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On the radial order of subharmonic functions.

F. W. GEHRING

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The purpose of this note is to show how the following theorem, due to Seidel and Walsh [2], can be deduced directly from an important maximal theorem of Hardy and Littlewood [1].

SEIDEL-WALSH THEOREM. *Suppose that $f(z)$ is analytic and univalent in $|z|<1$. Then, for almost all θ,*

$$f'(z)=o\{(1-|z|)^{-1/2}\}$$

uniformly as $z\rightarrow e^{i\theta}$ in each Stolz domain.

For $0<\alpha<\pi/2$ and $r>0$, let $S_{\alpha}(r, \theta)$ denote the open "tear drop" domain bounded by the two tangents, drawn from $re^{i\theta}$ to the circle $|z|=r\sin\alpha$, and the more distant part of the circle $|z|=r\sin\alpha$, between the points of contact. The Hardy-Littlewood theorem can be stated as follows.

HARDY-LITTLEWOOD THEOREM. *Suppose that $w(z)$ is non-negative and subharmonic in $|z|\leqq 1$, that $0<\alpha<\pi/2$, that $p>1$, and that*

$$W(\theta)=LUB\, w(z), \quad z\in S_{\alpha}(1, \theta).$$

Then

$$\int_{-\pi}^{\pi} W^{p}(\theta)d\theta\leqq C\int_{-\pi}^{\pi} w^{p}(e^{i\theta})d\theta,$$

where $C=C(\alpha, p)$ depends only on α and p.

We obtain the Seidel-Walsh theorem as a consequence of the following result.

THEOREM 1. *Suppose that $w(z)$ is non-negative and subharmonic in $|z|<1$, that $p>1$, and that*

$$\iint_{|z|<1} w^{p}(z)dxdy<\infty, \quad z=x+iy.$$

Then for almost all θ,

$$w(z)=o\{(1-|z|)^{-1/p}\}$$

uniformly as $z\rightarrow e^{i\theta}$ in each Stolz domain.

PROOF FOR THEOREM 1. It is sufficient to show, for each $0<\alpha<\pi/2$, that there exists a set $E=E(\alpha)$ of θ's with measure 2π such that, for θ in E,

$$(1-|z|)w^p(z)=o(1) \tag{1}$$

uniformly as $z\to e^{i\theta}$ in $S_\alpha(1,\theta)$.

Fix $0<\alpha<\pi/2$ and, for $1-\frac{1}{2}\cos\alpha=\delta<r<1$ and each θ, let

$$U(r,\theta)=LUB\,w(z)\,,$$

where the least upper bound is taken over all z subject to the restriction

$$z\in S_\alpha(1,\theta) \text{ and } |z-e^{i\theta}|\geqq 1-r\,. \tag*{2)}$$

Next pick $0<\beta<\pi/2$ and ρ so that

$$\tan\beta=2\tan\alpha, \quad (1-\rho)=\frac{1}{2}(1-r)\cos\alpha\,,$$

and let

$$W(\rho,\theta)=LUB\,w(z), \quad z\in S_\beta(\rho,\theta)\,.$$

Any z which satisfies condition 2) must lie in $S_\beta(\rho,\theta)$ and hence

$$U(r,\theta)\leqq W(\rho,\theta)$$

for all θ. From the Hardy-Littlewood theorem we obtain

$$\int_{-\pi}^{\pi}U^p(r,\theta)d\theta\leqq\int_{-\pi}^{\pi}W^p(\rho,\theta)d\theta\leqq C_1\int_{-\pi}^{\pi}w^p(\rho e^{i\theta})d\theta\,,$$

where $C_1=C(\beta,p)$, and integrating with respect to r we conclude that

$$\int_\delta^1\int_{-\pi}^{\pi}U^p(r,\theta)rdrd\theta\leqq C_2\int_0^1\int_{-\pi}^{\pi}w^p(\rho e^{i\theta})\rho d\rho d\theta<\infty\,,$$

where $C_2=2C_1/\cos\alpha$. From the Fubini theorem it follows that

$$\lim_{r\to1}\int_r^1U^p(r,\theta)rdr=0$$

for θ in $E=E(\alpha)$, a set with measure 2π. For each fixed θ, $U(r,\theta)$ is non-decreasing in r,

$$U^p(r,\theta)r(1-r)\leqq\int_r^1U^p(r,\theta)rdr\,,$$

and we conclude, for θ in E, that

$$\lim_{r\to 1}(1-r)U^p(r,\theta)=0\,. \tag{3}$$

Since 3) implies 1) the proof for Theorem 1 is complete.

PROOF FOR SEIDEL-WALSH THEOREM. A familiar argument [2] allows us to assume that the image of $|z|<1$ under $\zeta=f(z)$ has finite area or that

$$\iint_{|z|<1}|f'(z)|^2dxdy<\infty, \quad z=x+iy\,.$$

Set $w(z)=|f'(z)|$ and the desired conclusion follows from Theorem 1 with $p=2$.

The following result is a sharpened form of a theorem due to Tsuji [3].

THEOREM 2. *Suppose that $f(z)$ is analytic in $|z|<1$, that $p>0$, and that*

$$\iint_{|z|<1}|f(z)|^p dxdy<\infty, \quad z=x+iy\,.$$

Then, for almost all θ,

$$f(z)=o\{(1-|z|)^{-1/p}\}$$

uniformly as $z\to e^{i\theta}$ in each Stolz domain.

PROOF FOR THEOREM 2. Set $w(z)=|f(z)|^{p/2}$ and apply Theorem 1.

University of Michigan

References

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