Journal of the Mathematical Society of Japan Vol. 9, No. 1, January, 1957

On Skolem's theorem.

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(Received Nov. 10, 1956)

In 1922, Th. Skolem proved the following famous theorem: If there exists a model of any cardinal number for a system of axioms (satisfying certain conditions), then there exists also a countable model for the system. The aim of the present paper is to formulate and prove a corresponding theorem from the finite stand point. Our theorem reads as follows:

MAIN THEOREM. If A, B, C, D, E in Gödel [2] are consistent, then A, B, C, D, E and the following axioms are consistent.

$$\forall x \exists y (y \in \omega \wedge f_0(y) = x)$$
$$\forall x \forall y (x = y \vdash f_0(x) = f_0(y)),$$

where f_0 is a function, which is not contained in axioms A—E, and ω has the same meaning as in Gödel [2].

Our proof depends on results of our former paper [6], [7] which, in turn, is based on [8]. In [8], we have generalized LK (Logistischer klassischer Kalkül) of Gentzen [4] to GLC (Generalized logic calculus). Especially we shall make use of the "restriction theory" (§ 7) of [8]. In [6] we have treated in detail G^1LC, a specialization of GLC, and established the theorem: The fundamental conjecture holds for normal proof-figure. (Both terms: "fundamental conjecture" and "normal proof-figure" are defined in [6].) We shall now call $\widetilde{LK}$, a logical system obtained from G^1LC by restricting it as follows:

In every $\forall$ left on f-variable of the form

$$\frac{F(H), \Gamma \rightarrow \Delta}{\forall \varphi F(\varphi), \Gamma \rightarrow \Delta}$$

$F(\alpha)$ is not allowed to have any $\forall$ on f-variable. And the beginning sequence of $\widetilde{LK}$ is not allowed to have any logical symbol. We see that every proof-figure of $\widetilde{LK}$ is normal (in the sense defined in [6]).

Now we introduce two definitions in the system LK of Gentzen.

DEFINITION 1. A formula in *LK* is called ***normal***, if and only if it is of the form $\forall x_1 \cdots \forall x_i F(x_1, \cdots, x_i)$, where $F(x_1, \cdots, x_i)$ contains neither $\forall$ nor $\exists$.

DEFINITION 2. Let Γ_0 be a system of axioms in *LK*. We say that Γ_0 *satisfies the equality axioms* (with regard to $=$), if and only if the following condition is satisfied:

Let $A(a)$ be an arbitrary formula in *LK* and any function or any predicate contained in $A(a)$ be also contained in Γ_0. Then the following sequences are probable

$$\Gamma_0 \to a=a$$

and

$$\Gamma_0, a=b \to A(a) \vdash\dashv A(b)\,.$$

Now, we have the following theorem.

THEOREM 1. *Let Γ_0 be normal consistent axioms and satisfy the equality axioms. Moreover, let the following axioms be provable under Γ_0.*

1.1 $\forall x \forall y(e(x) \wedge e(y) \vdash x<y \vee x=y \vee y<x)$

1.2 $\forall x \forall y \forall z(e(x) \wedge e(y) \wedge e(z) \wedge x<y \wedge y<z \vdash x<z)$

1.3 $\forall x \forall y(e(x) \vdash \neg(x=y \wedge x<y))$

1.4 $\forall x \forall y(e(x) \wedge e(y) \wedge x<y \vdash x'<y \vee x'=y)$

1.5 $\forall x(e(x) \vdash x<x')$

1.6 $\forall x(e(x) \vdash 0<x \vee 0=x)$

1.7 $e(0)$

1.8 $\forall x(e(x) \vdash e(x'))$

1.9 $\forall x \forall y(e(x) \wedge e(y) \vdash e(x+y))$

1.10 $\forall x(e(x) \vdash x+0=x)$

1.11 $\forall x \forall y(e(x) \wedge e(y) \vdash x+y'=(x+y)')$

1.12 $\forall x \forall y(e(x) \wedge e(y) \vdash x+y=y+x)$

1.13 $\forall x \forall y \forall z(e(x) \wedge e(y) \wedge e(z) \vdash (x+y)+z=x+(y+z))$

1.14 $\forall x \forall y(e(x) \wedge e(y) \vdash (x<y \vdash\dashv \exists z(e(z) \wedge 0<z \wedge x+z=y)))$

1.15 $\forall x \forall y(e(x) \wedge e(y) \vdash e(x \cdot y))$

1.16 $\forall x(e(x) \vdash x \cdot 0'=x)$

1.17 $\forall x \forall y \forall z(e(x) \wedge e(y) \wedge e(z) \vdash (x+y) \cdot z=x \cdot z+y \cdot z)$

1.18 $\forall x \forall y(e(x) \wedge e(y) \vdash x \cdot y=y \cdot x)$

1.19 $\forall x \forall y \forall z(e(x) \wedge e(y) \wedge e(z) \vdash (x \cdot y) \cdot z=x \cdot (y \cdot z))$

1.20 $\forall x(e(x) \vdash e(g_1(x)) \wedge e(g_2(x)))$

1.21 $\forall x(e(x) \vdash j(g_1(x), g_2(x)) = x)$

1.22 $\forall x \forall y(e(x) \wedge e(y) \vdash g_1(j(x, y)) = x \wedge g_2(j(x, y)) = y)$

1.23 $\forall x(e(x) \vdash g_2(x) < x)$

1.24 $\forall x(e(x) \wedge 0' < x \vdash g_1(x) < x)$

1.25 $\forall x \forall y(e(x) \wedge e(y) \wedge y < x \vdash j(x, y) = x \cdot x + y)$

1.26 $\forall x \forall y(e(x) \wedge e(y) \wedge x \leqq y \vdash j(x, y) = y \cdot y + y + x)$

Then, Γ_0 and the following axioms are consistent.

$$\forall x \exists y(e(y) \wedge f_0(y) = x)$$

$$\forall x \forall y(x = y \vdash f_0(x) = f_0(y)),$$

where f_0 is not contained in Γ_0.

PROOF. By [7], we have only to prove that Γ_0 and the following axioms are consistent.

2.1 $\forall x \exists y(e(y) \wedge \mathrm{abz}(x, y))$

2.2 $\forall x \forall y \forall z(\mathrm{abz}(x, z) \wedge \mathrm{abz}(y, z) \vdash x = y)$

2.3 $\forall x \forall y \forall z(\mathrm{abz}(x, y) \wedge y = z \vdash \mathrm{abz}(x, z))$

2.4 $\forall x \forall y \forall z(\mathrm{abz}(y, x) \wedge y = z \vdash \mathrm{abz}(z, x))$,

where abz is not contained in Γ_0. ("abz" is taken from "abzählen". $\mathrm{abz}(x, y)$ will mean substantially, that "y-th element is x").

Now the above cited result of [6] assures that Γ_0 is consistent in $\widetilde{LK}$. By the restriction theory of [8], we see, moreover that Γ_0 and $\forall \varphi \forall x \forall y(x = y \vdash (\varphi[x] \vdash\dashv \varphi[y]))$ are consistent. Γ_0 and $\forall \varphi \forall x \forall y(x = y \vdash (\varphi[x] \vdash\dashv \varphi[y]))$ are shortly denoted by $\widetilde{\Gamma}_0$ and we use the abbreviated notation $\Gamma \to \Delta$ for $\widetilde{\Gamma}_0, \Gamma \to \Delta$, and $n(a)$ for $\forall \varphi(\varphi[0] \wedge \forall x(\varphi[x] \vdash \varphi[x']) \vdash \varphi[a])$. We have easily

$$\begin{aligned} &\to n(0) \\ n(a) &\to n(a') \\ n(a) &\to e(a) \\ e(a), a \leqq b, n(b) &\to n(a) \\ n(a), n(b) &\to n(a+b) \\ n(a), n(b) &\to n(a \cdot b) \\ n(a), n(b) &\to n(j(a, b)). \end{aligned}$$

Let us now assume first that Γ_0 consists of a finite number of axioms. Let all the special variables and all the functions in Γ_0 be $s_1, \cdots, s_m, f_1(*_1, \cdots, *_{i_1}), \cdots, f_n(*_1, \cdots, *_{i_n})$, and let k be $\max(i_1, \cdots, i_n)$. In utilizing g_1, g_2, j, we can now construct easily the functions $\tilde{g}_0(a), \cdots, \tilde{g}_k(a), \tilde{j}(a_0, a_1, \cdots, a_k)$ satisfying

$$e(a) \rightarrow e(\tilde{g}_0(a)) \wedge \cdots \wedge e(\tilde{g}_k(a))$$

$$e(a_0), \cdots, e(a_k) \rightarrow e(\tilde{j}(a_0, \cdots, a_k))$$

$$e(a) \rightarrow \tilde{j}(\tilde{g}_0(a), \cdots, \tilde{g}_k(a)) = a$$

$$e(a_0), \cdots, e(a_k) \rightarrow \tilde{g}_r(\tilde{j}(a_0, \cdots, a_k)) = a_r \ (r = 0, \cdots, k)$$

$$e(a) \rightarrow \tilde{g}_0(a) \leqq a \wedge \tilde{g}_1(a) < a \wedge \cdots \wedge \tilde{g}_k(a) < a$$

$$e(a), 0' < a \rightarrow \tilde{g}_0(a) < a$$

$$n(a_0), \cdots, n(a_k) \rightarrow n(\tilde{j}(a_0, \cdots, a_k)) .$$

Let $D(\alpha, b)$ be defined to be

$$\begin{aligned}
&\forall x(e(x) \wedge x \leqq b \vdash \\
&\quad (\tilde{g}_0(x) = 0 \vdash \forall y(\alpha[x, y] \vdash\dashv y = s_1)) \\
&\quad \wedge (\tilde{g}_0(x) = 1 \vdash \forall y(\alpha[x, y] \vdash\dashv y = s_1)) \\
&\quad \cdots\cdots\cdots\cdots \\
&\quad \wedge (\tilde{g}_0(x) = m \vdash \forall y(\alpha[x, y] \vdash\dashv y = s_m)) \\
&\quad \wedge (\tilde{g}_0(x) = m+1 \vdash \forall y(\alpha[x, y] \vdash\dashv \\
&\qquad \exists z_1 \cdots \exists z_{i_1}(\alpha[\tilde{g}_1(x), z_1] \wedge \cdots \wedge \alpha[\tilde{g}_{i_1}(x), z_{i_1}] \wedge y = f_1(z_1, \cdots, z_{i_1})))) \\
&\quad \cdots\cdots\cdots\cdots \\
&\quad \wedge (g_0(x) = m+n \vdash \forall y(\alpha[x, y] \vdash\dashv \\
&\qquad \exists z_1 \cdots \exists z_{i_n}(\alpha[\tilde{g}_1(x), z_1] \wedge \cdots \wedge \alpha[\tilde{g}_{i_n}(x), z_{i_n}] \wedge y = f_n(z_1, \cdots, z_{i_n})))) \\
&\quad \wedge (g_0(x) > m+n \vdash \forall y(\alpha[x, y] \vdash\dashv y = s_1))) .
\end{aligned}$$

We see easily

$$D(\alpha, b), D(\beta, c), n(b), n(c), d \leqq b, d \leqq c, e(d) \rightarrow \forall y(\alpha[d, y] \vdash\dashv \beta[d, y])$$

$$D(\alpha, b), n(b), d \leqq b, e(d) \rightarrow \exists y(\alpha[d, y]) \wedge \forall y \forall z(\alpha[d, y] \wedge \alpha[d, z] \vdash y = z) .$$

Moreover, we have easily

$$n(b) \rightarrow \exists \varphi D(\varphi, b).$$

abz(c, b) is now defined to be $n(b) \wedge \exists \varphi (D(\varphi, b) \wedge \varphi[b, c])$, and $cl(c)$ to be $\exists x\, \mathrm{abz}(c, x)$. We see then

$$\mathrm{abz}(c, b) \rightarrow n(b)$$

$$\mathrm{abz}(c, b) \rightarrow e(b)$$

$$\mathrm{abz}(c, b), \mathrm{abz}(d, b) \rightarrow c = d$$

$$n(b) \rightarrow \exists x\, \mathrm{abz}(x, b)$$

$$\rightarrow cl(s_1)$$

....

$$\rightarrow cl(s_m)$$

$$\mathrm{abz}(a_1, b_1), \cdots, \mathrm{abz}(a_{i_r}, b_{i_r}) \rightarrow \mathrm{abz}(f_r(a_1, \cdots, a_{i_r}), \tilde{j}(m+r, b_1, \cdots, b_{i_r})).$$

Hence, we have

$$cl(a_1), \cdots, cl(a_{i_r}) \rightarrow cl(f_r(a_1, \cdots, a_{i_r})) \quad (r = 1, \cdots, n)$$

Since $*'$ is $f_r(*_1, \cdots, *_{i_r})$ for suitable $r(1, \cdots, n)$, we see easily

$$n(a) \rightarrow cl(a).$$

Moreover, we have

$$\forall x(cl(x) \vdash \exists y(e(y) \wedge cl(y) \wedge \mathrm{abz}(x, y)))$$

$$\forall x \forall y \forall z(\mathrm{abz}(y, x) \wedge \mathrm{abz}(z, x) \vdash y = z)$$

$$\forall x \forall y \forall z(x = y \wedge \mathrm{abz}(x, z) \vdash \mathrm{abz}(y, z))$$

$$\forall x \forall y \forall z(x = y \wedge \mathrm{abz}(z, x) \vdash \mathrm{abz}(z, y)).$$

Therefore, by the restriction theory of [8], we have the desired result.

In case Γ_0 contains an infinite number of axioms, suppose that Γ_0 and 2.1—2.4 are not consistent. Then there exists a finite subsystem Γ_0^1 of Γ_0 such that Γ_0^1 and 2.1—2.4 are not consistent, in contradiction to what was proved.

By Hilbert-Bernays [3] or Maehara [5] and by theorem 1, we have the following theorem.

THEOREM 2. *Let Γ_0 be consistent axioms satisfying the equality axioms under which* 1.1—1.26 *are provable. Then,* Γ_0 and *the following axioms are consistent*

$$\forall x \exists y (e(y) \wedge f_0(y) = x)$$

$$\forall x \forall y (x = y \vdash f_0(x) = f_0(y)),$$

where f_0 is not contained in Γ_0.

Our Main Theorem is obviously a special case of this theorem.

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