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The local and global covariant variations of differential forms under an infinitesimal conformal transformation.

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The problems of variation in the potential theory are of great interest both in pure and applied mathematics. (Courant and Hilbert [1], Bergman and Schiffer [2]) As far as the author knows, it is scarce to construct an invariant theoretical method for variation of differential form, when the metric of a Riemannian space is changed. In short, it is a sort of variation problem of elliptic differential equation, when its coefficients are changed.

The purpose of this paper is to make a beginning for such a problem stated above, and the results obtained are quite formal as yet. We shall confine ourselves only to conformal variation of the metric of a Riemannian space.

Two kinds of variations, δ_l and δ_g, are defined for an infinitesimal conformal transformation of a domain with boundary. If either δ_l or δ_g of a form is zero under the transformation, the norm of the differential form is constant. In virture of this property, we shall call δ_l and δ_g the covariant variations, after the covariant differentiation in Riemannian space. They induce norm preserving transformations in the Hilbert space of differential forms, if their covariant variations are zero.

Although the *local* covariant variation δ_l of a differential form at a point depends only on the variation of the metric at that point, the *global* one δ_g depends on the variation of the metric in the whole domain. Moreover, contrary to the local covariant variation, the global one has a special property that *if a differential form remains harmonic* (*exact*, *derived*) *during the transformation, the global covariant variation of the form is harmonic* (*exact*, *derived*) *also.*

Since the domain under consideration has its boundary, the terms, harmonic, exact and derived, are understood in relative sense.

The bases of our discussions are the well-known theorems of harmonic forms (theorems 1, 2, 3 and 4 in § 0). Theorem 1 is the existence theorem of harmonic forms. (de Rham and Bidal [4], [5]; Duff and Sencer [6]) Theorem 2 is the decomposition theorem of forms. (Conner [3]) Theorems 3 and 4 are the characterization of decomposed parts and it is an immediate consequence of theorems 1 and 2.

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Some unusual notations are used in this paper to denote the covariant variations with δ, hence we shall compare the notations:

notations in this paper	usual notations
$\triangle$	d
∇	$-\delta$
$\square$	$-\triangle$

§ 0. Preliminaries.

This paragraph is devoted to explanations of notations and some well-known results and theorems due to P. E. Conner [3], which are somewhat modified for our use.

Let $\boldsymbol{M}$ be an n-dimensional orientable Riemannian space of C^∞ with the line element $ds^2=g_{ij}dx^i dx^j$, $\boldsymbol{D}$ be a bounded domain in $\boldsymbol{M}$ with boundary $\boldsymbol{B}$, which is the union of a finite number of disjoint closed manifolds of class C^∞, i. e., we assume that $\boldsymbol{D}$ is a finite manifold.

Let α_p and β_q denote p-form $A_{i_1\cdots i_p}dx^{i_1}\cdots dx^{i_p}$ and q-form $B_{i_1\cdots i_q}dx^{i_1}\cdots dx^{i_q}$ defined on the neighbourhood of $\boldsymbol{B}+\boldsymbol{D}$. (All functions are assumed to be of class C^∞ in the following.)

We shall abopt the following notations as usual

$$*\alpha=\frac{1}{p!}A_{i_1\cdots i_p}\varepsilon^{i_1\cdots i_p}{}_{i_{p+1}\cdots i_n}dx^{i_{p+1}}\cdots dx^{i_n}\,, \tag{0.1.1}$$

$$\alpha\wedge\beta=\frac{1}{p!\,q!}A_{k_1\cdots k_p}B_{l_1\cdots l_q}\delta^{k_1\cdots k_p l_1\cdots l_q}_{i_1\cdots i_p j_1\cdots j_q}dx^{i_1}\cdots dx^{i_p}dx^{j_1}\cdots dx^{j_q}\,. \tag{0.1.2}$$

Now we shall define the multiplication $\vee$ of two forms α and β by the following formula

(0.2) $$*(\alpha\wedge\beta)=*\beta\vee*\alpha\,.$$

Let D_i denote the covariant differential operators and let N_i be the components of the unit tangential covariant vector to $\boldsymbol{B}$ and N^i those of the outwards normal contravariant vector to $\boldsymbol{B}$.

Put

(0.3.1) $$D=dx^iD_i\,,$$

(0.3.2) $$\triangle\alpha=D\wedge\alpha\,,$$

(0.3.3) $$\nabla\alpha=*D\vee\alpha\,,$$

(0.3.4) $$\nu=N_idx^i\,,$$

(0.3.5) $$\perp\alpha=\nu\wedge\alpha\,,$$

(0.3.6) $$\top\alpha=*\nu\vee\alpha\,.$$

$\triangle\alpha$ and $\nabla\alpha$ are usually denoted by $d\alpha$ and $-\delta\alpha$ respectively.

From (0.1), (0.2) and (0.3) we get the following identities at once

(0.4.1) $$*\nabla\alpha=(-1)^{p+1}\triangle*\alpha,\qquad \nabla*\alpha=(-1)^p*\triangle\alpha\,,$$

(0.4.2) $$*\top\alpha=(-1)^{p+1}\perp*\alpha,\qquad \top*\alpha=(-1)^p*\perp\alpha\,,$$

(0.4.3) $$\triangle(\alpha\wedge\beta)=\triangle\alpha\wedge\beta+(-1)^p\alpha\wedge\triangle\beta\,,$$

(0.4.4) $$\nabla(\alpha\vee\beta)=\nabla\alpha\vee\beta+(-1)^{n-p}\alpha\vee\nabla\beta\,,$$

(0.4.5) $$\alpha=\perp\top\alpha+\top\perp\alpha\,,$$

(0.4.6) $$\perp\perp\alpha=\top\top\alpha=0\,,$$

(0.4.7) $$\perp\top\perp\alpha=\perp\alpha,\qquad \top\perp\top\alpha=\top\alpha\,.$$

$\perp\top\alpha$ and $\top\perp\alpha$ are usually called the normal and tangential parts of α respectively. The operators $\perp$ and $\top$ have invariant meaning only on the boundary $\boldsymbol{B}$. Let α and β be p-forms. It is easy to prove that if $\alpha=\beta$ (on $\boldsymbol{B}$), i. e., $\alpha=\beta$ for all dx tangential to $\boldsymbol{B}$, then $\perp\alpha=\perp\beta$ for all dx, and conversely. Hence $\perp\alpha=\perp\beta$ is equivalent to $\alpha=\beta$ (on $\boldsymbol{B}$). Dually $\top\alpha=\top\beta$ is equivalent to $*\alpha=*\beta$ (on $\boldsymbol{B}$).

Since if $\alpha=0$ (on $\boldsymbol{B}$) then $\triangle\alpha=0$ (on $\boldsymbol{B}$), it follows that

(0.5.1) $$\text{if}\quad \perp\alpha=0,\quad \text{then}\quad \perp\triangle\alpha=0\,.$$

Dually

(0.5.2) $$\text{if}\quad \top\alpha=0,\quad \text{then}\quad \top\nabla\alpha=0\,.$$

Put

$$\int_D \alpha \wedge *\beta = (\alpha \cdot \beta), \tag{0.6.1}$$

where α and β are p-forms.

Put

$$\oint_B \alpha \wedge *\beta = \{\alpha \cdot \beta\},$$

where α and β are p-form and $(p+1)$-form respectively.

We have well-known formulae in our notations

$$\{\alpha \cdot \beta\} = (\triangle\alpha \cdot \beta) + (\alpha \cdot \nabla\beta), \tag{0.7.1}$$

$$\{\nabla\alpha \cdot \beta\} = (\triangle\nabla\alpha \cdot \beta) + (\nabla\alpha \cdot \nabla\beta), \tag{0.7.2}$$

$$\{\alpha \cdot \triangle\beta\} = (\triangle\alpha \cdot \triangle\beta) + (\alpha \cdot \nabla\triangle\beta), \tag{0.7.3}$$

$$\{\nabla\alpha \cdot \beta\} + \{\beta \cdot \triangle\alpha\} = (\square\alpha \cdot \beta) + (\triangle\alpha \cdot \triangle\beta) + (\nabla\alpha \cdot \nabla\beta), \tag{0.7.4}$$

$$\{\nabla\alpha \cdot \alpha\} + \{\alpha \cdot \triangle\alpha\} = (\square\alpha \cdot \alpha) + (\triangle\alpha \cdot \triangle\alpha) + (\nabla\alpha \cdot \nabla\alpha), \tag{0.7.5}$$

where

$$\square\alpha = (\triangle\nabla + \nabla\triangle)\alpha. \tag{0.7.6}$$

$\square\alpha$ is usually denoted by $-\triangle\alpha$.

If $\perp\alpha = 0$ or $\top\beta = 0$, then $\{\alpha \cdot \beta\} = 0$; and if $\{\alpha \cdot \beta\} = 0$ for all β, then $\perp\alpha = 0$; and if $\{\alpha \cdot \beta\} = 0$ for all α, then $\top\beta = 0$ by (0.6.2) and (0.4.2).

We shall call

a form α such that $\triangle\alpha = 0$ ($\nabla\alpha = 0$) absolutely exact or closed (co-exact or co-closed),

a form α such that $\alpha = \triangle\beta$ ($\alpha = \nabla\beta$) abs. derived or homologous to zero (co-derived or co-homologous to zero),

a form α such that $\perp\alpha = 0$ ($\top\alpha = 0$) relative (co-relative),

a form α such that $\perp\alpha = \triangle\alpha = 0$ ($\top\alpha = \nabla\alpha = 0$) rel. exact (rel. co-exact),

a form α such that $\alpha = \triangle\beta$, $\perp\beta = 0$ ($\alpha = \nabla\beta$, $\top\beta = 0$) rel. derived (rel. co-derived),

From (0.5.1) and (0.5.2) it follows that if α is rel. derived (co-derived) then α is rel. (co-rel.).

Let us denote respectively

set of all p-forms by $\boldsymbol{C}_a^p$,

set of all rel. (co-rel.) p-forms by $\boldsymbol{C}_r^p(\overline{\boldsymbol{C}}_r^p)$,

set of all abs. exact (co-exact) p-forms by $\boldsymbol{Z}_a^p(\overline{\boldsymbol{Z}}_a^p)$,

set of all abs. derived (co-derived) p-forms by $\boldsymbol{B}_a^p(\overline{\boldsymbol{B}}_a^p)$,
set of all rel. exact (co-exact) p-forms by $\boldsymbol{Z}_r^p(\overline{\boldsymbol{Z}}_r^p)$,
set of all rel. derived (co-derived) p-forms by $\boldsymbol{B}_r^p(\overline{\boldsymbol{B}}_r^p)$.

Put

$$\boldsymbol{Z}_a^p \cap \overline{\boldsymbol{Z}}_r^p = \boldsymbol{H}_a^p, \qquad \overline{\boldsymbol{Z}}_r^p \cap \boldsymbol{Z}_a^p = \overline{\boldsymbol{H}}_r^p,$$

$$\boldsymbol{Z}_r^p \cap \overline{\boldsymbol{Z}}_a^p = \boldsymbol{H}_r^p, \qquad \overline{\boldsymbol{Z}}_a^p \cap \boldsymbol{Z}_r^p = \overline{\boldsymbol{H}}_a^p.$$

Evidently $\boldsymbol{H}_a^p = \overline{\boldsymbol{H}}_r^p$, $\boldsymbol{H}_r^p = \overline{\boldsymbol{H}}_a^p$, and $\boldsymbol{H}_a^p \approx \overline{\boldsymbol{H}}_a^{n-p} \approx \boldsymbol{H}_r^{n-p}$ as vector spaces.

A p-form α in $\boldsymbol{H}_r^p = \overline{\boldsymbol{H}}_a^p$ may be called a rel. harmonic or abs. co-harmonic form, it is characterized by

$$\triangle\alpha = 0, \quad \perp\alpha = 0, \quad \nabla\alpha = 0. \tag{0.8}$$

A p-form α in $\boldsymbol{H}_a^p = \overline{\boldsymbol{H}}_r^p$ may be called an abs. harmonic or rel. co-harmonic form, it is characterized by

$$\nabla\alpha = 0, \quad \top\alpha = 0, \quad \triangle\alpha = 0, \tag{0.8$'$}$$

If the domain $\boldsymbol{D}$ is compact, i. e., the boundary is empty, the above definitions of harmonic forms coincide with the ordinary definition of harmonic forms.

By (0.7.5), *the equations* (0.8) *and* (0.8)$'$ *are respectively equivalent to the following equations*

$$\square\alpha = 0, \quad \perp\alpha = 0, \quad \perp\nabla\alpha = 0; \tag{0.9}$$

$$\square\alpha = 0, \quad \top\alpha = 0, \quad \top\triangle\alpha = 0. \tag{0.9$'$}$$

The following theorems are quite analogous to the theorems which hold when the domain is compact. The theorems are proved by P. E. Conner [3].

They may be also proved by the method of parametrics used by Bidal and de Rham [4], [5] with some modifications under the conditions that the domain $\boldsymbol{D}$ is bounded, i. e., $\boldsymbol{B}+\boldsymbol{D}$ is compact.

THEOREM 1. *The equations*

$$\square\beta = \alpha, \quad \perp\beta = 0, \quad \perp\nabla\beta = 0 \tag{0.10}$$

are consistent, if and only if α *is orthogonal to the space* $\boldsymbol{H}_r$, *i. e.,* $(\alpha\cdot\gamma) = 0$ *for all* γ *such that* $\square\gamma = \perp\gamma = \perp\nabla\gamma = 0$.

The equations

$$\square\beta = \alpha, \quad \top\beta = 0, \quad \top\triangle\beta = 0 \tag{0.10$'$}$$

are consistent, if and only if α *is orthogonal to the space* $\boldsymbol{H}_a$.

THEOREM 2.

$$C_a = B_r \oplus H_r \oplus \overline{B}_a, \tag{0.11}$$

$$C_a = B_a \oplus H_a \oplus \overline{B}_r. \tag{0.11$'$}$$

We shall only deal with the decomposition (0.11) in the following, since (0.11)$'$ is dual to (0.11). Let $H\alpha$, $K\alpha$ and $L\alpha$ be the orthogonal projections of α into $\boldsymbol{H}_r$, $\boldsymbol{B}_r$ and $\overline{\boldsymbol{B}}_a$ in turn. Analogously to the theorems where the domain is compact, we see that the equations

$$\square\beta = (E-H)\alpha, \tag{0.12.1}$$

$$\perp\beta = \perp\triangledown\beta = 0, \tag{0.12.2}$$

$$H\beta = 0 \tag{0.12.3}$$

have *the unique solution* β *for an arbitrarily given* α, where E is the unit operator. β is given by

$$\beta = G\alpha \tag{0.13}$$

or

$$\beta(x) = (G(x\cdot y)\cdot\alpha(y))$$

where $G(x\cdot y)$ is Green's form.

From (0.12.2) and (0.13) we have

$$\perp G\alpha = \perp\triangledown G\alpha = 0. \tag{0.14}$$

From (0.12), (0.13) and (0.14) we have also

$$\alpha = \triangle\triangledown G\alpha + H\alpha + \triangledown\triangle G\alpha, \tag{0.15.1}$$

$$K = \triangle\triangledown G, \tag{0.15.2}$$

$$L = \triangledown\triangle G, \tag{0.15.3}$$

$$E = H + K + L, \tag{0.15.4}$$

$$\square G = (E-H). \tag{0.15.5}$$

It is easy to prove that

$$\triangle G\alpha = G\triangle\alpha, \text{ if } \perp\alpha = 0, \tag{0.16.1}$$

$$\triangledown G\alpha = G\triangledown\alpha, \tag{0.16.2}$$

$$\square G\alpha = G\square\alpha, \text{ if } \perp\alpha = \perp\triangledown\alpha = 0. \tag{0.16.3}$$

We also have the following identities by the definitions of the operators H, K and L:

$$K\triangle\alpha = \triangle\alpha, \text{ if } \perp\alpha = 0, \tag{0.17.1}$$

$$\triangle K\alpha = 0, \tag{0.17.2}$$

$$K\nabla\alpha = 0, \tag{0.17.3}$$

$$L\nabla\alpha = \nabla\alpha, \tag{0.17.4}$$

$$\nabla L\alpha = 0, \tag{0.17.5}$$

$$L\triangle\alpha = 0, \text{ if } \perp\alpha = 0, \tag{0.17.6}$$

$$\triangle H\alpha = 0, \tag{0.17.7}$$

$$\nabla H\alpha = 0, \tag{0.17.8}$$

$$H\triangle\alpha = 0, \text{ if } \perp\alpha = 0, \tag{0.17.9}$$

$$H\nabla\alpha = 0, \tag{0.17.10}$$

$$\Box H\alpha = 0, \tag{0.17.11}$$

$$H\Box\alpha = 0, \text{ if } \perp\nabla\alpha = 0, \tag{0.17.12}$$

$$\perp K\alpha = 0, \tag{0.17.13}$$

$$\perp H\alpha = 0, \tag{0.17.14}$$

$$\perp L\alpha = \perp\alpha. \tag{0.17.15}$$

The last is the consequence of (0.15.4).

THEOREM 3. *$L\alpha$ is characterised by the equations*

$$L\alpha = \nabla\xi, \tag{0.18.1}$$

$$\triangle\nabla\xi = \triangle\alpha, \tag{0.18.2}$$

$$\perp\nabla\xi = \perp\alpha. \tag{0.18.3}$$

(That is, for arbitrarily given α there exists a ξ satisfying (0.18.2) and (0.18.3); $\nabla\xi$ is unique and it is equal to $L\alpha$.)

PROOF. The equations $\triangle\nabla\xi = \perp\nabla\xi = 0$ have the unique solution $\nabla\xi = 0$ by (0.7.2), hence if (0.18.2) and (0.18.3) are consistent, $\nabla\xi$ is unique. Put $\xi = \triangle G\alpha$.

$$\begin{aligned} \triangle\nabla\xi &= \triangle\nabla\triangle G\alpha \\ &= \triangle\Box G\alpha \\ &= \triangle(E-H)\alpha \\ &= \triangle\alpha \end{aligned}$$

by (0.7.6), (0.15.5) and (0.17.7).

$$\perp\nabla\xi=\perp\nabla\triangle G\alpha=\perp L\alpha=\perp\alpha$$

by (0.15.3) and (0.17.15).

$$\nabla\xi=\nabla\triangle G\alpha=L\alpha\,.$$

Hence the theorem is proved.

From (0.18.2), we have at once

COROLLARY

(0.19) $$\triangle L\alpha=\triangle\alpha\,.$$

We have analogously the following theorem:

THEOREM 4. *$K\alpha$ is characterised by the equations*

(0.18.1)′ $$K\alpha=\triangle\xi\,,$$

(0.18.2)′ $$\nabla\triangle\xi=\nabla\alpha\,,$$

(0.18.3)′ $$\perp\xi=0\,.$$

COROLLARY

(0.19)′ $$\nabla K\alpha=\nabla\alpha\,.$$

§ 1. Local covariant variation under an infinitesimal conformal transformation.

Let

(1.1) $$vg_{ij}=2\mu g_{ij}$$

be an infinitesimal conformal transformation of the metric tensor g_{ij}, where v denotes the variation and μ is a function of C^∞. Tensor or differential form with certain properties, e. g., harmonic form, may change under the transformation, and its variation is denoted by v. (The variation of dx are zero.)

We have at once

(1.2.1) $$vg^{ij}=-2\mu g^{ij}\,,$$

(1.2.2) $$vN_i=\mu N_i\,,$$

(1.2.3) $$v\nu=\mu\nu\,,$$

(1.2.4) $$v\perp\alpha=\perp v\alpha+\mu\perp\alpha,\quad v\top\alpha=\top v\alpha-\mu\top\alpha\,,$$

(1.2.5) $$v\triangle\alpha_p=\triangle v\alpha_p\,,$$

(1.2.6) $$v*\alpha_p = *[v\alpha_p + (n-2p)\alpha_p],$$

(1.2.7) $$v\nabla\alpha_p = \nabla v\alpha_p + (n-2p)\nabla\mu\alpha_p - (n-2p+2)\mu\nabla\alpha_p.$$

We shall define an operator δ_l by the formula

(1.3) $$\delta_l\alpha_p = v\alpha_p + M\alpha_p,$$

where M is an operator defined by

(1.4) $$M\alpha_p = \left(\frac{n}{2} - p\right)\mu\alpha_p.$$

We shall call $\delta_l\alpha$ the local covariant variation of a form α, and we shall denote δ_l by $'$ for simplicity.

Making use of (1.2), we have by direct calculations

(1.5.1) $$(*\alpha)' = *\alpha',$$

(1.5.2) $$(\perp\alpha)' = \perp\alpha', \quad (\top\alpha)' = \top\alpha',$$

(1.5.3) $$(\triangle\alpha)' = \triangle\alpha' + M\triangle\alpha - \triangle M\alpha,$$

(1.5.4) $$(\nabla\alpha)' = \nabla\alpha' - M\nabla\alpha + \nabla M\alpha,$$

(1.5.5) $$v(\alpha\cdot\beta) = (\alpha'\cdot\beta) + (\alpha\cdot\beta'),$$

(1.5.6) $$\text{if } \perp\alpha = 0, \text{ then } \perp M\alpha = 0,$$

(1.5.7) $$\text{if } \top\alpha = 0, \text{ then } \top M\alpha = 0.$$

Let us define operators $\delta_l\triangle = \triangle'$ and $\delta_l\nabla = \nabla'$ by

(1.6.1) $$(\delta_l\triangle)\alpha = \triangle'\alpha = (\triangle\alpha)' - \triangle\alpha', \quad \text{i. e. } (\triangle\alpha)' = \triangle'\alpha + \triangle\alpha',$$

(1.6.2) $$(\delta_l\nabla)\alpha = \nabla'\alpha = (\nabla\alpha)' - \nabla\alpha', \quad \text{i. e. } (\nabla\alpha)' = \nabla'\alpha + \nabla\alpha'$$

respectively. We shall call $\delta_l\triangle$ and $\delta_l\nabla$ the local covariant variations of the operators $\triangle$ and ∇ respectively.

It follows that

(1.7.1) $$\triangle'\alpha = M\triangle\alpha - \triangle M\alpha,$$

(1.7.2) $$\nabla'\alpha = -M\nabla\alpha + \nabla M\alpha,$$

by (1.5.3), (1.5.4), (1.6.1) and (1.6.2).

Following theorem 3 L_α is characterized by the equations

(1.8) $$L\alpha = \nabla\xi,$$

(1.9) $$\triangle\nabla\xi = \triangle\alpha,$$

(1.10) $$\perp\nabla\xi = \perp\alpha.$$

Operating δ_l on (1.9) we have

$$\begin{aligned}
&\triangle'\nabla\xi+\triangle\nabla'\xi+\triangle\nabla\xi'=\triangle'\alpha+\triangle\alpha' ,\\
&\triangle\nabla\xi'=\triangle'(\alpha-\nabla\xi)+\triangle(\alpha'-\nabla'\xi)\\
&\qquad=(M\triangle-\triangle M)(\alpha-\nabla\xi)+\triangle(\alpha'-\nabla'\xi)\\
&\qquad=M\triangle(\alpha-\nabla\xi)-\triangle M(\alpha-\nabla\xi)+\triangle(\alpha'-\nabla'\xi)\\
&\qquad=-\triangle M(\alpha-\nabla\xi)+\triangle(\alpha'-\nabla'\xi) ;
\end{aligned}$$

$$\triangle\nabla\xi'=\triangle[-M(\alpha-\nabla\xi)+(\alpha'-\nabla'\xi)] , \tag{1.11}$$

by virtue of (1.6.1), (1.7.1) and (1.9).

Operating δ_l on (1.10), we have from (1.5.2) and (1.6.2)

$$\perp\nabla'\xi+\perp\nabla\xi'=\perp\alpha' ,$$

$$\perp\nabla\xi'=\perp(\alpha'-\nabla'\xi) . \tag{1.12}$$

Since $\perp(\nabla\xi-\alpha)=0$ by (1.10), we have $\perp M(\alpha-\nabla\xi)=0$ by (1.5.6). Hence (1.12) is equal to

$$\perp\nabla\xi'=\perp[-M(\alpha-\nabla\xi)+(\alpha'-\nabla'\xi)] . \tag{1.13}$$

(1.11) and (1.13) mean that

$$\nabla\xi'=L[-M(\alpha-\nabla\xi)+(\alpha'-\nabla'\xi)] \tag{1.14}$$

according to the theorem 3.

Let us write

$$(\delta_l L)\alpha=L'\alpha=(L\alpha)'-L\alpha' , \tag{1.15}$$

then we get

$$\begin{aligned}
&L'\alpha+L\alpha'=\nabla'\xi+\nabla\xi' ,\\
&L'\alpha=\nabla'\xi+\nabla\xi'-L\alpha'\\
&\qquad=L[-M(\alpha-\nabla\xi)+(\alpha'-\nabla'\xi)]+\nabla'\xi-L\alpha'\\
&\qquad=(E-L)\nabla'\xi-LM(\alpha-\nabla\xi)\\
&\qquad=(E-L)(-M\nabla+\nabla M)\xi-LM(\alpha-\nabla\xi)\\
&\qquad=-(E-L)M\nabla\xi-LM(\alpha-\nabla\xi)\\
&\qquad=-(E-L)ML\alpha-LM(\alpha-L\alpha) ;
\end{aligned}$$

$$L'\alpha=(-ML+2LML-LM)\alpha , \tag{1.16}$$

by (1.14), (1.7.2), (0.17.4) and (1.8).

According to theorem 4 $K\alpha$ is characterized by the equations

(1.17) $K\alpha = \triangle\xi$,

(1.18) $\nabla\triangle\xi = \nabla\alpha$,

(1.19) $\perp\xi = 0$.

Similarly to the above, we get

$$\nabla'\triangle\xi + \nabla\triangle'\xi + \nabla\triangle\xi' = \nabla'\alpha + \nabla\alpha' ,$$

$$\nabla\triangle\xi' = \nabla'(\alpha - \triangle\xi) + \nabla(\alpha' - \triangle'\xi)$$

$$= (-M\nabla + \nabla M)(\alpha - \triangle\xi) + \nabla(\alpha' - \triangle'\xi) ,$$

(1.20) $\nabla\triangle\xi' = \nabla[M(\alpha - \triangle\xi) + (\alpha' - \triangle'\xi)]$,

by (1.6.2) and (1.18);

(1.21) $\perp\xi' = 0$

by (1.19) and (1.5.2).
(1.20) and (1.21) mean that

(1.22) $\triangle\xi' = K[M(\alpha - \triangle\xi) + (\alpha' - \triangle'\xi)]$

according to the theorem 4.

Put

(1.23) $(\delta_l K)\alpha = K'\alpha = (K\alpha)' - K\alpha'$,

and we get

$$K'\alpha + K\alpha' = \triangle'\xi + \triangle\xi'$$

$$K'\alpha = \triangle'\xi + \triangle\xi' - K\alpha'$$

$$= \triangle'\xi + K[M(\alpha - \triangle\xi) + (\alpha' - \triangle'\xi)] - K\alpha'$$

$$= (E - K)(M\triangle - \triangle M)\xi + KM(\alpha - \triangle\xi) ;$$

(1.24) $K'\alpha = (E - K)(M\triangle - \triangle M)\xi + KM(\alpha - \triangle\xi)$.

Since $\perp\xi = \perp M\xi = 0$ by (1.5.6), it follows that $K\triangle M\xi = \triangle M\xi$ from (0.17.1). Hence, from (1.24),

$$K'\alpha = (E - K)MK\alpha + KM(\alpha - K\alpha) ,$$

(1.25) $K'\alpha = (MK - 2KMK + KM)\alpha$.

Since $H + K + L = E$, we have

(1.26.1) $H'\alpha = -(K' + L')\alpha = (ML - 2LML + LM - MK + 2KMK - KM)\alpha$, where

$$(1.26.2)\qquad H'\alpha=(H\alpha)'-H\alpha'.$$

THEOREM 5. *The local covariant variations of the operators H, K and L are given by* (1.26), (1.25) *and* (1.16) *respectively.*

Making use of the identities

$$(1.27.1)\qquad HH=H,\quad KK=K,\quad LL=L,$$

$$(1.27.2)\qquad HK=KH=0,$$

$$(1.27.3)\qquad KL=LK=0,$$

$$(1.27.4)\qquad LH=HL=0,$$

$$(1.27.5)\qquad E=H+K+L,$$

we have

$$(1.28.1)\qquad H'=HH'+H'H,$$

$$(1.28.2)\qquad K'=KK'+K'K,$$

$$(1.28.3)\qquad L'=LL'+L'L,$$

$$(1.28.4)\qquad HH'+KK'+LL'=-(H'H+K'K+L'L),$$

$$(1.28.5)\qquad HH'+KK'+LL'=(KMH-HMK)+(KML-LMK)+(HML-LMH),$$

from (1.16), (1.25) and (1.26).

§ 2. Global covariant variation under an infinitesimal conformal transformation.

Now let us define the global covariant variation δ_g by the formula

$$\delta_g\alpha=H\delta_l H\alpha+K\delta_l K\alpha+L\delta_l L\alpha.$$

Its geometrical meaning is obvious: *Project α to each of three spaces $\boldsymbol{H}_r$, $\boldsymbol{B}_r$ and $\overline{\boldsymbol{B}}_a$. Project each of local covariant variations of these projections to corresponding spaces once more. Sum of three projections is the global covariant variation of α.*

By (1.15), (1.23), (1.26) and (1.27.1), we have

$$(2.1)\qquad \delta_g\alpha=\delta_l\alpha+(HH'+KK'+LL')\alpha.$$

(2.1) is equivalent to

(2.2) $$\delta_g\alpha=\delta_l\alpha-(H'H+K'K+L'L)\alpha$$

by (1.28.4).

From the definition we get

$$\begin{aligned}\delta_g H\alpha&=\delta_l(H\alpha)-(H'H+K'K+L'L)H\alpha\\&=(H\alpha)'-H'H\alpha\\&=H'\alpha+H\alpha'-H'H\alpha\\&=H'\alpha-H'H\alpha+H[\delta_g\alpha-(HH'+KK'+LL')\alpha]\\&=(H'-H'H-HH')\alpha+H\delta_g\alpha\end{aligned}$$

by (1.27), and then from (1.28.1) we get finally $\delta_g H\alpha=H\delta_g\alpha$. Similarly we get $\delta_g K\alpha=K\delta_g\alpha$ and $\delta_g L\alpha=L\delta_g\alpha$.

Hence we obtain

THEOREM 6. *The operator δ_g is commutative with the operations H, K and L.*

Let us write $\delta_g\alpha=\dot{\alpha}$, $(\delta_g H)\alpha=\dot{H}\alpha=(H\alpha)^{\cdot}-H\dot{\alpha}$, etc., for simplicity. We shall call $\delta_g H$ the global covariant variation of the operator H, etc.

In this terminology, theorem 6 is equivalent to

COROLLARY. *The global covariant variations of the operators H, K and L are zero.*

THEOREM 7. *If α is in C_r (i. e., if α remains in C_r during the transformation), then $\delta_g\alpha$ is in C_r too.*

PROOF. If α is in C_r, it follows that $\perp L'\alpha=0$ and $\perp LL'\alpha=0$ by (0.17.15) and (1.5.6). From (2.1) we have

$$\begin{aligned}\perp\delta_g\alpha&=\perp\delta_l\alpha+\perp(HH'+KK'+LL')\alpha\\&=\perp\delta_l\alpha=\perp\alpha',\end{aligned}$$

since $\perp H\alpha=\perp K\alpha=0$ by (0.17.13) and (0.17.14). On the other hand $\perp\alpha=0$ implies $\perp\alpha'=0$ by (1.5.2). Hence $\perp\delta_g\alpha=0$, and $\delta_g\alpha$ is in C_r.

THEOREM 8. *If α is in H_r then $\delta_g\alpha$ is in H_r.*
If α is in B_r then $\delta_g\alpha$ is in B_r.
If α is in $\bar{B}_a$ then $\delta_g\alpha$ is in $\bar{B}_a$.

PROOF. If α is in H_r, then $K\alpha=L\alpha=0$. Hence

$$\delta_g K\alpha=K\delta_g\alpha=0,\quad \delta_g L\alpha=L\delta_g\alpha=0,$$

by theorem 6. That is $\delta_g\alpha$ is in H_r. The others are similarly proved.

THEOREM 9. $v(\alpha\cdot\beta)=(\dot{\alpha}\cdot\beta)+(\alpha\cdot\dot{\beta})$.

PROOF. Making use of (2.1) and (1.28.5), we get

$$\begin{aligned}(\dot{\alpha}\cdot\beta)&=(\alpha'\cdot\beta)+((HH'+KK'+LL')\alpha\cdot\beta)\\&=(\alpha'\cdot\beta)+((KMH-HMK+KML-LMK+HML-LMH)\alpha\cdot\beta)\\&=(\alpha'\cdot\beta)+(MH\alpha\cdot K\beta)-(MK\alpha\cdot H\beta)+(ML\alpha\cdot K\beta)\\&\quad-(MK\alpha\cdot L\beta)+(ML\alpha\cdot H\beta)-(MH\alpha\cdot L\beta)\\&=(\alpha'\cdot\beta)+(H\alpha\cdot MK\beta)-(K\alpha\cdot MH\beta)+(L\alpha\cdot MK\beta)\\&\quad-(K\alpha\cdot ML\beta)+(L\alpha\cdot MH\beta)-(H\alpha\cdot ML\beta)\,,\end{aligned}$$

since $(M\alpha\cdot\beta)=(\alpha\cdot M\beta)$, $(H\alpha\cdot\beta)=(\alpha\cdot H\beta)$, etc. for arbitrarily given α and β. Similarly

$$\begin{aligned}(\alpha\cdot\dot{\beta})=(\alpha\cdot\beta')&+(K\alpha\cdot MH\beta)-(H\alpha\cdot MK\beta)+(K\alpha\cdot ML\beta)\\&-(L\alpha\cdot MH\beta)+(H\alpha\cdot ML\beta)-(L\alpha\cdot MH\beta)\,.\end{aligned}$$

From (1.5.5) it follows that $v(\alpha\cdot\beta)=(\dot{\alpha}\cdot\beta)+(\alpha\cdot\dot{\beta})$.

COROLLARY. *If* $\dot{\alpha}=\dot{\beta}=0$, *then* $(\alpha\cdot\beta)$ *is constant.*

Because of the definition (2.2), we have

$$\delta_g(\triangle\alpha)=(\triangle\alpha)'-(H'H+K'K+L'L)\triangle\alpha\,.$$

If $\perp\alpha=0$, we get $H\triangle\alpha=L\triangle\alpha=0$ and $K\triangle\alpha=\triangle\alpha$ from (0.17.1), (0.17.6) and (0.17.9). Thus we get

$$\begin{aligned}\delta_g\triangle\alpha&=(\triangle\alpha)'-K'\triangle\alpha\\&=\triangle'\alpha+\triangle\alpha'-K'\triangle\alpha\,.\end{aligned}$$

According to the definition (2.1),

$$\triangle(\delta_g\alpha)=\triangle[\alpha'+(HH'+KK'+LL')\alpha]=\triangle\alpha'+\triangle L'\alpha$$

from (0.17.7), (0.17.2) and (0.19). Accordingly, if $\perp\alpha=0$,

$$\begin{aligned}\delta_g\triangle\alpha-\triangle\delta_g\alpha&=\triangle'\alpha-K'\triangle\alpha-\triangle L'\alpha\\&=(M\triangle-\triangle M)\alpha-(MK-2KMK+KM)\triangle\alpha\\&\quad-\triangle(-ML+2LML-LM)\alpha\\&=(KM\triangle-\triangle ML)\alpha\,,\end{aligned}$$

by (1.7.1), (1.25), (1.16), (0.17) and (0.19). Writing $\delta_g(\triangle\alpha)-\triangle\delta_g\alpha=\dot{\triangle}\alpha$ for simplicity, we have

(2.3) $$\dot{\triangle}\alpha=(KM\triangle-\triangle ML)\alpha, \quad \text{if} \ \perp\alpha=0.$$

Similarly we have

(2.4) $$\dot{\nabla}\alpha=(-LM\nabla+\nabla MK)\alpha.$$

Since $\square\alpha=(\triangle\nabla+\nabla\triangle)\alpha$, it follows that

(2.5) $$\begin{aligned}\dot{\square}\alpha=(&KM\triangle\nabla+\triangle\nabla MK-LM\nabla\triangle-\nabla\triangle ML\\ &-2\triangle M\nabla+2\nabla M\triangle)\alpha,\end{aligned}$$

if $\perp\alpha=\perp\nabla\alpha=0.$

Operating δ_g on the equations

(2.6.1) $\square\beta=(E-H)\alpha,$

(2.6.2) $\perp\beta=0,$

(2.6.3) $\perp\nabla\beta=0,$

(2.6.4) $H\beta=0,$

we get

(2.7.1) $\square\dot{\beta}=(E-H)\alpha-\dot{\square}\beta,$

(2.7.2) $\perp\dot{\beta}=0,$

(2.7.3) $\perp\nabla\dot{\beta}=-\perp\dot{\nabla}\beta,$

since $\dot{E}=\dot{H}=0$, and by theorem 7.

The equations with boundary conditions (2.7) must be consistent, since (2.6) are consistent and (2.7) are the variations of (2.6). The solution $\dot{\beta}$ is given by

(2.8) $$\dot{\beta}=G((E-H)\dot{\alpha}-\dot{\square}\beta)+\tilde{G}\dot{\nabla}\beta,$$

where $\tilde{G}$ is an operator defined by the formula

$$\tilde{G}\alpha=\{\alpha(y)\cdot G(x,y)\}=\oint_B\alpha(y)\wedge *G(x,y).$$

On the other hand, by (0.13)

(2.9) $\beta=G\alpha.$

Put $(G\alpha)^{\bullet}-G\dot{\alpha}=\dot{G}\alpha$, it follows that

(2.10) $$\overset{\bullet}{\beta}=\dot{G}\alpha+G\dot{\alpha}\,.$$

Form (2.8) and (2.10) we have

$$\dot{G}\alpha+G\dot{\alpha}=G((E-H)\dot{\alpha}-\overset{\bullet}{\square}\beta)+\tilde{G}\overset{\bullet}{\nabla}\beta$$
$$=G\dot{\alpha}-G\overset{\bullet}{\square}\beta+\tilde{G}\overset{\bullet}{\nabla}\beta\,,$$

since $GH=0$.

We get finally from (2.9)

(2.11) $$\dot{G}\alpha=-G\overset{\bullet}{\square}G\alpha+\tilde{G}\overset{\bullet}{\nabla}G\alpha\,,$$

or

$$\dot{G}=-G\overset{\bullet}{\square}G+\tilde{G}\overset{\bullet}{\nabla}G\,.$$

THEOREM 10. *The global covariant variation of the operator* G *is given by* (2.11).

For 0-forms (2.11) reduces to $\dot{G}=-G\overset{\bullet}{\square}G$. Analogous result for the ordinary variation of Green's function is given by S. Bergman and M. Schiffer [2] (Note that the infinitesimal transformation there differs from our transformation.)

§ 3. Infinitesimal point transformation.

We shall call an infinitesimal point transformation

(3.1) $$\bar{x}^i=x^i+\xi^i dt$$

is in domain $\boldsymbol{D}$, if the contravariant vector ξ is tangential to the boundary $\boldsymbol{B}$ at each point. The domain $\boldsymbol{D}$ remains unchanged as a whole under this transformation in $\boldsymbol{D}$. And the spaces of forms, $\boldsymbol{B}_r$, $\boldsymbol{H}_r$ and $\overline{\boldsymbol{B}}_a$ remain unchanged too.

Lie derivative Xg_{ij} of the metric tensor g_{ij} for the transformation (3.1) is given by

$$Xg_{ij}=\xi_{i,j}+\xi_{j,i}\quad (Xdx^i=0,\text{ by definition})\,.$$

If the transformation is conformal, we have

$$Xg_{ij}=2\mu g_{ij}\,,\quad [7]$$

and we shall identify Xg_{ij} with vg_{ij} in (1.1), then we get the covariant variations $\delta(\xi)$ with respect to the transformation ξ.

If a form α is in $\boldsymbol{H}_r$, its global covariant variation $\delta_g(\xi)\alpha$ is also in $\boldsymbol{H}_r$ by theorem 8, and it is a linear combination of the independent generators α_λ in $\boldsymbol{H}_r$ with constant coefficients. Let α_λ be a set of independent generators of $\boldsymbol{H}_r$ and let

$$\delta_g(\xi)\alpha_\lambda = K_\lambda^\mu \alpha_\mu\,,$$

where $\lambda, \mu = 1, \cdots, R$ (R is the Betti number)

Let $\xi_a^i (a=1, \cdots, r)$ be the set of independent infinitesimal generators of a conformal transformation group $\boldsymbol{G}$ in $\boldsymbol{D}$ with structural coefficients C_{ab}^c. It is well-known that

$$[\xi_a, \xi_b] = \xi_a^p \xi_{b,p}^i - \xi_b^p \xi_{a,p}^i = C_{ab}^c \xi_c^i\,, \tag{3.2}$$

$$C_{ab}^d C_{dc}^e + C_{bc}^d C_{da}^e + C_{ca}^d C_{db}^e = 0\,. \quad (a, b, c, d, e = 1, \cdots, r) \tag{3.3}$$

The group $\boldsymbol{G}$ induces a homomorphic linear transformation group in $\boldsymbol{H}_r$ by the global covariant variation. The homomorphism requires the condition

$$\delta_g([\xi_a, \xi_b]) = \delta_g(\xi_a)\delta_g(\xi_b) - \delta_g(\xi_b)\delta_g(\xi_a)\,. \tag{3.4}$$

Putting

$$\delta_g(\xi_a)\alpha_\lambda = K_\lambda{}^\mu{}_a \alpha_\mu\,, \tag{3.5}$$

we have the necessary conditions between the constants C and K by (3.5), (3.4) and (3.2):

$$K_\lambda{}^\mu{}_a K_\mu{}^\nu{}_b - K_\lambda{}^\mu{}_b K_\mu{}^\nu{}_a = C_{ab}^c K_\lambda{}^\nu{}_c\,. \tag{3.6}$$

We shall put

$$R_\lambda{}^\nu{}_{ab} = K_\lambda{}^\mu{}_a K_\mu{}^\nu{}_b - K_\lambda{}^\mu{}_b K_\mu{}^\nu{}_a\,, \tag{3.7}$$

R is something analogous to the curvature tensor in Riemannian space.

Let A_λ^μ be the components of a tensor in $\boldsymbol{H}_r$ with respect to the generators α_λ and let A_a^b be the components of a tensor in the parameter space of group $\boldsymbol{G}$ at its identity element with respect to the generators ξ_a. It is easy to define a mixed tensor A_λ^a.

Since Betti group $\boldsymbol{H}_r$ and the tangential vector space to the parameter space at its identity element admit only homogeneous linear transformations, we shall deal with tensors whose components are constant, and coefficients of a linear connection can be taken for components of a tensor.

Let us define covariant differentiation of a mixed tensor $A^\lambda{}_{\mu a}$ formally by the formula

$$A^{\lambda}{}_{\mu a;b} = K_{\sigma}{}^{\lambda}{}_{b} A_{\mu}{}^{\sigma}{}_{a} - K_{\mu}{}^{\sigma}{}_{b} A^{\lambda}{}_{\sigma a} - C^{c}_{ab} A^{\lambda}{}_{\mu c},$$

where K and C are taken for coefficients of a linear connection. But if we take K and C for tensor, we have

$$K_{\lambda}{}^{\mu}{}_{a;b} = 0,$$

$$C^{c}_{ab;d} = 0.$$

by (3.3) and (3.6). It follows that

$$R_{\lambda}{}^{\mu}{}_{ab,c} = 0.$$

THEOREM 11. *Covariant derivative of the curvature tensor defined by* (3.7) *for a conformal transformation group is zero.*

The dual theorems for the decomposition (0.11)′ *can be obtained similarly, since the local covariant variation is commutative with* * *by* (1.5.1).

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