(b+n),a)=A_1(n,a,b)$, we obtain a defining formula for $\tilde{A}_1$. The existence of the function $\tilde{A}_1$, and hence also of A_1 is then clear.

$A_2(a,b)$ is defined by $A_1(g^1(K(a,b)),g^2(K(a,b)),b)$.

$B_1(a,b)$ is defined by the following formula.

$$B_1(0,b)=b$$
$$\wedge\forall x((0<x\wedge x<\omega\vdash B_1(x,b)=B_1(\delta(x),b)+j(0,B_1(\delta(x),b),0))$$
$$\wedge(\omega\leqq x\vdash B_1(x,b)=0)).$$

$B_1(b)$ is defined by $\sup(\{x\}B_1(x, b), \omega)$

$$\omega \leqq b, a < B_1(b) \rightarrow \exists x(x < b \wedge a = A_3(x, b))$$

$\mathrm{Cp}(a, b)$ is defined by $\mathrm{Min}(x)(B_1(x, b) > a)$.

*We have

$$\forall x(x < B_1(b) \vdash \mathrm{Cp}(x, b) < \omega)$$

$$\forall x(\mathrm{Cp}(x, b) = 0 \wedge x < B_0(b) \vdash x < b)$$

$$\forall x(x < B_1(b) \wedge 0 < \mathrm{Cp}(x, b) \vdash$$
$$\exists y \exists z \exists u(u < 12 \wedge y < B_1(\delta(\mathrm{Cp}(x, b)), b) \wedge z < B(\delta(\mathrm{Cp}(x, b)), b)$$
$$\wedge x = B_1(\delta(\mathrm{Cp}(x, b)), b) + j(u, y, z))) .$$

$D(a, b)$ is defined by the following formula.

$$(\mathrm{Cp}(a, b) = 0 \vdash D(a, b) = 0)$$
$$\wedge (0 < \mathrm{Cp}(a, b) \vdash D(a, b) = a - B_1(\delta(\mathrm{Cp}(a, b)), b)) .$$

$E(a, b, c)$ is defined by the following formula.

$$E(0, b, c) = c$$
$$\wedge (a > 0 \wedge a < b \vdash E(a, b, c) = \delta_0(a))$$
$$\wedge (\mathrm{Cp}(a, b) > 0 \wedge B_1(b) > a \vdash (\tilde{g}_0(D(a, b)) < 9 \vdash$$
$$E(a, b, c) = j(\tilde{g}_0(D(a, b)), E(\tilde{g}_1(D(a, b)), b, c), E(\tilde{g}_2(D(a, b)), b, c)))$$
$$\wedge (\tilde{g}_0(D(a, b)) = 9 \vdash E(a, b, c) = C(E(\tilde{g}_1(D(a, b)), b, c))$$
$$\wedge (\tilde{g}_0(D(a, b)) = 10 \vdash E(a, b, c) = g_1(E(\tilde{g}_1(D(a, b)), b, c)))$$
$$\wedge (\tilde{g}_0(D(a, b)) = 11 \vdash E(a, b, c) = g_2(E(\tilde{g}_1(D(a, b)), b, c))))$$
$$\wedge (B_1(b) \leqq a \vdash E(a, b, c) = 0) .$$

$A(a, b, c)$ is defined by $E(A_2(a, b), b, c)$.

*The following sequences are provable.

$$\omega \leqq b \rightarrow \forall x \forall y \forall u \exists z(x < b \wedge y < b \wedge u < 9$$
$$\vdash z < b \wedge A(z, b, c) = j(u, A(x, b, c), A(y, b, c)))$$
$$\omega \leqq b \rightarrow \exists x(x < b \wedge A(x, b, c) = c)$$
$$\omega \leqq b \rightarrow \forall x \exists y(x < b \vdash y < b \wedge A(y, b, c) = x)$$
$$\omega \leqq b \rightarrow \forall x \exists y(x < b \vdash y < b \wedge A(y, b, c) = C(A(x, b, c)))$$
$$\omega \leqq b \rightarrow \forall x \exists y(x < b \vdash y < b \wedge A(y, b, c) = g_1(A(x, b, c)))$$
$$\omega \leqq b \rightarrow \forall x \exists y(x < b \vdash y < b \wedge A(y, b, c) = g_2(A(x, b, c))) .$$

$f(a)$ is defined by the following formula.

$$\forall x(f(x)=0 \mapsto \exists y(y<b \wedge A(y,b,c)=x)).$$

*We have

$$\omega \leqq b \rightarrow \mathrm{clos}(f) \wedge \forall x(x<b \vdash f(x)=0) \wedge f(c)=0.$$

Hence we see

$$\omega \leqq b \rightarrow \mathrm{clos}(f) \wedge \forall x(x<b \vdash f(x)=0) \wedge f(c)=0$$
$$\wedge \forall x \exists y(f(x)=0 \vdash A(y,b,c)=x \wedge y<b).$$

Therefore the proposition is proved.

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