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On certain cohomological operations.

By Tsuneyo YAMANOSHITA

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Introduction

Let A, B be given abelian groups and m, n fixed non-negative integers. Then Serre [7] has defined as follows the cohomology operation relative to (A, B, m, n). It is a mapping C defined for each CW-complex K of the m-th cohomology group $H^m(K, A)$ into $H^n(K, B)$, such that the following diagram is commutative

$$\begin{array}{ccc} H^m(K, A) & \xrightarrow{f^*} & H^m(K', A) \\ \downarrow C & & \downarrow C \\ H^n(K, B) & \xrightarrow{f^*} & H^n(K', B) , \end{array}$$

where K' is another CW-complex, f^* the homomorphism of the cohomology group of K into that of K' induced by a simplicial mapping $f: K' \rightarrow K$. In generalizing this notion, we shall now consider operations of the following kind. Our mapping C has as its domain of definition a subgroup S of $H^m(K, A)$ and as its range a factor group $H^n(K, B)/M$ of $H^n(K, B)$. Once C is given, an subgroup $S=S(K)$ of $H^m(K, A)$ and the subgroup $M=M(K)$ of $H^n(K, B)$ are thus defined by K; we postulate now

$$S(K') \subset f^*(S(K)) ,$$

$$M(K') \subset f^*(M(K))$$

for every simplicial mapping $f: K' \rightarrow K$. C will be then called cohomological operation if the following diagram is commutative

$$\begin{array}{ccc} H^m(K, A) \supset S & \xrightarrow{f^*} & S' \subset H^m(K, A') \\ \downarrow C & & \downarrow C \\ H^n(K, B)/M & \xrightarrow{f^*} & H^n(K', B)/M' , \end{array}$$

whence $S'=S(K')$, $M'=M(K')$. For example, the Adem operation Φ introduced in [1] is a cohomological operation in our sense.

More generally, X being any topological space, and the cohomology group $H^*(X, G)$ of X with an abelian group G as coefficient being the singular one, we can correspondingly define the cohomological operation in an obvious manner.

Z means as usual the additive group of integers; Z_n the group Z mod n; p a prime number. The meanings of $F_h, G_h, f_h, g_h, f'_h, g'_h$ will be clear from the following exact sequences

$$(\mathrm{I}_h) \qquad 0 \longrightarrow Z \xrightarrow{F_h} Z \xrightarrow{G_h} Z_{p^h} \longrightarrow 0,$$

$$(\mathrm{II}_h) \qquad 0 \longrightarrow Z_{p^h} \xrightarrow{f_h} Z_{p^{h+1}} \xrightarrow{g_h} Z_p \longrightarrow 0,$$

$$(\mathrm{III}_h) \qquad 0 \longrightarrow Z_p \xrightarrow{f'_h} Z_{p^{h+1}} \xrightarrow{g'_h} Z_{p^h} \longrightarrow 0.$$

The coboundary operators associated with (I_h), (II_h), (III_h) (cf. § 1) are denoted by $1/_{p^h}\delta$, δ_h, δ'_h respectively.

§ 1 contains algebraic preliminaries and topological meanings of Bockstein operators (cf. § 1 Theorem 2.1.) and in particular of $1/_{p^h}\delta$, δ_h, δ'_h.

In § 2, we define the operations Δ^i_p, $i=1, \cdots$ and give their fundamental properties. Δ^1_p is nothing other than $\delta_1=\delta'_1$; Δ^2_p maps $\mathrm{Ker}\,\Delta^1_p \cap H^n(X, Z_p)$ into $H^{n+1}(X, Z_p)/\mathrm{Im}\,\delta'_1$ homomorphically; in general, Δ^i_p, $i \geqq 1$, maps $\mathrm{Ker}\,\Delta^{i-1}_p \cap H^n(X, Z_p)$ into $H^{n+1}(X, Z_p)/\mathrm{Im}\,\delta'_{h-1}$ homomorphically. The knowledge of the effect of Δ^i_p, $i=1, 2, \cdots$ will suffice to determine the p-primary component of $H^*(X, Z)$. (Theorem 1.1 in § 2).

Furthermore, if $E \supset F$, the knowledge on the effects of Δ^i_p on $H^*(E, F; Z_p)$ and on $H^*(E, Z_p)$ will give us some information on the effect of Δ^i_p on $H^*(F, Z_p)$. These circumstances, useful in the computation of homotopy groups of spheres and CW-complexes, are expounded in § 3. Professor H. Toda has kindly communicated to me, that Professor H. Cartan and himself have also obtained the same results as our Theorems 3.2, 3.5, 3.7 and utilized them to compute the stable homotopy groups of spheres.

In § 4, we define the operations P^h_p, $1/_p P^h_p$ and allied operations. P^1_2 is nothing but the Pontrjagin square [11, 15]. $1/_p P^1_p$ is useful in giving the generators of $H^*(\Omega(S^n), Z_p)$, where $\Omega(S^n)$ is the loop space of the n-sphere S^n.

The author intends to publish in a forthcoming paper the applications of the results of this paper to the homotopy theory. He wishes to express his hearty thanks to his friends N. Yoneda, Y. Saito, T. Nakamura, A. Hattori, who have given him valuable suggestions through kind criticisms and discussions and also to Professor S. Iyanaga for his constant encouragement during the preparation of this paper.

§ 1. Algebraic preliminaries

1. Torsion products and extension groups [14]. Let A, B be (abelian) groups. If we represent A as a factor group of a free (abelian) group F, then the kernel R of the epimorphism $F \to A$ is also free. Let T be the kernel of the homomorphism $R \otimes B \to F \otimes B$ and H the cokernel of the homomorphism $\mathrm{Hom}(F, B) \to \mathrm{Hom}(R, B)$. These are invariants of the pair (A, B), and are denoted respectively by $A * B$ (torsion product of A and B) and $\mathrm{Ext}(A, B)$ (extension group of B by A). $A * B$ is a covariant functor in A and in B. $\mathrm{Ext}(A, B)$ is a contravariant functor in A and a covariant functor in B. We list here some properties of torsion products and extension groups.

1.1. $A * B \approx B * A$.

1.2. Let

$$(S) \qquad 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

be an exact sequence and G a group. Then we have an exact sequence

$$0 \longrightarrow A * G \longrightarrow B\ G \longrightarrow C * G \longrightarrow A \otimes G \longrightarrow B \otimes G \longrightarrow C \otimes G \longrightarrow 0 .$$

1.3. The functor $A * B$ commutes with the formation of direct sum.

1.4. We have

$$Z * A = 0 \quad \text{and} \quad Z_n * A = {}_nA ,$$

where ${}_nA$ is the subgroup of A consisting of a with $na = 0$.

1.5. Under the same hypothesis as in 1.2, we have the following exact sequences:

$$0 \longleftarrow \operatorname{Ext}(A, G) \longleftarrow \operatorname{Ext}(B, G) \longleftarrow \operatorname{Ext}(C, G) \longleftarrow \operatorname{Hom}(A, G) \longleftarrow$$
$$\operatorname{Hom}(B, G) \longleftarrow \operatorname{Hom}(C, G) \longleftarrow 0\,,$$
$$0 \longrightarrow \operatorname{Hom}(G, A) \longrightarrow \operatorname{Hom}(G, B) \longrightarrow \operatorname{Hom}(G, C) \longrightarrow \operatorname{Ext}(G, A) \longrightarrow$$
$$\operatorname{Ext}(G, B) \longrightarrow \operatorname{Ext}(G, C) \longrightarrow 0\,.$$

1.6. The functor $\operatorname{Ext}(A, B)$ commutes with the formation of direct sum (finite).

1.7. We have

$$\operatorname{Ext}(Z, A) = 0 \quad \text{and} \quad \operatorname{Ext}(Z_n, A) \approx A/nA\,,$$

where nA is the subgroup of A consisting of the elements $na, a \in A$.

2. Coboundary operators associated with exact sequences of coefficients. When we have an exact sequence (S), we have clearly the following exact sequence of singular cohomology groups of a space X:

$$\cdots \longrightarrow H^n(X, A) \longrightarrow H^n(X, B) \longrightarrow H^n(X, C) \xrightarrow{\ \delta\ } H^{n+1}(X, A) \longrightarrow \cdots\,,$$

whence δ is the coboundary operator associated with (S). δ is defined in the following way. We have the following commutative diagram for cochain groups:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^{n+2}(X, A) & \xrightarrow{f} & C^{n+2}(X, B) & \xrightarrow{g} & C^{n+2}(X, C) & \longrightarrow & 0 \\
 & & \uparrow d & & \uparrow & & \uparrow d & & \\
0 & \longrightarrow & C^{n+1}(X, A) & \xrightarrow{f} & C^{n+1}(X, B) & \xrightarrow{g} & C^{n+1}(X, C) & \longrightarrow & 0 \\
 & & \uparrow d & & \uparrow & & \uparrow d & & \\
0 & \longrightarrow & C^{n}(X, A) & \xrightarrow{f} & C^{n}(X, B) & \xrightarrow{g} & C^{n}(X, C) & \longrightarrow & 0 \\
 & & \uparrow d & & \uparrow & & \uparrow d & & \\
0 & \longrightarrow & C^{n-1}(X, A) & \xrightarrow{f} & C^{n-1}(X, B) & \xrightarrow{g} & C^{n-1}(X, C) & \longrightarrow & 0
\end{array}$$

Diagram 1.

Let c be a representative cocycle of class $\{c\} \in H^n(X, C)$. From the above diagram, it is easily seen $\{f^{-1} \circ d \circ g^{-1}(c)\}$ is uniquely determined for the class $\{c\}$ as an element of $H^{n+1}(X, A)$. We denote this by $\delta\{c\}$. Obviously $\delta : H^n(X, C) \rightarrow H^{n+1}(X, A)$ is a homomorphism.

We shall determine the kernel and the image of δ. Denote by L_n the n-th homology group $H_n(X, Z)$. Then, as is well-known, we have

$$H^n(X, G) \approx \mathrm{Ext}\,(L_{n-1}, G) \oplus \mathrm{Hom}\,(L_n, G)$$

and with regards to the sequence (S), we obtain a commutative diagram

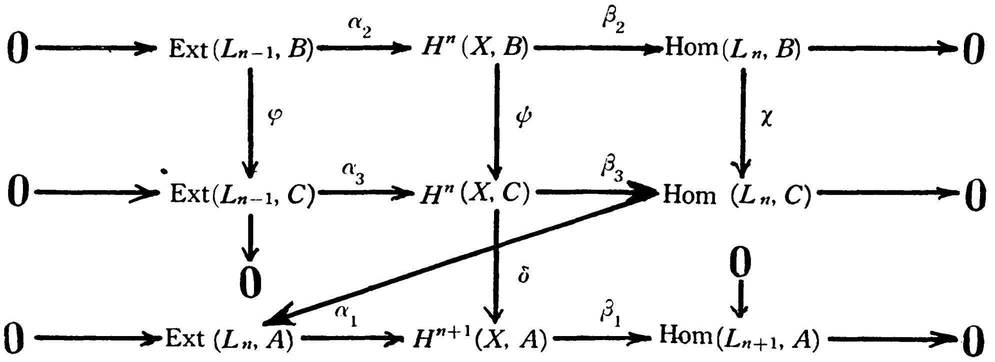

Diagram 2.

In this diagram (2), the δ kernel is the ψ image by the exactness of the sequence

$$H^n(X, B) \xrightarrow{\psi} H^n(X, C) \xrightarrow{\delta} H^{n+1}(X, A).$$

Since φ is epimorphic by 1.5, the φ image of $H^n(X, B)$ contains the α_3 image of $\mathrm{Ext}\,(L_{n\ \ 1}, C)$. Also, since β_2 is epimorphic, the β_3 image of ψ $H^n(X, B)$ coincides with the χ image of $\mathrm{Hom}\,(L_n, B)$. Noting that β_3 kernel is the α_3 image of $\mathrm{Ext}\,(L_{n-1}, C)$, we obtain

$$(H^n(X, C) \supset) \delta^{-1}(0) \approx \mathrm{Ext}\,(L_{n-1}, C) \oplus \chi \mathrm{Hom}\,(L_n, B).$$

Next, since the β_3 image of ψ $H^n(X, B)$ coincides with the $\chi \circ \beta_2$ image of $H^n(X, B)$ and is contained in the χ image of $\mathrm{Hom}\,(L_n, B)$, β_3 induces the homomorphism

$$\beta_3^* : H^n(X, C)/\psi H^n(X, B) \rightarrow \mathrm{Hom}\,(L_n, C)/\chi \mathrm{Hom}\,(L_n, B).$$

β_3^* is epimorphic, for β_3 is epimorphic. We shall show that β_3^* is a monomorphism. Let b and c be respectively elements of $\mathrm{Hom}\,(L_n, B)$ and $H^n(X, C)$ such that $\beta_3(c) = \chi(b)$. As β_2 is epimorphic, there exists an element b' of $H^n(X, B)$ with $\beta_2(b')=b$. So we have $\beta_3\circ\psi(b')=\chi\circ\beta_2(b')=\beta_3(c)$. Therefore there is an element $c'\in \operatorname{Ext}(L_{n-1}, C)$ with $\alpha_3(c')=\psi(b')=c$. Furthermore φ is epimorphic, and so we have an element b'' in $\operatorname{Ext}(L_{n-1}, B)$ such that $\varphi(b'')=c'$. Then it follows from $\alpha_3\circ\varphi(b'')=\varphi\circ\alpha_2(b'')=\varphi(b')-c$ that $c=\varphi(b'-\alpha_2(b''))$. This shows that β_3^* is monomorphic. Hence we obtain:

THEOREM 2.1. *Let*

$$(S)\qquad 0\longrightarrow A\longrightarrow B\overset{g}{\longrightarrow} C\longrightarrow 0$$

be an exact sequence, and δ be the coboundary operator associated with this sequence. Then we have

$$(H^n(X, C)\supset)\delta^{-1}(0)\approx \operatorname{Ext}(L_{n-1}, C)\oplus g\operatorname{Hom}(L_n, B)\,,$$

$$\delta H^n(X, C)\approx \operatorname{Hom}(L_n, C)/g\operatorname{Hom}(L_n, B),\quad (L_i=H_i(X, Z))\,.$$

REMARK 2.2. In the diagram (1) $\delta H^n(X, C)$ is the λ image of $\operatorname{Hom}(L_n, C)$.

REMARK 2.3. As to the boundary operator δ_* in homology groups associated with the above sequence (S), we have

$$(H_n(X, C)\supset)\delta_*^{-1}(0)\approx g(L_n\otimes B)\oplus L_{n-1}*C\,,$$

$$\delta_*H_n(X, C)\approx L_n\otimes C/g(L_n\otimes B)\,.$$

EXAMPLE 2.4. Let (S) be the exact sequence

$$0\longrightarrow Z_p\overset{f_h'}{\longrightarrow} Z_{p^{h+1}}\overset{g_h'}{\longrightarrow} Z_{p^h}\longrightarrow 0\,.$$

Then we have

$$\delta_h' H^n(X, Z_{p^h})\approx \operatorname{Hom}(L_n, Z_{p^h})/g_h'\operatorname{Hom}(L_n, Z_{p^{h+1}})\,.$$

Suppose that $H_*(X, Z)$ is of finite type in all degrees, then L_n admits a direct sum decomposition into cyclic groups. If the number of summands in this decomposition whose orders are powers of p with exponents $\leqq h$ is exactly n', then our δ_h' image is a vector space with dimension n' over Z_p.

In the same way, we have

$$(H_n(X, Z_{p^h})\supset)\delta_h'^{-1}(0)\approx \operatorname{Ext}(L_{n-1}, Z_{p^h})\oplus g_h'\operatorname{Hom}(L_n, Z_{p^{h+1}})\,.$$

In particular $\delta_1'^{-1}(0)$ $(\subset H^n(X, Z_p))$ is a vector space with the same dimension as (Tor $L_{n-1}\otimes Z_p)\oplus$((the free part of $L_n)\otimes Z_p)\oplus((\sum$ the cyclic direct summands of order $p^{\nu}(\nu\geqq 2)$ of $L_n)\otimes Z_p)$ over Z_p. Namely $\delta_1'^{-1}(0)$ is the so-called $\left(\frac{1}{p}\delta \bmod p\right)$ kernel.

EXAMPLE 2.5. Let (S) be the exact sequence

$$0\longrightarrow Z_{p^h}\xrightarrow{f_h} Z_{p^{h+1}}\xrightarrow{g_h} Z_p\longrightarrow 0\,.$$

Then we have

$$(H^n(X, Z_p)\supset)\delta_h^{-1}(0)\approx \operatorname{Ext}(L_{n-1}, Z_p)\oplus g_h \operatorname{Hom}(L_n, Z_{p^{h+1}})\,.$$

Hence follows that $\delta_h^{-1}(0)$ has the same dimension as ($\sum$ the p-primary direct summands of $L_{n-1})\otimes Z_p)\oplus$((the free part of $L_n)\otimes Z_p)\oplus((\sum$ the cyclic direct summands of order $p^{\nu}(\nu\geqq h+1))\otimes Z_p)$.

Of course $\delta_1^{-1}(0)$ is equal to $\delta_1'^{-1}(0)$. $\delta_h H^n(X, Z_p)$ is of dimension n' (see Example 2.4).

EXAMPLE 2.6. Consider the coboundary operator $\frac{1}{p}\delta$ associated with the exact sequence

$$0\longrightarrow Z\longrightarrow Z\longrightarrow Z_p\longrightarrow 0\,.$$

Then $\frac{1}{p}\delta$ kernel has the same dimension as (($\sum$ the p-primary direct summands of $L_{n-1})\otimes Z_p)\oplus$((the free part of $L_n)\otimes Z_p)$.

The $\frac{1}{p}\delta$ image has the same dimension as ($\sum$ the p-primary direct summands of $L_n)\otimes Z_p$.

PROPOSITION 2.7. *The coboundary operator δ associated with the exact sequence*

$$(S)\qquad 0\longrightarrow A\xrightarrow{f} B\xrightarrow{g} C\longrightarrow 0$$

commutes with the coboundary homomorphism Δ of the cohomology sequence up to sign:

$$\delta\Delta=-\Delta\delta\,.$$

PROOF. For an exact sequence of chain complexes consisting of free groups,

$$0 \longleftarrow R \overset{\lambda}{\longleftarrow} F \overset{\mu}{\longleftarrow} Q \longleftarrow 0\,,$$

we have for any group G

$$0 \longrightarrow C^i(R, G) \overset{\lambda^*}{\longrightarrow} C^i(F, G) \overset{\mu^*}{\longrightarrow} C^i(Q, G) \longrightarrow 0\,,$$

where $C^i(R, G), C^i(F, G), C^i(Q, G)$ denote the i-th cochain groups with the coefficient G. Furthermore commutativity holds in the following diagram:

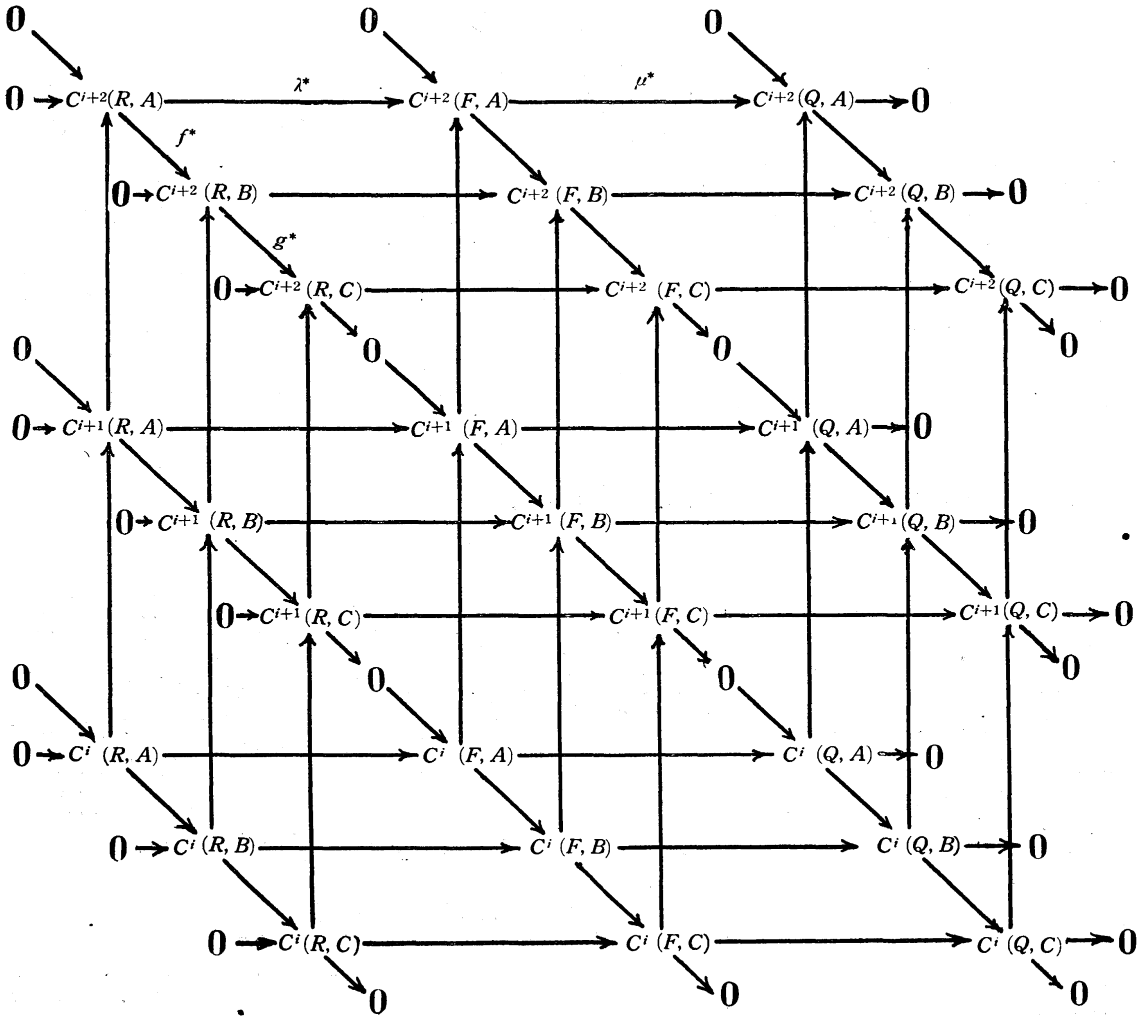

Diagram 3.

We shall denote by d any of the coboundary operators in any of the above cochain groups unless no confusion is likely to occur. Take a cocycle q_c^i of $C^i(Q, C)$. Then as μ^* is epimorphic, there exists an element $f_c^i \in C^i(F, C)$ such that the μ^* image of f_c^i is q_c^i. Take $f_c^{i+1} = d(f_c^i)$. This is annuled by μ^*. Therefore there exists an element r_c^{i+1} of $C^{i+1}(R, C)$ whose λ^* image is f_c^{i+1}. Further g is epimorphic, and so we have $g(r_b^{i+1}) = r_c^{i+1}$ with some $r_b^{i+1} \in C^{i+1}(R, B)$. On the other hand, we have

$$g \circ d(r_b^{i+1}) = d \circ g(r_b^{i+1}) = 0 .$$

This implies the existence of $r_a^{i+2} \in C^{i+2}(R, A)$ such that

$$f^*(r_a^{i+2}) = d(r_b^{i+1}) .$$

Now g and μ^* being epimorphic, there exists an element $f_b^i \in C^i(F, B)$ whose g image is f_c^i. Since

$$g(df_b^i - \lambda^* r_b^{i+1}) = d \circ g(f_b^i) - \lambda^* \circ g(r_b^{i+1}) = f_c^{i+1} - f_c^{i+1} = 0 ,$$

there is an element f_a^{i+1} of $C^{i+1}(F, A)$ whose f^* image is $d(f_b^i) - \lambda^*(r_b^{i+1})$. Then we have

$$f \circ \mu^*(f_a^{i+1}) = \mu^* \circ (d(f_b^i) - \lambda^*(r_b^{i+1})) = \mu^* \circ d(f_b^i) = d \circ \mu^*(f_b^i) .$$

Moreover we have

$$\begin{aligned} f(d(f_a^{i+1}) + \lambda^*(r_a^{i+2})) &= d \circ f(f_a^{i+1}) + \lambda^* \circ f(r_a^{i+2}) \\ &= -d \circ \lambda^*(r_b^{i+1}) + \lambda^* \circ d(r_b^{i+1}) = 0 \end{aligned}$$

and f^* is monomorphic. Therefore we have $d(f_a^{i+1}) = -\lambda^*(r_a^{i+1})$, also we have $\mu^* \circ d(f_a^{i+1}) = 0$. Therefore there exists $r_a'^{i+2} = C^{i+2}(R, A)$ with the property $\lambda^*(r_a'^{i+2}) = d(f_a^{i+1})$ $(= \lambda^*(-r_a^{i+2}))$. That is $r_a'^{i+2} = -r_a^{i+2}$ as λ^* is epimorphic. This implies that the cohomology class $\{r_a'^{i+2}\}$ of $r_a'^{i+2}$ representing $\Delta \circ \delta\{q_c^i\}$ is the same as the cohomology class $-\{r_a^{i+2}\}$ of r_a^{i+2} representing $-\delta \circ \Delta\{q_c^i\}$. Q. E. D.

PROPOSITION 2.9. *In the exact sequence of groups*

$$0 \longrightarrow A \overset{i}{\longrightarrow} B \overset{k}{\longrightarrow} B/iA \longrightarrow 0 ,$$

we assume that a commutative ring structure is given in each of $A, B, B/iA$ in such a way that

(i) *k is a ring homomorphism and*

(ii) *for the ideal iA in B, we have $(iA)^2 = 0$.*

Then we can define a bilinear multiplication of A and B/iA into A induced by the natural multiplication of iA and B, whence a bilinear pairing of $H^*(X, A)$ and $H^*(X, B/iA)$ can be defined in an obvious way.

Let δ be the coboundary operator associated with the above exact sequence. For $\alpha \in H^j(X, B/iA)$, $\beta \in H^{j'}(X, B/iA)$, we have then the following equality

$$\delta(\alpha\beta) = \delta(\alpha)\beta + (-1)^j \alpha\delta(\beta) .$$

Proof is left to the reader.

PROPOSITION 2.10. *The coboundary operator δ'_h is obtained in composing the coboundary operator associated with the exact sequence*

$$0 \longrightarrow Z \xrightarrow{F_h} Z \xrightarrow{G_h} Z_{p^h} \longrightarrow 0$$

with the homomorphism $H^*(X, Z) \to H^*(X, Z_p)$ induced by the natural homomorphism $Z \to Z_p(\to 0)$, and the coboundary operator δ_h is obtained in composing the coboundary operator associated with the exact sequence

$$0 \longrightarrow Z \xrightarrow{F_1} Z \xrightarrow{G_1} Z_p \longrightarrow 0$$

with the homomorphism $H^*(X, Z) \to H^*(X, Z_{p^h})$ induced by the natural homomorphism $Z \to Z_{p^h}(\to 0)$.

PROOF. It follows from the following commutative diagram:

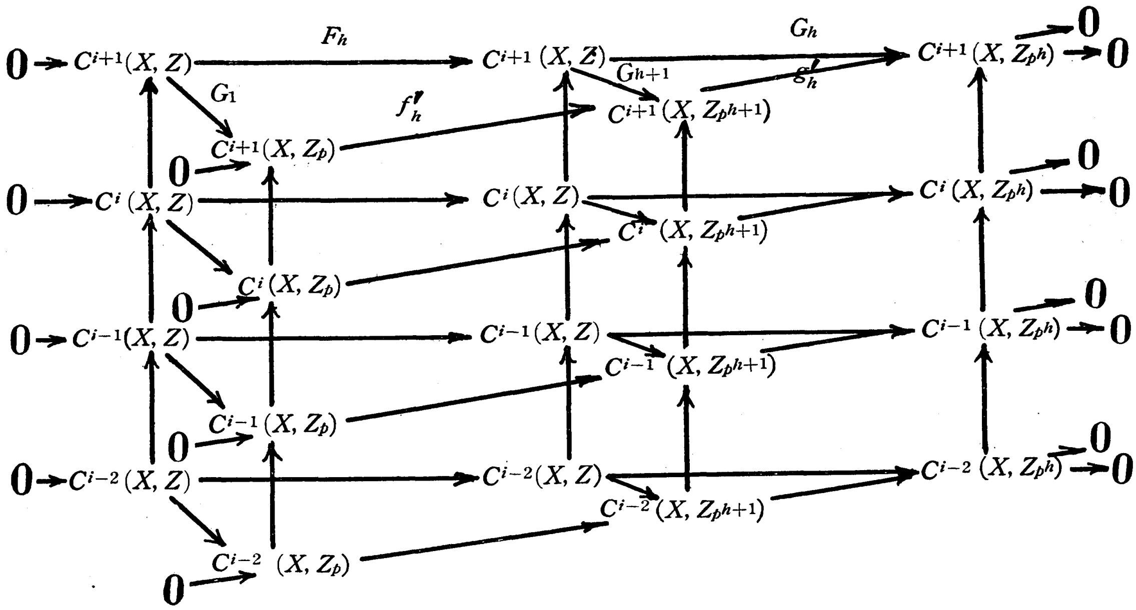

Diagram 4.

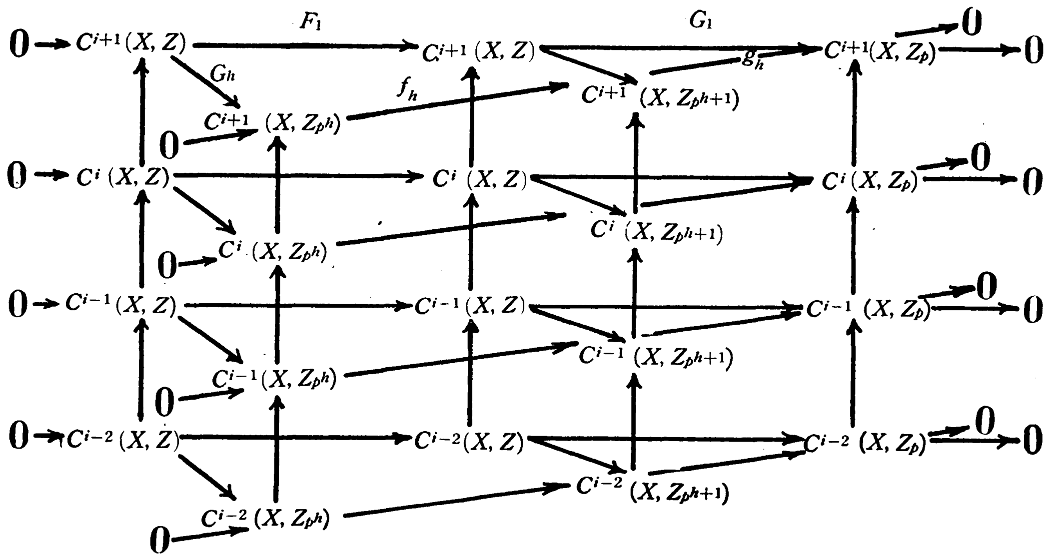

Diagram 5.

§ 2. Definition of Δ_p^i and its properties

1. Definition of Δ_p^i. The cohomological operations Δ_p^i $(i=1, 2,\cdots)$ is inductively defined as follows.

Operation Δ_p^1 is defined as the coboundary operator associated with the exact sequence

$$0 \longrightarrow Z_p \longrightarrow Z_{p^2} \longrightarrow Z_p \longrightarrow 0 .$$

Thus Δ_p^1 is nothing else than δ_1 of Example § 1, 2. 5.

Assume the operation Δ_p^i are defined $i=1,\cdots, h-1, (h>1)$ and the Δ_p^{h-1} kernel coincides with the δ_{h-1} kernel. Then we shall define as follows

$$\Delta_p^h : \Delta_p^{h-1} \text{ kernel } (\subset H^{n-1}(X, Z_p)) \rightarrow H^n(X, Z_p) \text{ mod } \delta'_{h-1} \text{ image.}$$

Obviously we have the following commutative diagram:

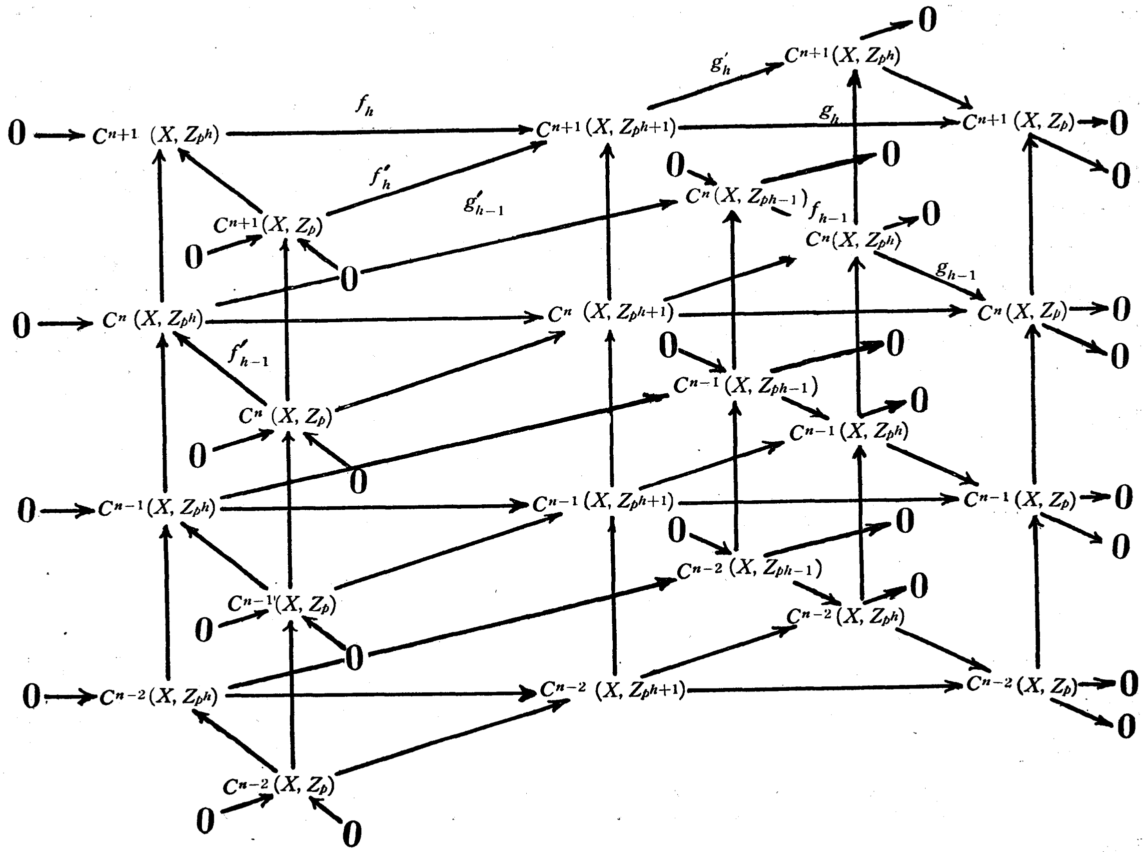

Diagram 6.

Let $\mathfrak{c}$ be an element of $H^{n-1}(X, Z_p)$ whose Δ_p^{h-1} image is zero. Then by the exactness of the following sequence

$$\cdots\longrightarrow H^{n-1}(X, Z_{p^{h-1}})\xrightarrow{f_{h-1}} H^{n-1}(X, Z_{p^h})\xrightarrow{g_{h-1}} H^{n-1}(X, Z_p)\xrightarrow{\delta_{h-1}}$$
$$H^n(X, Z_{p^{h-1}})\longrightarrow\cdots,$$

$\mathfrak{c}$ is contained in the g_{n-1} image of $H^{n-1}(X, Z_{p^h})$. That is, take a representative cocycle c of class $\mathfrak{c}$, then there exists a cocycle $c_1 \in C^{n-1}(X, Z_{p^h})$ such that $g_{h-1}(c_1)=c$. g_h' is epimorphic, and so we have $g_h'(c_2)=c_1$ for some cochain c_2 of $C^{n-1}(X, Z_{p^{h+1}})$. Then $g_h'\circ\delta(c_2)=\delta\circ g_h'(c_2)=\delta(c_1)=0$. Since f_h' is monomorphic, there exists a unique element c_3 of $C^n(X, Z_p)$ such that $f_h'(c_3)=\delta(c_2)$. Clearly c_3 is a cocycle. Then $\Delta_p^h(\mathfrak{c})$ is defined as the cohomology class of mod. δ_{h-1}' image. The class $\{c_3\}$ does not depend on the choice of c_2 out of $C^{n-1}(X, Z_{p^{h+1}})$, and, of course on the choice of c, but depends on the class c_1. c beeing given, we can replace $\{c_1\}$ by $\{c_1+c_1'\}$, where $c_1'=f_{h-1}(c_1''), c_1''$ being a cocycle in $C^{n-1}(X, Z_{p^{h-1}})$. Since g'_{h-1} is epimorphic, there exists a cochain $c_2''\in C^{n-1}(X, Z_{p^h})$ whose g'_{h-1} image coincides with c_1''. Since $g'_{n-1}\circ\delta(c_2'')=0$, there is an element $c_3''\in C^n(X, Z_p)$ whose f'_{h-1} image is $\delta(c_2'')$. Thus the δ'_{h-1} image of the class $\{c_1'\}$ is the class $\{c_3''\}$. This completes the definition of the operation Δ_p^h.

Furthermore we prove now that the Δ_p^h kernel is the δ_h kernel. First, let us show the δ_h kernel is contained in the Δ_p^h kernel. Let c be a cocycle of $C^{n-1}(X, Z_p)$ such that $\delta_h(\{c\})=0$, then there exists a cocycle $c'\in C^{n-1}(X, Z_{p^{h-1}})$ with $g_h(c')=c$ by the exactness of the following sequence

$$\cdots\longrightarrow H^{n-1}(X, Z_{p^h})\xrightarrow{f_h} H^{n-1}(X, Z_{p^{h+1}})\xrightarrow{g_h} H^{n-1}(X, Z_p)\longrightarrow H^n(X, Z_{p^h})\longrightarrow\cdots.$$

As $\delta(c')=0$, we have $\Delta_p^h(\{c\})=0$.

Second, we prove that the Δ_p^h kernel is contained in the δ_h kernel. In using the above notations, we may write $c_3=c_3''+\delta(b)$, $b\in c^{n-1}(X, Z_p)$.

Then we have

$$f_h'(c_3)=\delta(c_2)=f_h'(c_3'')+f_h'\circ\delta(b)=\delta\circ f_h(c_2'')+\delta\circ f_h'(b),$$

hence

$$\delta(c_2-f_h(c_2'')-f_h'(b))=0.$$

Clearly

$$\begin{aligned}&g_h(c_2-f_h(c_2'')-f_h'(b))\\&=g_{h-1}\circ g_h'(c_2)-g_h\circ f_h(c_2'')-g_h\circ f_h\circ f_{h-1}'(b)=c.\end{aligned}$$

This implies $\delta_h\{c\}=0$.

Thus the cohomological operations Δ_p^i are now defined for all $i=1, 2, 3,\cdots$.

From the definition of Δ_p^h and § 1 Example 2.4, we have

THEOREM 1.1. *Let $H_*(X, Z)$ be of finite type in all degrees, and n' be the number of cyclic direct summands of $H_{n-1}(X, Z)$ with order p^i. Then the Δ_p^i image in $H^n(X, Z_p)$ mod δ'_{i-1} image has the dimension n' over Z_p. The Δ_p^i kernel is equal to the δ_i kernel.*

REMARK 1.2. The operations Δ_p^i can be defined in relative cohomology groups in the same manner.

REMARK 1.3. Similarly we can define corresponding operations in homology groups.

REMARK 1.4. We can define $\Delta^1_{p^\lambda}$ as the coboundary operator associated with the exact sequence

$$0 \longrightarrow Z_{p^\lambda} \longrightarrow Z_{p^{2\lambda}} \longrightarrow Z_{p^\lambda} \longrightarrow 0\,,$$

and define inductively $\Delta^i_{p^\lambda}$ $(i=2, 3, \cdots)$ similarly as above.

2. The fundamental properties of the operations Δ^i_p

From the definition of Δ^i_p follows immediately

2.1. The operations Δ^i_p commute with the homomorphism of cohomology groups induced by mapping of spaces:

$$\begin{array}{ccc} \delta_{i-1} \text{ kernel } (\subset H^n(X, Z_p)) & \xrightarrow{\Delta^i_p} & H^{n+1}(X, Z_p)/\delta'_{i-1} \text{ image} \\ f^* \uparrow & & f^* \uparrow \\ \delta_{i-1} \text{ kernel } (\subset H^n(Y, Z_p)) & \xrightarrow{\Delta^i_p} & H^{n+1}(Y, Z_p)/\delta'_{i-1} \text{ image.} \end{array}$$

From Proposition 2.7 in § 1 follows further

2.2. The operations Δ^i_p commutes with the coboundary homomorphism Δ of cohomology sequence up to sign:

$$\Delta^i_p \circ \Delta = -\Delta \circ \Delta^i_p\,.$$

2.1 and 2.2 imply

2.3. The operations Δ^i_p commute with the transgression τ up to sign:

$$\Delta^i_p \circ \tau = -\tau \circ \Delta^i_p\,.$$

Now we have

2.4. $\Delta^j_p \circ \Delta^i_p$ has the natural meaning and is

$$\Delta^j_p \circ \Delta^i_p = 0\,.$$

Especially $$\Delta^i_p \circ \Delta^i_p = 0\,.$$

Since Δ^i_p image $=\delta'_i$ image$/\delta'_{i-1}$ image, this follows from the following Lemma.

LEMMA 2.5. $\delta_j \circ \delta'_i = 0$.

PROOF. The δ'_i image of $H^{n-1}(X, Z_{p^i})$ is contained in the G_1 image of $H^n(X, Z)$ by § 1, Prop. 2.10 and the G_1 image of $H^n(X, Z)$ is contained in the g_j image of $H^n(X, Z_{p^{j+1}})$ by $G_1 = g_j \circ G_{j+1}$. Then $\delta_j \circ \delta'_i = 0$ follows immediately from the exactness of the following sequence.

$$H^n(X, Z_{p^{j-1}}) \xrightarrow{g_j} H^n(X, Z_p) \xrightarrow{\delta_j} H^{n+1}(X, Z_{p^j}) .$$

We have furthermore

2.6. Let α and β be respectively elements of $H^m(X, Z_p)$ and $H^n(X, Z_p)$ satisfying $\delta_{i-1}(\alpha)=\delta_{i-1}(\beta)=0$, then $\Delta_p^i(\alpha\beta)$ is definable, as $\delta_{i-1}(\alpha\beta)=0$ follows from the assumption $\delta_{i-1}(\alpha)=\delta_{i-1}(\beta)=0$. That $\Delta_p^i(\alpha)\beta$ and $\alpha\Delta_p^i(\beta)$ have the natural meanings follows from

$$(\delta_{i-1} \text{ kernel}) (\delta'_{i-1} \text{ image}) \subset \delta'_{i-1} \text{ image}.$$

Finally the equality

$$\Delta_p^i(\alpha\beta)=\Delta_p^i(\alpha)\beta+(-1)^m\alpha\Delta_p^i(\beta) ,$$

follows from § 1 Proposition 2.9.

Let us put δ_h kernel $=J_h$, δ'_h image $=J'_h$ $(h=1, 2,\cdots)$ $\bigcup_h J'_h=J'_\infty$, $\bigcap_h J_h=J_\infty$. Then we have

PROPOSITION 2.7.

$$J'_1\subset J'_2\subset\cdots\subset J'_h\subset\cdots\subset J'_\infty\subset J_\infty\subset\cdots\subset J_h\subset\cdots J_2\subset J_1 .$$

PROOF. $J_h=\delta_h$ kernel $\supset\delta_{h+1}$ kernel $=J_{h+1}$ is clear from the definition of the operations Δ_p^i.

Now let u be a cocycle of $C^{n-1}(X, Z_{p^h})$, then the δ'_h image of the class $\{u\}$ is equal to the δ'_{h+1} image of the class $\{f_h(u)\}$. (See diagram (5)). Hence follows $J'_{h+1}=\delta'_{h+1}$ image $\supset\delta'_h$ image $=J'_h$.

PROPOSITION 2.8. *We have*

(1) *if* $h\leqq k$ $\quad J_h\cdot J_k\subset J_h$

(2) $\quad J'_h\cdot J_k\subset J_k$ *for arbitrary* h, k,

(3) *if* $h\leqq k$ $\quad J'_h\cdot J_k\subset J'_h$,

(4) *if* $h\leqq k$ $\quad J'_h\cdot J'_k\subset J'_h$.

PROOF. (1) follows from 2.6 and Theorem 1.1, and (2) from (1) and Proposition 2.7.

As for (3), let α'', α', α and β be respectively a cocycle of $C^r(X, Z_{p^h})$, a cochain of $C^r(X, Z_{p^{h+1}})$, a cochain of $C^{r+1}(X, Z_p)$ and a cocycle of $C^s(X, Z_{p^k})$ such that $g'_h(\alpha')=\alpha''$, $f'_h(\alpha)=\delta(\alpha')$ and $\delta_k(\{\beta\})=0$. Then β is the g_k image of λ which is a cocycle, by the exactness of the sequence

$$\cdots\longrightarrow H^s(X, Z_{p^k}) \xrightarrow{f_k} H^s(X, Z_{p^{k+1}}) \xrightarrow{g_k} H^s(X, Z_p) \xrightarrow{\delta_k} H^{s+1}(X, Z_{p^k}) \longrightarrow\cdots$$

and by the commutativity of the following diagram

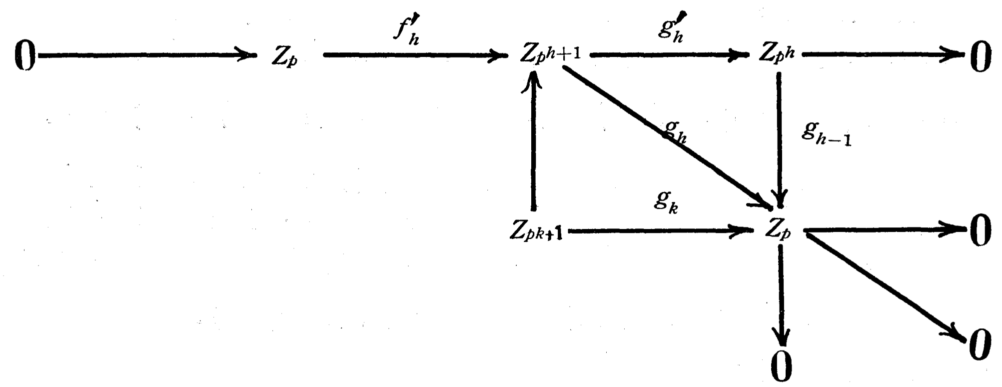

Diagram 7.

Take a cocycle $\alpha'' g_h'(\lambda)$ of $C^{r+s}(X, Z_{p^h})$, then we have

$$\alpha'' g_h'(\lambda) = g_h'(\alpha') g_h'(\lambda) = g_h'(\alpha'\lambda),$$

and

$$\delta(\alpha'\lambda) = \delta(\alpha')\lambda + (-1)^r \alpha'\delta(\lambda) = f_h'(\alpha)\lambda$$
$$= f_h'(\alpha g_h(\lambda)) = f_h'(\alpha\beta).$$

This completes the proof of (3), (4) follows immediately from (3), and Prop. 2.7 Q. E. D.

REMARK 2.9. The relations (1), (2), (3), (4) are also induced by a certain filtration of $C(X, Z)$.

From Proposition 2.8 follows

PROPOSITION 2.10. *J_h/J_h' is an algebra over Z_p with a differential operator $\bar{\Delta}_p^{h+1}$ for each h. Here $\bar{\Delta}_p^{h+1}$ is the operator induced by Δ_p^{h+1} and has a property $\bar{\Delta}_p^{h+1} \circ \bar{\Delta}_p^{h+1} = 0$.*

The structure of these algebras for Eilenberg-MacLane complex $K(\Pi, n)$ has been determined by T. Nakamura.

PROPOSITION 2.11. *Let v be an element of the $H^r(X, Z_2)$ with $\delta_{h-1}(v) = 0$ (δ_0 may be defined as the zero operator), then we have*

$$\Delta_2^{h+1}(v^2) = \begin{cases} Sq^r \circ \Delta_2^1(v) + v\Delta_2^1(v) & \text{if } r \text{ is even and } h = 1 \\ v\Delta_2^h(v) & \text{if } r \text{ is even and } h > 1. \\ 0 & \text{if } r \text{ is odd.} \end{cases}$$

PROOF. If r is odd, then $v^2 = Sq^r v = Sq^1 Sq^{r-1} v = \Delta_2^1(Sq^{r-1} v)$, and so the above result is clear by 2.4.

If r is even, we have $\Delta_2^h(v^2)=0$ by 2.6. So we can compute $\Delta_2^{h+1}(v^2)$ as follows. Take a representative cocycle v' of v, then v' is the g_{h-1} image of a cocycle $v''\in C^r(X, Z_{2^h})$ by the exactness of the sequence

$$\cdots\to H^r(X, Z_{2^{h-1}})\to H^r(X, Z_{2^h})\to H^r(X, Z_2)\to H^{r+1}(X, Z_{2^{h-1}})\to\cdots .$$

We consider a cochain $u\smile_0 u+u\smile_1\delta(u)$, where $\smile_i$ is Steenrod's i-product, and u is an integral cochain whose G_h image is v'' (see § 1 Proposition 2.10). Then Steenrod's coboundary formula shows that

$$\delta(u\smile_0 u+u\smile_1\delta(u))=2u\smile_0\delta(u)+\delta u\smile_1\delta(u) .$$

Therefore we have

$$\begin{aligned}&F_{h+1}(u\smile_0 F_h^{-1}\circ\delta(u)+2^{h-1}F_h^{-1}\circ\delta(u)\smile_1 F_h^{-1}\circ\delta(u))\\&=2u\smile_0\delta(u)+\delta(u)\smile_1\delta(u) ,\end{aligned}$$

and

$$\begin{aligned}&G_1(u\smile_0 F_h^{-1}\circ\delta(u)+2^{h-1}F_h^{-1}\circ\delta(u)\smile_1 F_h^{-1}\circ\delta(u))\\&=\begin{cases}v'\smile_0 G_1\circ F_h^{-1}\circ\delta(u)+G_1\circ F_h^{-1}\circ\delta(u)\smile_1 G_1\circ F_h^{-1}\circ\delta(u) & \text{for } h=1\\ v'\smile_0 G_1\circ F_h^{-1}\circ\delta(u) & \text{for } h>1 .\end{cases}\end{aligned}$$

This proves our Proposition.

REMARK 2.12. The above proof shows that $\delta_2'\circ P_2^1(v)$ mod δ_1' image coincides with $\Delta_2^2(v^2)$, where $P_2^1(v)$ means the Pontryagin square of v. (See [11]).

PROPOSITION 2.13. *Let v be an element of $H^r(X, Z_p)$ with $\delta_{h-1}(v)=0$, and p be an odd prime, then we have*

$$\Delta_p^{h+1}(v^p)=v^{p-1}\Delta_p^h(v) .$$

PROOF. Let D_i be the Steenrod's D_i operator. Then we have

(1) $D_i\circ\delta(u)+(-1)^{i+1}\delta\circ D_i(u)=-D_{i-1}(u)+D_{i+1}\circ T(u)$ if i is odd,

(2) $D_i\circ\delta(u)+(-1)^{i+1}\delta\circ D_i(u)=D_{i-1}\circ(1+T+\cdots+T^{p-1})(u)$ if i is even,

where $T(u)(\sigma_1\times\cdots\times\sigma_{p-1}\times\sigma_p)=(-1)^{D(\sigma_p)\sum_{i=1}^{p-1}D(\sigma_i)}u(\sigma_p\times\sigma_1\times\cdots\times\sigma_{p-1})$, and $D(\sigma_j)$ denotes the degree of σ_j.

We have only to consider the case, where r is even. Then we have $\Delta_p^h(v^p)=0$ by 2.6. Let v' be a cochain such that $\{G_1(v')\}=v$ and $\delta(v')=p^h v''$. Then by (2) we have

$$\delta\circ D_0(v'\otimes\cdots\otimes v')=D_0\circ\delta(v'\otimes\cdots\otimes v')$$
$$=\sum_{j_1=1}^{p}D_0(v'\otimes\cdots\otimes\overset{(j_1)}{\delta(v')}\otimes\cdots\otimes v'),$$

where (j_1) on $\delta(v')$ indicates that $\delta(v')$ is at the j_1-th place in the product $v'\otimes\cdots\otimes\delta(v')\otimes\cdots\otimes v'$. The same notation will be used in the following.

Also by (1) we have

$$D_1\circ\delta(v'\otimes\cdots\otimes\overset{(j_1)}{\delta(v')}\otimes\cdots\otimes v')+\delta\circ D_1(v'\otimes\cdots\otimes\overset{(j_1)}{\delta(v')}\otimes\cdots\otimes v')$$
$$\begin{cases}=-D_0(v'\otimes\cdots\otimes\overset{(j_1)}{\delta(v')}\otimes\cdots\otimes v')+D_0(v'\otimes\cdots\otimes\overset{(j_1+1)}{\delta(v')}\otimes\cdots\otimes v') & \text{for } j_1\leqq p-1\\ =-D_0(v'\otimes\cdots\otimes\delta(v'))+D_0(\delta(v')\otimes v'\cdots\otimes v') & \text{for } j_1=p.\end{cases}$$

Furthermore the following equality holds

$$D_1\circ\delta(v'\otimes\cdots\otimes\overset{(j_1)}{\delta(v')}\otimes\cdots\otimes v')+\delta\circ D_1(v'\otimes\cdots\otimes\overset{(j_1)}{\delta(v')}\otimes\cdots\otimes v')$$
$$=D_1\Big(\sum_{1\leqq j_2<j_1}v'\otimes\cdots\otimes\overset{(j_2)}{\delta(v')}\otimes\cdots\otimes\overset{(j_1)}{\delta(v')}\otimes\cdots\otimes v'$$
$$-\sum_{j_1<j_2\leqq p}v'\otimes\cdots\otimes\overset{(j_1)}{\delta(v')}\otimes\cdots\otimes\overset{(j_2)}{(v')}\otimes\cdots\otimes v')\Big)+\delta\circ D_1(v'\otimes\cdots\otimes\overset{(j_1)}{\delta(v')}\otimes\cdots\otimes v').$$

Therefore we have

$$\delta\circ D_0(v'\otimes\cdots\otimes v')=-\sum_{k=1}^{p}kD_1\Big(\sum_{1\leqq j_2<k}v'\otimes\cdots\otimes\overset{(j_2)}{\delta(v')}\otimes\cdots\otimes\overset{(k)}{\delta(v')}\otimes$$
$$\cdots\otimes v'-\sum_{k<j_2}v'\otimes\cdots\otimes\overset{(k)}{\delta(v')}\otimes\cdots\otimes\overset{(j_2)}{\delta(v')}\otimes\cdots\otimes v')+\delta\circ D_1(v'\otimes\cdots$$
$$\otimes\overset{(k)}{\delta(v')}\otimes\cdots\otimes v')\Big)+pD_0(\delta(v')\otimes v'\otimes\cdots\otimes v')$$
$$=-D_1\sum_{1\leqq m<n\leqq p}(n-m)v'\otimes\cdots\otimes\overset{(m)}{\delta(v')}\otimes\cdots\otimes\overset{(n)}{\delta(v')}\otimes\cdots\otimes v'-\sum_{k=1}^{p}k\delta\circ D_1$$
$$(v'\otimes\cdots\otimes\overset{(k)}{\delta(v')}\otimes\cdots\otimes v')+pD_0(\delta(v')\otimes v'\otimes\cdots\otimes v').$$

Thus we have

(3) $$\delta[D_0(v'\otimes\cdots\otimes v')+\sum_{k=1}^{p}kD_1(v'\otimes\cdots\otimes\overset{(k)}{\delta(v')}\otimes\cdots\otimes v')]$$

$$=pD_0(\delta(v')\otimes v'\otimes\cdots\otimes v')-D_1\sum_{1\leqq m<n\leqq p}(n-m)v'\otimes\cdots\otimes\overset{(m)}{\delta(v')}\otimes\cdots$$
$$\otimes\overset{(n)}{\delta(v')}\otimes\cdots\otimes v'.$$

By (2) we have

$$D_2\circ\delta(v'\otimes\cdots\otimes\overset{(m)}{\delta(v')}\otimes\cdots\otimes\overset{(n)}{\delta(v')}\otimes\cdots\otimes v')-\delta\circ D_2(v'\otimes\cdots$$
$$\otimes\overset{(m)}{\delta(v')}\otimes\cdots\otimes\overset{(n)}{\delta(v')}\otimes\cdots\otimes v')$$
$$=D_1(\sum_{j=0}^{p-n}v'\otimes\cdots\otimes\overset{(m+j)}{\delta(v')}\otimes\cdots\otimes\overset{(n+j)}{\delta(v')}\otimes\cdots\otimes v'-\sum_{j=1}^{n-m}v'\otimes\cdots$$
$$\otimes\overset{(j)}{\delta(v')}\otimes\cdots\otimes\overset{p-n+m+j}{\delta(v')}\otimes\cdots\otimes v'+\sum_{j=1}^{m-1}v'\otimes\cdots\otimes\overset{(j)}{\delta(v')}\otimes\cdots\otimes\overset{(n-m+j)}{\delta(v')}\otimes\cdots\otimes v'$$
$$=D_1(\sum_{j=1}^{p+m-n}v'\otimes\cdots\otimes\overset{(j)}{\delta(v')}\otimes\cdots\otimes\overset{(n-m+j)}{\delta(v')}\otimes\cdots\otimes v'-\sum_{j=1}^{n-m}v'\otimes\cdots\otimes\overset{(j)}{\delta(v')}\otimes$$
$$\cdots\otimes\overset{(p-n+m+j)}{\delta(v')}\otimes\cdots\otimes v').$$

Therefore we have

$$\sum_{1\leqq m<n\leqq p}[D_2\circ\delta(v'\otimes\cdots\otimes\overset{(m)}{\delta(v')}\otimes\cdots\otimes\overset{(n)}{\delta(v')}\otimes\cdots\otimes v')-\delta\circ D_2(v'\otimes$$
$$\cdots\otimes\overset{(m)}{\delta(v')}\otimes\cdots\otimes\overset{(n)}{\delta(v')}\otimes\cdots\otimes v')]$$
$$=\sum_{1\leqq m<n\leqq p}D_1(\sum_{j=1}^{p+m-n}v'\otimes\cdots\otimes\overset{(j)}{\delta(v')}\otimes\cdots\otimes\overset{(n-m+j)}{\delta(v')}\otimes\cdots\otimes v'-\sum_{j=1}^{n-m}v'\otimes$$
$$\cdots\otimes\overset{(j)}{\delta(v')}\otimes\cdots\otimes\overset{(p-n+m+j)}{\delta(v')}\otimes\cdots\otimes v')$$
$$=D_1\sum_{1\leqq m<n\leqq p}[p-2(n-m)]v'\otimes\cdots\otimes\overset{(m)}{\delta(v')}\otimes\cdots\otimes\overset{(n)}{\delta(v')}\otimes\cdots\otimes v'$$
$$=pD_1\sum_{1\leqq m<n\leqq p}v'\otimes\cdots\otimes\overset{(m)}{\delta(v')}\otimes\cdots\otimes\overset{(n)}{\delta(v')}\otimes\cdots\otimes v'-2D_1\sum_{1\leqq m<n\leqq p}$$
$$(n-m)v'\otimes\cdots\otimes\overset{(m)}{\delta(v')}\otimes\cdots\otimes\overset{(n)}{\delta(v')}\otimes\cdots\otimes v'.$$

Finally we have

$$\delta[D_0(v'\otimes\cdots\otimes v')+\sum_{k=1}^{p}kD_1(v'\otimes\cdots\otimes\overset{(k)}{\delta(v')}\otimes\cdots\otimes v')] \tag{4}$$
$$=pD_0(\delta(v')\otimes v'\otimes\cdots\otimes v')+1/2\Big[\sum_{1\leqq m<n\leqq p}D_2\circ\delta(v'\otimes\cdots\otimes\overset{(m)}{\delta(v')}\otimes$$

$$\cdots\otimes\delta(\overset{(n)}{v'})\otimes\cdots\otimes v') - \sum_{1\leqq m<n\leqq p} \delta\cdot D_2(v'\otimes\cdots\otimes\delta(\overset{(m)}{v'})\otimes\cdots\otimes\delta(\overset{(n)}{v'})\otimes\cdots\otimes v')$$

$$-pD_1\sum_{1\leqq m<n\leqq p} v'\otimes\cdots\otimes\delta(\overset{(m)}{v'})\otimes\cdots\otimes\delta(\overset{(n)}{v'})\otimes\cdots\otimes v'\Big].$$

Obviously the G_{h+1} image of the left side of (4) is zero, and the G_1 image of

$$(5)\qquad D_0(v'\otimes\cdots\otimes v') + \sum_{k=1}^{p} kD_1(v'\otimes\cdots\otimes\delta(\overset{(k)}{v'})\otimes\cdots\otimes v')$$

coincides with $(G_1(v'))^p$. (4) shows that

$$\{G_1 1/p^{h+1}\delta[D_0(v'\otimes\cdots\otimes v') + \sum_{k=1}^{p} kD_1(v'\otimes\cdots\otimes\delta(\overset{(k)}{v'})\otimes\cdots\otimes v')]\}$$

$$= \Delta_p^{h+1}(v)v^{p-1}.$$

REMARK 2.13. (3) gives us a proof of Proposition 2.11 if $p=2$ and r is even.

THEOREM 2.14. *Let $x\in H^k(F, Z_p)$ be a transgressive element of even degree, and let $y\in H^{k+1}(B, Z_p)$ be an image of x under the transgression, then*

$$d_r\kappa_r^2(y\otimes x^{p-1}) = 0 \qquad \textit{for } 2\leqq r\leqq k(p-1)$$

and

$$d_s\kappa_s^2(y\otimes x^{p-1}) = \kappa_s^2(-\Delta_p^1\circ\mathfrak{P}^{k/2}(y)\otimes 1),$$

where $s=k(p-1)+1$.

This theorem has been proved independently by T. Kudo [16] and by T. Nakamura. Partly it has been also proved by A. Borel [7]. (On the notation, see [3]).

§ 3. The operation Δ_p^i and the cohomology sequence

Let $E/F=B$ be a fibering of a space E, and A a principal ring.

Then Serre [6] has proved: the sequences in the following commutative diagram

$$
3.1\quad
\begin{array}{ccccccccc}
H^{\lambda+\mu-1}(F,A) & \xleftarrow{i^*} & H^{\lambda+\mu-1}(E,A) & \xleftarrow{p^*} & H^{\lambda+\mu-1}(B,A) & \xleftarrow{\tau} & H^{\lambda+\mu-2}(F,A) & \leftarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
H^{\lambda+\mu-1}(F,A) & \xleftarrow{i^*} & H^{\lambda+\mu-1}(E,A) & \xleftarrow{j^*} & H^{\lambda+\mu-1}(E,F;A) & \xleftarrow{\Delta} & H^{\lambda+\mu-2}(F,A) & \leftarrow & \cdots
\end{array}
$$

$$
\begin{array}{ccccccccc}
\leftarrow H^2(B,A) & \longleftarrow & H^1(F,A) & \leftarrow & H^1(E,A) & \leftarrow & H^1(B,A) & \longleftarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\leftarrow H^2(E,F;A) & \leftarrow & H^1(F,A) & \leftarrow & H^1(E,A) & \leftarrow & H^1(E,F;A) & \leftarrow & 0
\end{array}
$$

is exact under the conditions that the local system formed by $H^i(F,A)$ is trivial for each $i\geqq 0$, and $H^i(B,A)=0$ for

$$0<i<\lambda \text{ and } H^i(F,A)=0 \text{ for } 0<i<\mu .$$

Hence we can obtain some informations about the effect of Δ^i_p. in $H^*(F,Z_p)$ if we know the effects of p^* and of Δ^i_p in $H^*(E,Z_p)$ and in $H^*(B,Z_p)$, in utilizing the lower exact sequence.

The notations i^*, j^* will keep the meaning in this lower sequence also in the sequel.

Now we have

THEOREM 3.2. *Let α and β be respectively elements of $H^s(E,Z_p)$ and of $H^{s+1}(E,F;Z_p)$ such that $\delta_{r-1}(\alpha)=0$ and $\Delta^r_p(\alpha)=j^*(\beta)$ mod δ'_{r-1} image. Then we have*

$$\Delta\circ\Delta^{r+}_p\circ i^*(\alpha)=-\Delta^1_p(\beta) \text{ mod } \Delta\circ\delta'_r H^s(F,Z_{p^r}) .$$

PROOF. (Cf. diagram (8)).

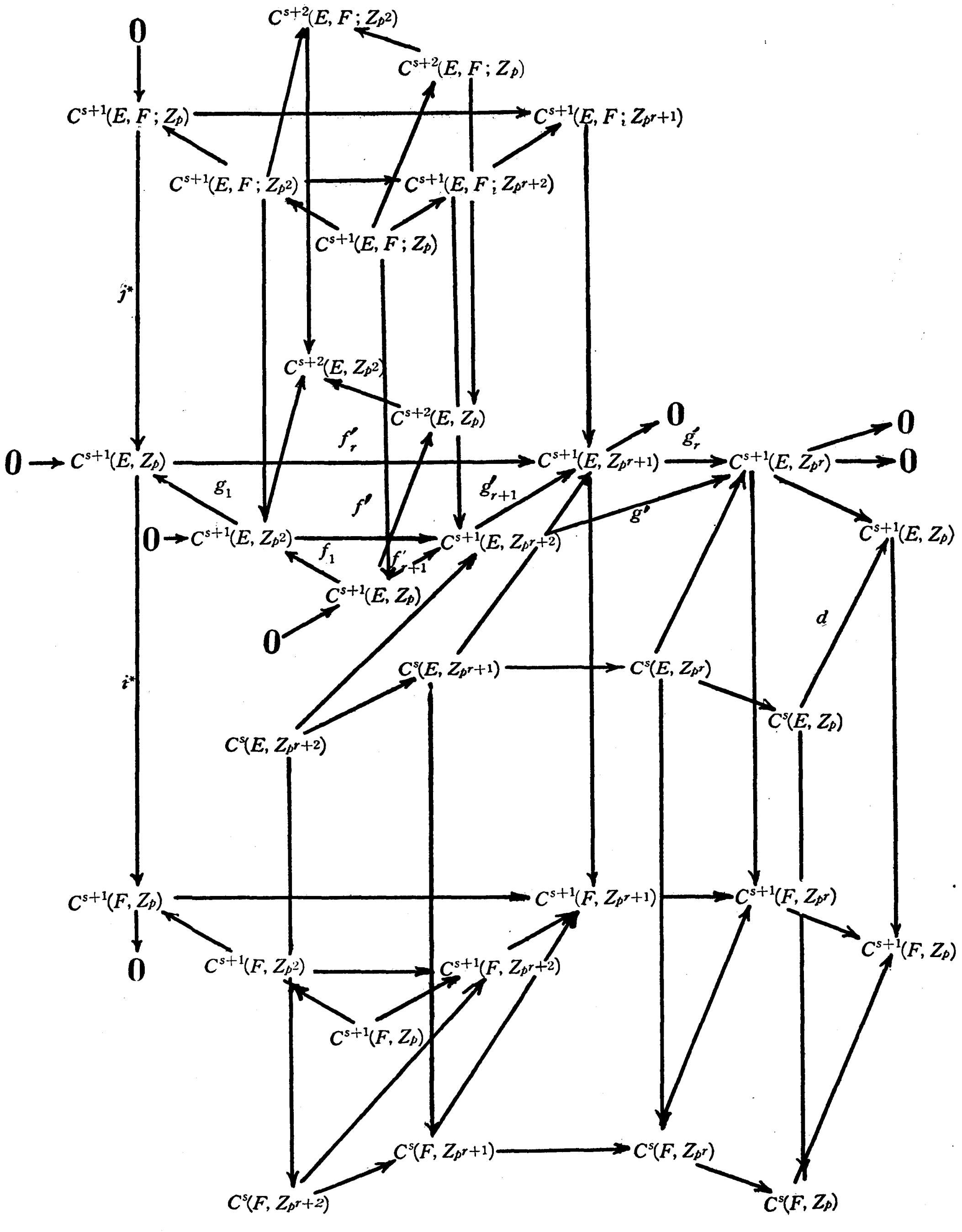

Diagram 8.

Let a, b, c, e and f be respectively a cocycle of $C^s(E, Z_p)$, a cocycle of $C^s(E, Z_{p^r})$, an element of $C^s(E, Z_{p^{r+1}})$, a cocycle of $C^{s+1}(E, F; Z_p)$ and an element of $C^{s+1}(E, F; Z_{p^2})$ such that $\{a\}=\alpha, g_{r-1}(b)=a, g_r'(c)=b, \{e\}=\beta, g_1(f)=e$. By a suitable choice of b and c, we may assume $j^*(e)=f_r'^{-1}\circ d(c)$.

Since $j^*(e)=f_r'^{-1}\circ d(c), i^*(c)$ is a cocycle. Let c' be an element of $C^{s+1}(E, Z_{p^{r+2}})$ whose g'_{r+1} image coincides with c. Since

$$g'_{r+1}\circ i^*\circ d(c')=g'_{r+1}\circ d\circ i^*(c')=d\circ g'_{r+1}\circ i^*(c')=d\circ i^*\circ g'_{r+1}(c')=d\circ i^*(c)=0$$

and i^* is epimorphic, there exists an element $g\in C^{s+1}(E, Z_p)$ whose i^* image is $f'^{-1}_{r+1}\circ i^*\circ d(c')$. The class $\{f'^{-1}_{r+1}\circ i^*\circ d(c')\}$ mod δ_r' image coincides with $\Delta_p^{r+1}\circ i^*(\alpha)$. Obviously we have $i^*\circ f'_{r+1}(g)=i^*\circ d(c')$, that is $i^*(f'_{r+1}(g)-d(c'))=0$. Therefore there exists an element $h\in C^{s+1}(E, F; Z_{p^{r+2}})$ such that

$$j^*(h)=f'_{r+1}(g)-d(c')\,.$$

Further

$$\begin{aligned} g'_{r+1}\circ j^*(h) &= g'_{r+1}(f'_{r+1}(g)-d(c')) = -g'_{r+1}\circ d(c') \\ &= -d\circ g'_{r+1}(c') = -d(c) = -g'_{r+1}\circ f'\circ j^*(f) \end{aligned}$$

because

$$g'_{r+1}\circ f'\circ j^*(f)=f'_r\circ g_1\circ j^*(f)=f'_r\circ j^*\circ g_1(f)=f'_r\circ j^*(e)=d(c)\,.$$

This implies

$$g'_{r+1}(h+f'(f))=0\,.$$

Therefore we have $k\in C^{s+1}(E, F; Z_p)$ such that

$$f'_{r+1}(k)=f'\circ f_1(k)=h+f'(f)\,.$$

The above relations show that

$$f'(f-f_1(k))+h=0\,.$$

Thus

$$\begin{aligned} f'\circ j^*(f-f_1(k)) &= j^*\circ f'(f-f_1(k)) \\ &= -j^*(h) = -(f'_{r+1}(g)-d(c'))\,. \end{aligned}$$

This implies

$$\begin{aligned} d\circ f'\circ j^*(f-f_1(k)) &= f'\circ j^*\circ d(f-f_1(k)) \\ &= -d\circ f'_{r+1}(g) = -f'\circ f_1\circ d(g)\,. \end{aligned}$$

Thus

$$j^*\circ d(f-f_1(k)=j^*\circ f_1\circ f_1^{-1}\circ d(f-f_1(k))$$
$$=f_1\circ j^*\circ f_1^{-1}\circ d(f-f_1(k))=-f_1\circ d(g)\,,$$

because f' is monomorphism. In other words,

$$j^*\circ f_1^{-1}\circ d(f-f_1(k))=-d(g)\,.$$

This shows that

$$\Delta\{i^*(g)\}=\Delta\{f'^{-1}_{r+1}\circ i^*\circ d(c')\}=\Delta\circ\Delta_q^{r+1}\circ i^*(\alpha)$$
$$=-\{f_1^{-1}\circ d(f-f_1(k))\}=-\{f_1^{-1}\circ d(f)\}=-\Delta_p^1(\beta)\,.$$

Q. E. D.

REMARK 3.3. We have also a corresponding theorem to Theorem 2.2 on homology groups. The same remark holds also on the following theorems in § 3. (We give here only theorems on cohomology groups, as we need them solely for our purpose.)

REMARK 3.4. Theorem 3.2 gives an information on the effect of Δ_p^r $(r=1,2,\cdots)$ on $H^*(F,Z_p)$ when we have the knowledge on the effects of j^* on $H^*(E,F;Z_p)$, and of Δ_p^r $(r=1,2,\cdots)$ on $H^*(E,F;Z_p)$ and on $H^*(E,Z_p)$. We can prove in a similar way the following proposition giving an information on the effect of Δ_p^r $(r=1,2,\cdots)$ on $H^*(E,Z_p)$ from the knowledge on the effects of Δ on $H^*(F,Z_p)$, and of Δ_p^r $(r=1,2,\cdots)$ on $H^*(F,Z_p)$ and $H^*(E,F;Z_p)$. (The same remark holds also on the following theorems in § 3).

PROPOSITION. Let α and β be respectively elements of $H^s(E,F;Z_p)$ and of $H^s(F,Z_p)$ such that $\delta_{r-1}(\alpha)=0$ and $\Delta(\beta)=\Delta_p^r(\alpha)$ mod. δ'_{r-1} image. Then we have

$$i^*\circ\Delta_p^{r+1}\circ j^*(\alpha)=-\Delta_p^1(\beta) \quad \bmod i^*\circ\delta'_r H^s(E,Z_{p^r})\,.$$

REMARK 3.5. Theorem 3.3 can be also generalized in the following form:

Let α and β be respectively elements of $H^s(E,Z_p)$ and of $H^{s+1}(E,F;Z_p)$ such that $\delta_{r-1}(\alpha)=0$, $\delta_{t-1}(\beta)=0$ and $\Delta_p^r(\alpha)=j^*(\beta)$ mod δ_{r-1} image. Assume that the elements of $H^{s+1}(E,Z_{p^t})$, each of which has the g_{t-1} image coinciding with $\Delta_p^r(\alpha)$, are contained in the j^* image of $H^{s-1}(E,F;Z_{p^t})$ mod $\delta'_{r-1}H^s(E,Z_{p^r})$. Then we have

$$\Delta\circ\Delta_p^{r+t}\circ i^*(\alpha)=-\Delta_p^t(\beta) \quad \bmod \Delta\cdot\delta'_{r+t-1}H^s(F,Z_{p^{r+t-1}})\,.$$

THEOREM 3.6. *Let α and β be elements of $H^s(E, Z_p)$ and of $H^s(E, F; Z_p)$ such that $\delta_{r-1}(\alpha)=0$ $(r\geqq 2)$, $j^*(\beta)=\alpha$. Then there exists an element $\gamma\in H^s(F, Z_p)$ such that*

$$\Delta(\gamma) = -\Delta_p^1(\beta)\,,$$

$$\Delta_p^{r-1}(\gamma) = i^*\circ\Delta_p^r(\alpha) \bmod i^*\circ\delta'_{r-1}H^s(E, Z_{p^{r-1}}) + \delta'_{r-2}H^s(F, Z_{p^{r-2}})\,.$$

PROOF. (Cf. diagram (9)).

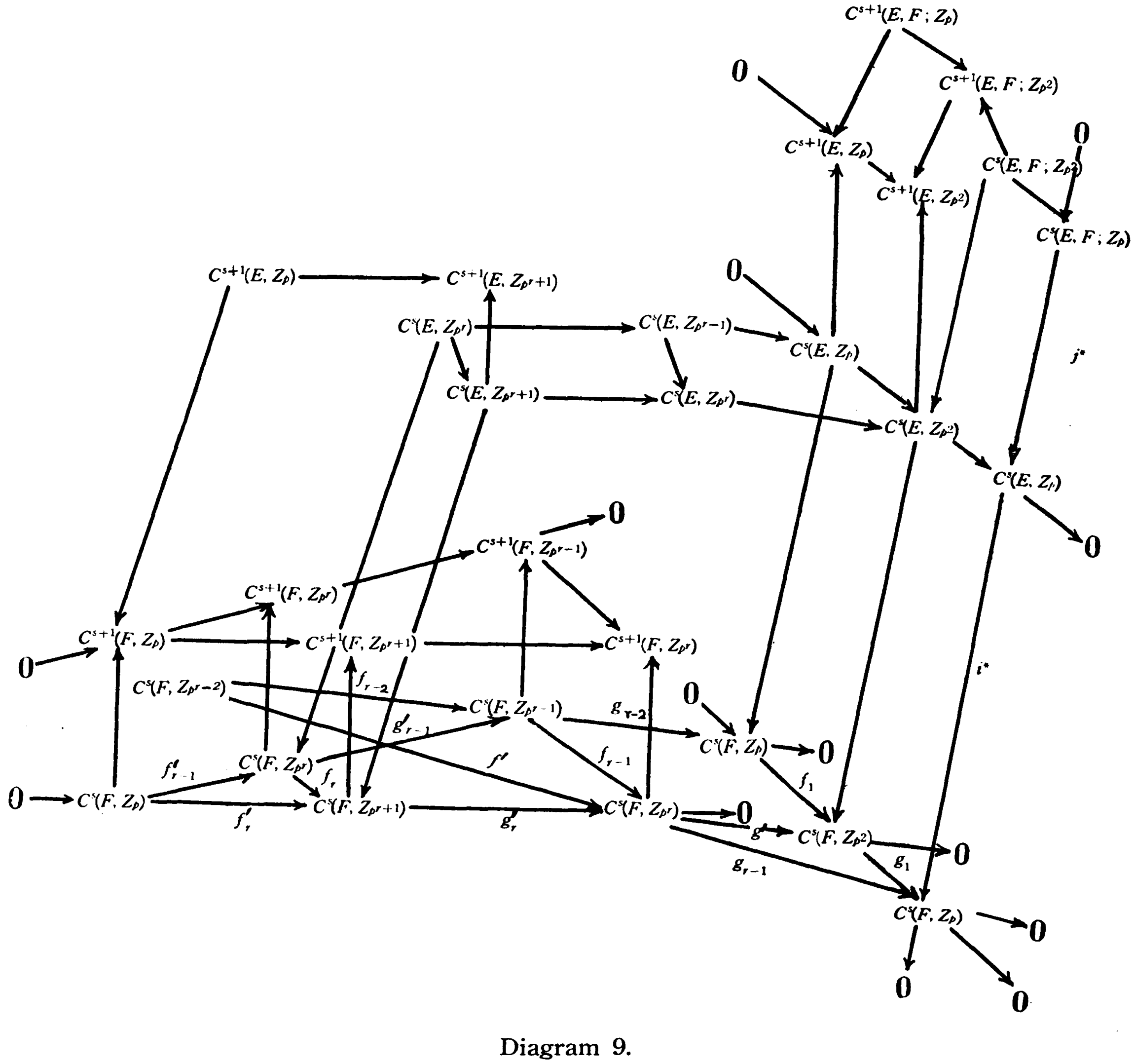

Diagram 9.

Let a, b, c, e and f be respectively a cocycle of $C^s(E, Z_p)$ a cocycle of $C^s(E, Z_{p^r})$, an element of $C^s(E, Z_{p^{r-1}})$, a cocycle of $C^s(E, F; Z_p)$ and an element of $C^s(E, F; Z_{p^2})$ such that $\{a\}=\alpha$, $g_{r-1}(b)=a$, $g'_r(c)=b$, $\{e\}=\beta$, $g_1(f)$

$=e$. Also we may assume without loss of gnerality that the j^* image of e is a. Then there exists an element $g \in C^s(E, Z_p)$ such that $f_1(g) = g'(b) - j^*(f)$, for we have

$$g_{r-1}(b) = g_1 \circ g'(b) = a = j^* \circ g_1(f) = g_1 \circ j^*(f) .$$

Obviously $g'(b)$ is a cocycle. Therefore

$$f_1 \circ d(g) = d \circ f_1(g) = -d \circ j^*(f) = -j^* \circ d(f)$$
$$= -j^* \circ f_1 \circ f_1^{-1} \circ d(f) = -f_1 \circ j^* \circ f_1^{-1} \circ d(f) .$$

Since f_1 is monomorphic, this implies $d(g) = -j^* \circ f_1^{-1} d(f)$. Of course, $\{f_1^{-1} \circ d(f)\}$ coincides with $\Delta_p^1(\beta)$. Therefore we have

$$\Delta\{i^*(g)\} = -\Delta_p^1\{e\} = -\Delta_p^1(\beta) .$$

On the other hand, we have

$$i^* \circ f_1(g) = f_1 \circ i^*(g) = i^*(g'(b) - j^*(f))$$
$$= i^* \circ g'(b) = g' \circ i^*(b) .$$

Since g_{r-2} is epimorphic, there exists an element $a' \in C^s(F, Z_{p^{r-1}})$ such that $g_{r-2}(a') = i^*(g)$. Then we have

$$g' \circ f_{r-1}(a') = f_1 \circ g_{r-2}(a') = f_1 \circ i^*(g) = g' \circ i^*(b) .$$

Therefore there exists an element $b' \in C^s(F, Z_{p^{r-2}})$ such that

$$f'(b') = i^*(b) - f_{r-1}(a') .$$

This implies

$$f'(b') + f_{r-1}(a') = i^*(b) - f_{r-1}(a') + f_{r-1}(a') = i^*(b) .$$

Namely

$$f_{r-1}(f_{r-2}(b') + a') = i^*(b) ,$$

and

$$g_{r-2}(f_{r-2}(b') + a') = g_{r-2}(a') = i^*(g) .$$

Thus

$$d \circ f_{r-1}(f_{r-2}(b') + a') = d \circ i^*(b) = i^* \circ d(b) = 0 .$$

This shows that $f_{r-2}(b') + a'$ is a cocycle, for f_{r-1} is monomorphic. As g'_{r-1} is epimorphic, there exists an element $c' \in C^s(F, Z_{p^r})$ such that

$$g'_{r-1}(c') = f_{r-2}(b') + a' .$$

And so we have

$$g'_r \circ f_r(c') = f_{r-1} \circ g'_{r-1}(c') = f_{r-1}(f_{r-2}(b') + a')$$
$$= i^*(b) = i^* \circ g'_r(c) = g'_r \circ i^*(c) .$$

Therefore there exists an element $e' \in C^s(F, Z_p)$ such that

$$f'_r(e') = i^*(c) - f_r(c') .$$

This implies

$$f'_r(e') = f_r \circ f'_{r-1}(e') = i^*(c) - f_r(c') ,$$

and so

$$f_r(f'_{r-1}(e') + c') = i^*(c) .$$

Further we have

$$f_r \circ d \circ (f'_{r-1}(e') + c') = d \circ f_r(f'_{r-1}(e') + c')$$
$$= d \circ i^*(c) = i^* \circ d(c) = f'_r \circ f'^{-1}_r \circ i^* \circ d(c)$$
$$= f'_r \circ i^* \circ f'^{-1}_r \circ d(c) = f_r \circ f'_{r-1} \circ i^* \circ f'^{-1}_r \circ d(c) .$$

Since f_r is monomorphic, we have

$$d \circ (f'_{r-1}(e') + c') = f'_{r-1} \circ i^* \circ f'^{-1}_r \circ d(c) .$$

This implies that

$\{i^* \circ f'^{-1}_r \circ d(c)\}$ coincides with $\Delta_p^{r-1}\{i^*(g)\}$ mod $i^* \circ \delta'_{r-1}$ image $+ \delta'_{r-2}$ image . Q. E. D.

REMARK 3.7. Theorem 3.6 can be also generalized in the following form:

Let α and β be respectively elements of $H^s(E, Z_p)$ and of $H^s(E, F; Z_p)$ such that $\delta_{r-1}(\alpha) = 0$, $\delta_{t-1}(\beta) = 0$, $r > t > 0$ and $j^*(\beta) = \alpha$. Assume further that the elements of $H^s(E, Z_{p^t})$, each of which has the g_{t-1} image coinciding with α, are contained in the j^* image of $H^s(E, F; Z_{p^t})$. Then there exists an element $\gamma \in H^s(F, Z_p)$ such that

$$\Delta(\gamma) = \Delta_p^t(\beta) ,$$
$$\Delta_p^{r-t}(\gamma) = -i^* \circ \Delta_p^r(\alpha) \mod i^* \circ \delta'_{r-1} H^s(E, Z_{p^{r-1}}) + \delta'_{r-t-1} H^s(F, Z_{p^{r-t-1}}) .$$

THEOREM 3.8. *Let α, β and γ be respectively elements of $H^s(E, Z_p)$, of $H^{s+1}(E, F; Z_p)$ and of $H^s(E, F; Z_p)$ such that*

$$\Delta_p^r(\alpha) = j^*(\beta) \ (r \geqq 2) \text{ and } \alpha = j^*(\gamma) .$$

Then there exists an element ε *of* $H^s(F, Z_p)$ *with the following properties:*

$$\Delta(\varepsilon)=\Delta^1_p(\gamma)\,,$$
$$\Delta\circ\Delta^r_p(\varepsilon)=\Delta^1_p(\beta) \quad \mathrm{mod}\ \Delta\circ\delta'_{r-1}H^s(F, Z_{p^{r-1}})\,.$$

PROOF is left to the reader. (Cf. diagram (10)). This theorem is a sort of combination of Theorem 3.2 and Theorem 3.6.

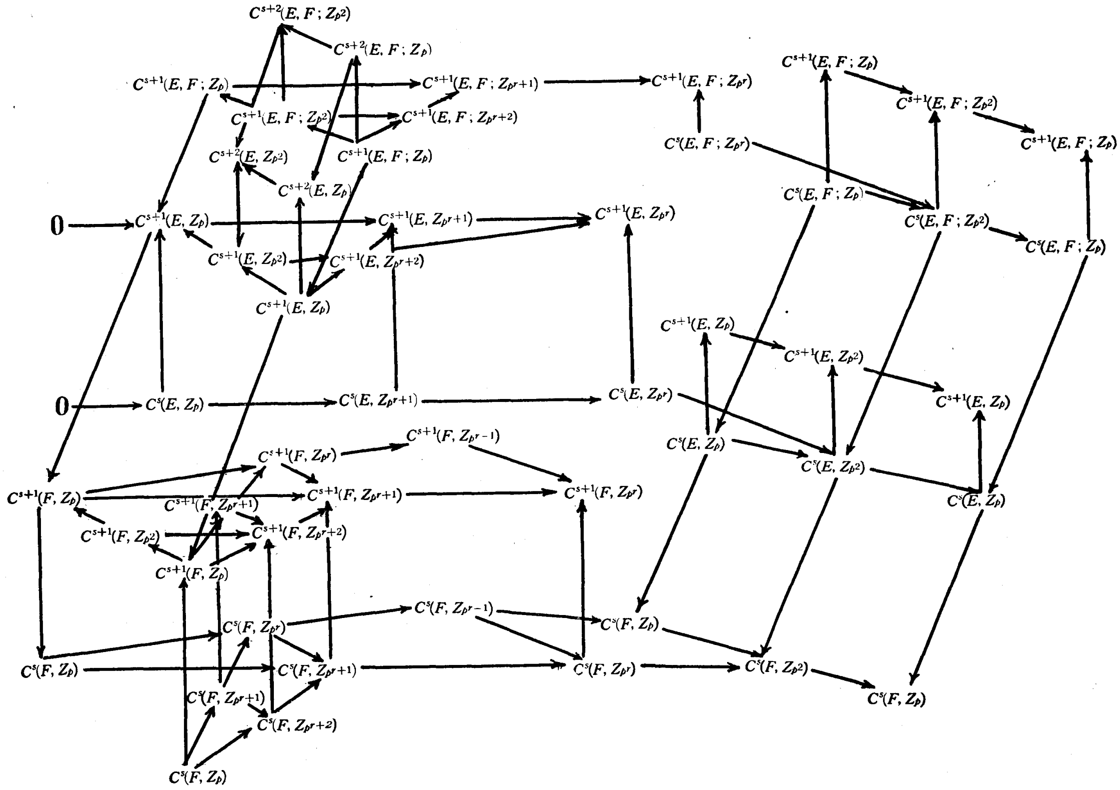

Diagram 10.

REMARK 3.9. Theorem 3.8 can be also generalized like Theorem 3.3 and Theorem 3.6. (Cf. Remark 3.5 and Remark 3.7).

THEOREM 3.10. *Let* α *and* β *be elements of* $H^s(E, Z_p)$ *and of* $H^s(E, F; Z_p)$ *such that* $\delta_{r-1}(\alpha)=0$, $\delta_r(\beta)=0$, $j^*\circ\delta'_rH^s(E, F; Z_{p^r})\subset\delta'_{r-1}H^s(E, Z_p)$ *and* $j^*\circ\Delta^{r+1}_p(\beta)=\Delta^r_p(\alpha)$.

Moreover we assume that $j^*(\beta)=0$, *then there exists an element* γ *of* $H^{s-1}(F, Z_p)$ *satisfying the following conditions*

$$\Delta(\gamma)=\beta\,,$$
$$\Delta^1_p(\gamma)=-i^*(\alpha) \quad \mathrm{mod}\, i^*\circ\delta_r^{-1}(0) \ \ (\subset i^*H^s(E, Z_p))$$

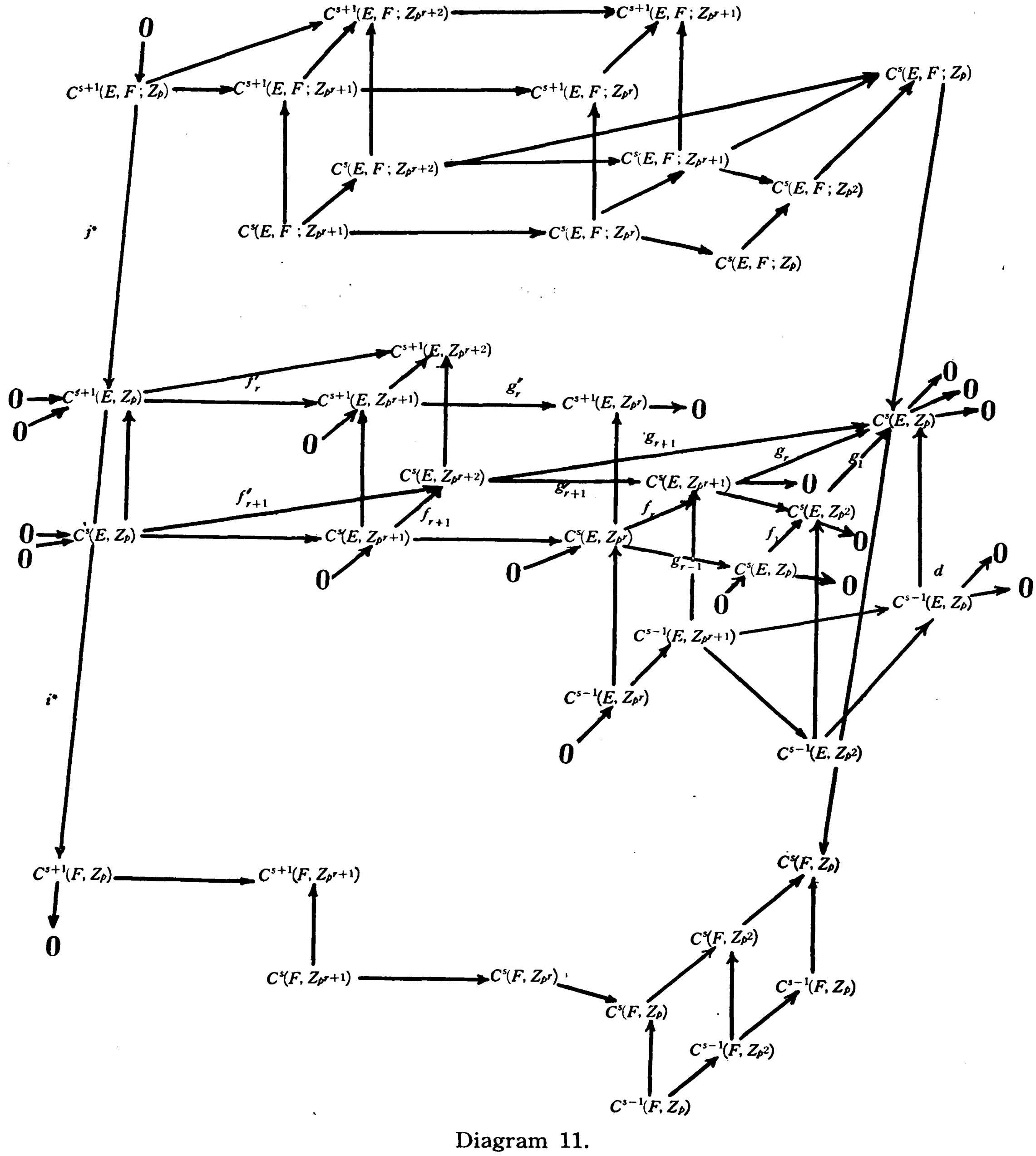

Diagram 11.

PROOF. (Cf. diagram (11)). Let a be a representative cocycle of α such that there exists a cocycle $b \in C^s(E, Z_{p^r})$ with $g_{r-1}(b) = a$. Moreover we assume that b is a g'_r image of $c \in C^s(E, Z_{p^{r+1}})$. On the other hand, let a', b' and c' be respectively an element of $C^s(E, F; Z_p)$ which represents β, a cocycle of $C^s(E, F; Z_{p^{r+1}})$ and an element of $C^s(E, F;$

$Z_{p^{r+2}})$ such that $g_r(b')=a'$ and $g'_{r+1}(c')=b'$.

Also we can assume without loss of generally

$$j^* \circ f'^{-1}_{r+1} \circ d(c') = f'^{-1}_r \circ d(c)\,,$$

then there exists an element $e \in C^{s-1}(E, Z_{p^{r+2}})$ such that $d \circ g_{,+1}(e) = j^*(a')$, because $j^*(\beta)=0$ and g_{r+1} is epimorphic. Then we have

$$\begin{aligned} &g_{r+1}(j^*(c') - f_{r+1}(c) - d(e)) \\ &= j^* \circ g_{r+1}(c') - d \circ g_{r+1}(e) = j^*(a') - j^*(a') = 0\,. \end{aligned}$$

Therefore there exists an element $c' \in C^s(E, Z_{p^{r+1}})$ such that

$$j^*(c') = f_{r+1}(c + c'') + d(e)\,.$$

Then

$$j^* \circ d(c') = d \circ j^*(c') = d \circ f_{r+1}(c) + d \circ f_{r+1}(c'')\,.$$

On the other hand, we have

$$\begin{aligned} j^* \circ d(c') &= f'_{r+1} \circ j^* \circ f'^{-1}_{r+1} \circ d(c') \\ &= f'_{r+1} \circ f'^{-1}_r \circ d(c) = f_{r+1} \circ d(c)\,. \end{aligned}$$

Since f_{r+1} is monomorphic, this implies that c'' is a cocycle. Obviously $\Delta^r_p\{g_r(c'')+a\}$ coincides with $\Delta^r_p\{a\} = \Delta^r_p(\alpha)$. Thus we can take $g_r(c'')+a$ for a, so we shall write a for $g_r(c'')+a$ from now on.

On the other hand, we have

$$\begin{aligned} g' \circ g'_{r+1} \circ j^*(c') &= g' \circ g'_{r+1} \circ f_{r+1}(c + c'') + g' \circ g'_{r+1} \circ d(e) \\ &= f_1(a) + g' \circ g'_{r+1} \circ f_{r+1}(c'') + d \circ g' \circ g'_{r+1}(e)\,, \end{aligned}$$

and therefore

$$\begin{aligned} i^* \circ g' \circ g'_{r+1} \circ j^*(c') &= i^* \circ g' \circ g'_{r+1} \circ f_{r+1}(c + c') + i^* \circ d \circ g' \circ g'_{r+1}(e)\,, \\ 0 &= i^* \circ f_1(a) + i^* \circ d \circ g' \circ g'_{r+1}(e)\,. \end{aligned}$$

This implies

$$i^*\{(a)\} = -\Delta^1_p\{i^* \circ g_{r+1}(e)\}\,.$$

Furthermore we have

$$g_1 \circ d \circ g' \circ g'_{r+1}(e) = d \circ g_1 \circ g' \circ g'_{r+.}(e) = j^*(a')\,.$$

This shows that

$$\Delta\{i^* \circ g_{r+1}(e)\} = \{a'\}\,. \qquad \text{Q. E. D.}$$

REMARK 3.11. Theorem 3.10 can be also generalized in the following form:

Let α and β be elements of $H^s(E, Z_p)$ and of $H^s(E, F; Z_p)$ such that $\delta_{r-1}(\alpha)=0$, $\delta_{r+r'-1}(\beta)=0$ $(r'\geqq 1)$, $\delta'_{r-1}H^s(E, Z_{p^{r-1}})\supset j^*\circ\delta'_{r+r'-1}H^s(E, F; Z_{p^{r+r'-1}})$ and $j^*\circ\Delta_p^{r+r'}(\beta)=\Delta_p^r(\alpha)$. Then there exists an element γ of $H^{s-1}(F, Z_p)$ such that

$$\Delta(\gamma)=\beta$$

$$\Delta_p^{r'}(\gamma)=-i^*(\alpha) \mod \delta'_{r'-1}H^{s-1}(F, Z_{p^{r'-1}})+i^*\circ\delta_r^{-1}(0)$$

THEOREM 3.12. *Let α and β be elements of $H^s(E, Z_p)$ and of $H^s(E, F; Z_p)$ such that $(H^s(E, Z_p)\supset)\delta_r^{-1}(0)\subset j^*$ image, $\delta_{r-1}(\alpha)=0$, $\delta_r(\beta)=0$, $j^*\circ\delta'_rH^s(E, F; Z_{p^r})\subset\delta'_{r-1}H^s(E, Z_{p^{r-1}})$, $j^*\circ\Delta_p^{r+1}(\beta)=\Delta_p^r(\alpha)$, $j^*(\beta)=0$ and $j^*(\gamma)=\alpha$, then there exists an element ε of $H^{s-1}(F, Z_p)$ satisfying the following conditions*

$$\Delta(\varepsilon)=\beta$$

$$\Delta\circ\Delta_p^2(\varepsilon)=\Delta_p^1(\gamma) \quad \mathit{mod}\ \Delta\circ\Delta_p^1H^s(F, Z_p)\,.$$

PROOF. This is obtained in combining the proofs of Theorem 3.6 and of Theorem 3.10. Complete proof is left to the reader.

REMARK 3.13. In combining Remark 3.7 and Remark 3.11, we can generalize also the Theorem 3.12.

§ 4. A generalization of the Pontrjagin square operation and auxiliary operations

1. A generalization of the Pontrjagin square operation.

J. H. C. Whitehead [11, 15, 17] defined the Pontrjagin square operation

$$P_2^1:\ H^n(X, Z_2)\rightarrow H^{2n}(X, Z_4)\,,$$

with the following properties

(1) if n is even $P_2^1(u+v)=P_2^1(u)+P_2^1(v)+f_1(uv)$,

if n is odd $P_2^1(u+v)=P_2^1(u)+P_2^1(v)$.

(2) if n is even $2P_2^1(u)=f_1(u^2)$,

if n is odd $2P_2^1(u)=0$.

We shall now generalize this operation. Denoting by θ the homomorphism $H^{pn}(X, Z_{p^{h-1}}) \to H^{pn}(X, Z_{p^{h-1}})$ induced by the injection $Z_{p^{h-1}} \to Z_{p^{h+1}}$, we define an operation

$$P_p^h: \operatorname{Ker} \delta_{h-1} \cap H^n(X, Z_p) \to H^{pn}(X, Z_{p^{h+1}})/\theta H^{pn}(X, Z_{p^{h-1}})$$

in the following manner.

case $p=2$.

Let $\alpha \in \operatorname{Ker} \delta_{h-1} \cap H^n(X, Z_2)$ be represented by a cocycle $u' \in C^n(X, Z_2)$, and let u be an element of $C^n(X, Z)$ such that $G_1(u)=u'$ and $\delta(u)=2^h u''$. As is shown in the proof of Proposition 2.11 in § 2, $G_{h+1}(u \smile_0 u + u \smile_1 \delta(u))$ is a cocycle of $C^{2n}(X, Z_{2^{h+1}})$. Now we put

$$P_2^h(\alpha) = \{G_{h+1}(u \smile_0 u + u \smile_1 \delta(u))\} \bmod \theta H^{2n}(X, Z_{2^{h-1}})\,.$$

In fact $\{G_{h+1}(u \smile_0 u + u \smile_1 \delta(u))\}$ is determined by α only, depends neither on the choices of u' and u, nor on the ways of performing $\smile_0, \smile_1$.

We shall verify this in the following paragraphs $1^\circ \sim 3^\circ$.

1°. $\{G_{h+1}(u \smile_0 u + u \smile_1 \delta(u))\}$ is independent of the ways of performing $\smile_i$, the choice of u being already made.

Let $\smile_i^0$ and $\smile_i^1$ denote Steenrod's i-products performed in two ways. Then Steenrod [9] has proved

$$\delta(u_{\vee_0} u) = u \smile_0^1 u - u \smile_0^0 u - [\delta(u)_{\vee_0} u + (-1)^n u_{\vee_0} \delta(u)]$$

and

$$\delta(u_{\vee_1} \delta(u)) = u \smile_1^1 \delta(u) - u \smile_1^0 \delta(u) - u_{\vee_0} \delta(u) + \delta(u)_{\vee_0} u - \delta(u)_{\vee_1} \delta(u)\,.$$

(For the meaning of $\vee^i$ see [9]). Therefore we have

$$\delta(u_{\vee_0} u + u_{\vee_1} \delta(u)) = u \smile_0^1 u + u \smile_1^1 \delta(u) - (u \smile_0^0 u + u \smile_1^0 \delta(u))$$
$$- [1 + (-1)^n] u_{\vee_0} 2^h u'' - 2^{2h} u''_{\vee_1} u''\,,$$

and so

$$G_{h+1} \circ \delta(u_{\vee_0} u + u_{\vee_1} \delta(u)) = G_{h+1}(u \smile_0^1 u + u \smile_1^1 \delta(u))$$
$$- G_{h+1}(u \smile_0^0 u + u \smile_1^0 \delta(u))\,.$$

2°. $\{G_{h+1}(u \smile_0 u + u \smile_1 \delta(u))\}$ is independent of the choice of u (for a fixed u').

If $G_1(u) = G_1(v) = u'$ and $\delta(u) = 2^h u''$, $\delta(v) = 2^h v''$, then we have $v = u + 2\lambda$, $\lambda \in C^n(X, Z)$, $\delta(\lambda) = 2^{h-1} \lambda''$. Now we have

$$(u+2\lambda)\smile_0(u+2\lambda)+(u+2\lambda)\smile_1\delta(u)+2\delta(\lambda))$$
$$=u\smile_0 u+u\smile_1\delta(u)+2(u\smile_0\lambda+\lambda\smile_0 u+2\lambda\smile_0\lambda+2\lambda\smile_1\delta(\lambda)$$
$$+u\smile_1\delta(\lambda)+\lambda\smile_1\delta(u))$$
$$=u\smile_0 u+u\smile_1\delta(u)+2[-\delta(u\smile_1\lambda)+[1+(-1)^n]\lambda\smile_0 u$$
$$+\delta(u)\smile_1\lambda+[1+(-1)^n]u\smile_1\delta(\lambda)+2\lambda\smile_0\lambda+2\lambda\smile_1\delta(\lambda)+\lambda\smile_1\delta(u)]\,,$$

because the following equality holds

$$\delta(u\smile_1\lambda)=-u\smile_0\lambda+(-1)^n\lambda\smile_0 u+\delta(u)\smile_1\lambda+(-1)^n u\smile_1\delta(\lambda)\,.$$

Obviously $G_{h-1}(\lambda\smile_0 u+\lambda\smile_0\lambda)$ is a cocycle in $C^{2n}(X, Z_{2^{h-1}})$. Therefore we have

$$\{G_{h+1}(v\smile_0 v+v\smile_1\delta(v))\}$$
$$=\{G_{h+1}[(u+2\lambda)\smile_0(u+2\lambda)+(u+2\lambda)\smile_1(\delta(u)+2\delta(\lambda))]\}$$
$$=\{G_{h+1}(u\smile_0 u+u\smile_1\delta(u))\}+\{\theta\cdot G_{h-1}[[1+(-1)^n]/2\lambda\smile_0 u$$
$$+\lambda\smile_0\lambda]\}=G_{h+1}(u\smile_0 u+u\smile_1\delta(u)\}\ \bmod \theta H^{2n}(X, Z_{2^{h-1}})\,.$$

3°. $\{G_{h+1}(u\smile_0 u+u\smile_1\delta(u))$ is independent of the choice of u'.

If $\{u'\}=\{v'\}=\alpha$, $G_1(u)=u'$, $\delta(u)=2^h u''$, then we have $v'=u'+\delta(w')$, $w'\in C^{n-1}(X, Z_2)$. Taking any $w\in C^{n-1}(X, Z)$ with $G_1(w)=w'$ and putting $v=u+\delta(w)$, we now have $G_1(v)=v'$, and further

$$v\smile_0 v+v\smile_1\delta(v)$$
$$=(u+\delta(w))\smile_0(u+\delta(w))+(u+\delta(w))\smile_1\delta(u)$$
$$=u\smile_0 u+u\smile_1\delta(u)+u\smile_0\delta(w)+\delta(w)\smile_0 u+\delta(w)\smile_0\delta(w)$$
$$+\delta(w)\smile_1\delta(u)$$
$$=u\smile_0 u+u\smile_1\delta(u)-\delta(u\smile_1\delta(w))+[1+(-1)^n]\delta(w)\smile_0 u$$
$$+\delta(w)\smile_0\delta(w)+\delta(u)\smile_1\delta(w)+\delta(w)\smile_1\delta(u)$$
$$=u\smile_0 u+u\smile_1\delta(u)-\delta(u\smile_1\delta(w))+[1+(-1)^n]\delta(w\smile_0 u)$$
$$+[1+(-1)^n]w\smile_0\delta(u)+\delta(w\smile_0\delta(w))-\delta(\delta(u)\smile_2\delta(w))\,,$$

because of the equalities:

$$\delta(u\smile_1\delta(w))=-u\smile_0\delta(w)+(-1)^n\delta(w)\smile_0 u+\delta(u)\smile_1\delta(w)\,,$$
$$\delta(w\smile_0 u)=\delta(w)\smile_0 u+(-1)^{n-1}w\smile_0\delta(u)\,,$$

and

$$\delta(\delta(u)\smile_2\delta(w)) = -\delta(u)\smile_1\delta(w)-\delta(w)\smile_1\delta(u).$$

Thus

$$\{G_{h+1}(v\smile_0 v+v\smile_1\delta(v))\}=\{G_{h+1}(u\smile_0 u+u\smile_1\delta(u))\}.$$

Case $p>2$.

Let $\alpha\in \operatorname{Ker}\delta_{h-1}\cap H^n(X, Z_p)$ be represented by a cocycle $u'\in C^n(X, Z_p)$ and let u be an element of $C^n(X, Z)$ such that

$$G_1(u)=u' \text{ and } \delta(u)=p^h u'' \text{ for some } u''\in C^{n-1}(X,Z).$$

First let n be even. As is shown in the proof of Proposition 2.13 in § 2,

$$G_{h+1}(D_0(u\otimes\cdots\otimes u)+\sum_{k=1}^{p} kD_1(u\otimes\cdots\otimes\overset{(k)}{\delta(u)}\otimes\cdots\otimes u))$$

is a cocycle of $C^{pn}(X, Z_{p^{h+1}})$. We set

$$P_p^h(\alpha)=\{G_{h+1}(D_0(u\otimes\cdots\otimes u)+\sum_{k=1}^{p} kD_1(u\otimes\cdots\otimes\overset{(k)}{\delta(u)}\otimes\cdots\otimes u))\}$$
$$\mod \theta H^{pn}(X, Z_{p^{h-1}}).$$

This is again determined only by α, independently of D_i operators and of the choices of u', u. In fact:

1°. $$\{G_{h+1}(D_0(u\otimes\cdots\otimes u)+\sum_{k=1}^{p} kD_1(u\otimes\cdots\otimes\overset{(k)}{\delta(u)}\otimes\cdots\otimes u))\}$$

is independent of the choice of D_i.

Let E_i be the Steenrod's notation [10], then there exists the following properties.

$E_0=0$ and

$$E_{i+1}\circ\delta+(-1)^i\delta\circ E_{i+1}=D_i-D_i'-E_i(-1+T) \text{ if } i \text{ is odd,}$$

$$E_{i+1}\circ\delta+(-1)^i\delta\circ E_{i+1}=D_i-D_i'-E_i(1+T+\cdots+T^{p-1}) \text{ if } i \text{ is even.}$$

Now we have

$$E_1\circ\delta(u\otimes\cdots\otimes u)+\delta\circ E_1(u\otimes\cdots\otimes u)$$
$$=E_1\sum_{j=1}^{p}(u\otimes\cdots\otimes\overset{(j)}{\delta(u)}\otimes\cdots\otimes u)+\delta\circ E_1(u\otimes\cdots\otimes u)$$
$$=(D_0-D_0')(u\otimes\cdots\otimes u),$$

and

$$E_2\circ\delta(u\otimes\cdots\otimes\overset{(k)}{\delta(u)}\otimes\cdots\otimes u)-\delta\circ E_2(u\otimes\cdots\otimes\overset{(k)}{\delta(u)}\otimes\cdots\otimes u)$$

$$\begin{cases} =(D_1-D'_1)\,(u\otimes\cdots\otimes\overset{(k)}{\delta(u)}\otimes\cdots\otimes u)-E_1(-u\otimes\cdots\otimes\overset{(k)}{\delta(u)}\otimes\cdots\otimes u \\ \qquad +u\otimes\cdots\otimes\overset{(k+1)}{\delta(u)}\otimes\cdots\otimes u) \text{ for } k\neq p \\ =(D_1-D'_1)\,(u\otimes\cdots\otimes\overset{(k)}{\delta(u)}\otimes\cdots\otimes u)-E_1(-u\otimes\cdots\otimes\delta(u)+\delta(u)\otimes \\ \qquad \cdots\otimes u) \text{ for } k=p. \end{cases}$$

Thus we have

$$E_2\sum_{k=1}^{p}k\delta(u\otimes\cdots\otimes\overset{(k)}{\delta(u)}\otimes\cdots\otimes u)-\delta\sum_{k=1}^{p}kE_2(u\otimes\cdots\otimes\overset{(k)}{\delta(u)}\otimes\cdots\otimes u)$$

$$=\sum^{p}k(D_1-D'_1)\,(u\otimes\cdots\otimes\overset{(k)}{\delta(u)}\otimes\cdots\otimes u)+E_1\sum^{p}(u\otimes\cdots\otimes\overset{(k)}{\delta(u)}\otimes$$

$$\cdots\otimes u)-pE_1(\delta(u)\otimes u\otimes\cdots\otimes u)$$

$$=\sum_{k=1}^{p}k(D_1-D'_1)\,(u\otimes\cdots\otimes\overset{(k)}{\delta(u)}\otimes\cdots\otimes u)+(D_0-D'_0)\,(u\otimes\cdots\otimes u)$$

$$-\delta E_1(u\otimes\cdots\otimes u)-pE_1(\delta(u)\otimes u\otimes\cdots\otimes u)\,.$$

This implies

$$\{G_{h+1}[D_0(u\otimes\cdots\otimes u)+\sum_{k=1}^{p}kD_1(u\otimes\cdots\otimes\overset{(k)}{\delta(u)}\otimes\cdots\otimes u)-(D'_0(u\otimes\cdots$$

$$\otimes u)+\sum_{k=1}^{p}kD'_1(u\otimes\cdots\otimes\overset{(k)}{\delta(u)}\otimes\cdots\otimes u))]\}$$

$$=\{G_{h+1}\circ\delta(\sum_{k=1}^{p}kE_2(u\otimes\cdots\otimes\overset{(k)}{\delta(u)}\otimes\cdots\otimes u)+E_1(u\otimes\cdots\otimes u))\}=0\,,$$

proving the stated independence.

$$2^\circ.\qquad \{G_{h+1}(D_0(u\otimes\cdots\otimes u)+\sum_{k=1}^{p}kD_1(u\otimes\cdots\otimes\overset{(k)}{\delta(u)}\otimes\cdots\otimes u))\}$$

is independent of the choice of u (for a fixed u').

If $G_1(u)=G_1(v)=u'$ and $\delta(u)=p^h u''$, $\delta(v)=p^h v''$, then we have $v=u+p\lambda$ $\lambda\in C^n(x, Z)$, $\delta(\lambda)=p^{h-1}\lambda''$. Now we have

$$D_0((u+p\lambda)\otimes\cdots\otimes(u+p\lambda))+\sum_{k=1}^{p}kD_1((u+p\lambda)\otimes\cdots$$

$$\otimes\overset{(k)}{\delta(u+p\lambda)}\otimes\cdots\otimes(u+p\lambda))$$

$$=D_0(u\otimes\cdots\otimes u)+\sum_{k=1}^{p}kD_1(u\otimes\cdots\otimes\overset{(k)}{\delta(u)}\otimes\cdots\otimes u)$$

$$+\sum_{r=1}^{p}\sum_{i_1<\cdots<i_r}p^rD_0(u\otimes\cdots\otimes\overset{(i_1)}{\lambda}\otimes\cdots\otimes\overset{(i_r)}{\lambda}\otimes\cdots\otimes u)+\sum_{r=1}^{p-1}\sum_{\substack{i_1<\cdots<i_r\\ i_\mu\neq k}}$$

$$\sum_{k=1}^{p}kp^rD_1(u\otimes\cdots\otimes\overset{(i_1)}{\lambda}\otimes\cdots\otimes\overset{(k)}{\delta(u)}\otimes\cdots\otimes\overset{(i_r)}{\lambda}\otimes\cdots\otimes u)$$

$$+\sum_{r=1}^{p-1}\sum_{\substack{i<\cdots<i_r\\ i_\mu\neq k}}\sum_{k=1}^{p}kp^{r+1}D_1(u\otimes\cdots\otimes\overset{(i_1)}{\lambda}\otimes\cdots\otimes\overset{(k)}{\delta(\lambda)}\otimes\cdots\otimes\overset{(i_r)}{\lambda}\otimes\cdots\otimes u)$$

$$+\sum_{k=1}^{p}kpD_1(u\otimes\cdots\otimes\overset{(k)}{\delta(\lambda)}\otimes\cdots\otimes u)$$

A similar computation as in (3) in the proof of Proposition 2.13, § 2 yields

$$\sum_{i_1=1}^{p}D_0(u\otimes\cdots\otimes\overset{(i_1)}{\lambda}\otimes\cdots\otimes u)=-\sum_{i_1=1}^{p}i_1D_1\,(\sum_{\substack{j=1\\ j\neq i_1}}^{p}u\otimes\cdots\otimes\overset{(i_1)}{\lambda}\otimes\cdots$$

$$\otimes\overset{(j)}{\delta(u)}\otimes\cdots\otimes u+u\otimes\cdots\otimes\overset{(i_1)}{\delta(\lambda)}\otimes\cdots\otimes u)-\sum_{i_1=1}^{p}i_1\delta\circ D_1(u\otimes\cdots$$

$$\otimes\overset{(i_1)}{\lambda}\otimes\cdots\otimes u)+pD_0(\lambda\otimes\cdots\otimes u)\,.$$

Therefore

$$\sum_{i_1=1}^{p}pD_0(u\otimes\cdots\otimes\overset{(i_1)}{\lambda}\otimes\cdots\otimes u)+\sum_{\substack{i_1=1\\ i_1\neq k}}^{p}\sum_{k=1}^{p}kpD_1(u\otimes\cdots\otimes\overset{(k)}{\delta(u)}\otimes$$

$$\cdots\otimes\overset{(i_1)}{\lambda}\otimes\cdots\otimes u)+\sum_{k=1}kpD_1(u\otimes\cdots\otimes\overset{(k)}{\delta(\lambda)}\otimes\cdots\otimes u)$$

$$=p\sum_{\substack{i_1=1\\ i_1\neq k}}^{p}\sum_{k=1}^{p}(k-i_1)D_1(u\otimes\cdots\otimes\overset{(i_1)}{\lambda}\otimes\cdots\otimes\overset{(k)}{\delta(u)}\otimes\cdots\otimes u)-\sum_{i_1=1}^{p}$$

$$i_1p\delta\circ D_1(u\otimes\cdots\otimes\overset{(i_1)}{\lambda}\otimes\cdots\otimes u)+p^2D_0(\lambda\otimes u\otimes\cdots\otimes u)\,.$$

Thus we have

$$\{G_{h+1}[D_0((u+p\lambda)\otimes\cdots\otimes(u+p\lambda))+\sum_{k=1}^{p}kD_1((u+p\lambda)\otimes\cdots\otimes(u\overset{(k)}{+}p\lambda)\otimes\cdots\otimes(u+p\lambda))]\}$$
$$=\{G_{h+1}[D_1(u\otimes\cdots\otimes u)+\sum_{k=1}^{p}kD_1(u\otimes\cdots\otimes\overset{(k)}{\delta(u)}\otimes\cdots\otimes u)]\}$$
$$+\theta\{G_{h-1}[D_0(\lambda\otimes u\otimes\cdots\otimes u)+\sum_{r=2}^{p}\sum_{i_1<\cdots<i_r}p^{r-1}D_0(u\otimes\cdots\otimes\overset{(i_1)}{\lambda}\otimes\cdots\otimes\overset{(i_r)}{\lambda}\otimes\cdots\otimes u)]\}.$$

Obviously

$$G_{h-1}[D_0(\lambda\otimes u\otimes\cdots\otimes u)+\sum_{r=2}^{p}\sum_{i_1<\cdots<i_r}p^{r-2}D_0(u\otimes\cdots\otimes\overset{(i_1)}{\lambda}\otimes\cdots\otimes\overset{(i_r)}{\lambda}\otimes\cdots\otimes u)]$$

is a cocycle of $C^{pn}(X, Z_{p^{h-1}})$. This proves our assertion.

3°. $\{G_{h+1}[D_0(u\otimes\cdots\otimes u)+\sum_{k=1}^{p}kD_1(u\otimes\cdots\otimes\overset{(k)}{\delta(u)}\otimes\cdots\otimes u)]\}$

is independent of the choice of u'.

If $\{u'\}=\{v'\}=\alpha$, $G_1(u)=u'$ and $\delta(u)=p^h u''$, then we have $v'=u'+\delta(w')$, $w'\in C^{n-1}(X, Z_p)$, and we can find $v=u+\delta(w)$, $w\in C^{n-1}(X, Z)$ such that $G_1(v)=v'$. We have immediately

$$D_0((u+\delta(w))\otimes\cdots\otimes(u+\delta(w)))+\sum_{k=1}^{p}kD_1((u+\delta(w))\otimes\cdots\otimes\overset{(k)}{\delta(u)}\otimes\cdots\otimes(u+\delta(w))) \tag{3}$$
$$=D_0(u\otimes\cdots\otimes u)+\sum_{k=1}^{p}kD_1(u\otimes\cdots\otimes\overset{(k)}{\delta(u)}\otimes\cdots\otimes u)+\sum_{r=1}^{p}\sum_{i_1<\cdots<i_r}D_0(u\otimes\cdots\otimes\overset{(i_1)}{\delta(w)}\otimes\cdots\otimes\overset{(i_r)}{\delta(w)}\otimes\cdots\otimes u)+\sum_{k=1}^{p}k\sum_{r=1}^{p-1}\sum_{\substack{i_1<\cdots<i_r\\ i_\mu\neq k}}D_1(u\otimes\cdots\otimes\overset{(i_1)}{\delta(w)}\otimes\cdots\otimes\overset{(k)}{\delta(u)}\otimes\cdots\otimes\overset{(i_r)}{\delta(w)}\otimes\cdots\otimes u).$$

Let $(i'_1,\cdots,i'_r)$ be any increasing subsequence of $(1,2,\cdots,p)$ with $i'_1=1$.

We denote by $(j'_1,\cdots,j'_{p-r})$ the subsequence of $(1,2,\cdots,p)$ complementary to $(i'_1, \cdots, i'_r)$. Then the following equalities can be easily verified:

$$(4)\qquad \sum_{j=0}^{p-1} D_0 T^j(\overset{(i'_1)}{\delta(w)}\otimes\cdots\otimes\overset{(i'_r)}{\delta(w)}\otimes\cdots\otimes u)$$

$$= pD_0(\overset{(i'_1)}{\delta(w)}\otimes\cdots\otimes\overset{(i'_r)}{\delta(w)}\otimes\cdots\otimes u) - \sum_{j=0}^{p-1}[(j+1)D_1\circ\delta\circ T^j(\overset{(i'_1)}{\delta(w)}\otimes$$

$$\cdots\otimes\overset{(i'_r)}{\delta(w)}\otimes\cdots\otimes u) - \sum_{j=1}^{p-1}(j+1)\delta\circ D_1\circ T^j(\overset{(i'_1)}{\delta(w)}\otimes\cdots\otimes\overset{(i'_r)}{\delta(w)}\otimes\cdots\otimes u)],$$

$$\delta T^j(\overset{(i'_1)}{\delta(w)}\otimes\cdots\otimes\overset{(i'_r)}{\delta(w)}\otimes\cdots\otimes u)$$

$$= T^j\sum_{k=1}^{p-1}(\overset{(i'_1)}{\delta(w)}\otimes\cdots\otimes\overset{(i'_k)}{\delta(u)}\otimes\cdots\otimes\overset{(i'_r)}{\delta(w)}\otimes\cdots\otimes u).$$

With regard to the difference between the second summand in the right hand side of (4) and the corresponding terms in the last summand of the right hand side of (3), we have

$$(4)\qquad -\sum_{j=0}^{p-1}(j+1)D_1T^j\sum_{m=1}^{p-r}(\overset{(i'_1)}{\delta(w)}\otimes\cdots\otimes\overset{(j'_m)}{\delta(u)}\otimes\cdots\otimes\overset{(i'_r)}{\delta(w)}\otimes\cdots\otimes u)$$

$$+\sum_{m=1}^{p-r}\sum_{j=0}^{p-j'_m}(j'_m+j)D_1T^j(\overset{(i'_1)}{\delta(w)}\otimes\cdots\otimes\overset{(j'_m)}{\delta(u)}\otimes\cdots\otimes\overset{(i'_r)}{\delta(w)}\otimes\cdots\otimes u)$$

$$+\sum_{m=1}^{p-r}\sum_{j=p-j'_m+1}^{p-1}(j'_m+j-p)D_1T^j(\overset{(i'_1)}{\delta(w)}\otimes\cdots\otimes\overset{(j'_m)}{\delta(u)}\otimes\cdots\otimes\overset{(i'_r)}{\delta(w)}\otimes\cdots\otimes u)$$

$$=\sum_{m=1}^{p-r}\sum_{j=0}^{p-j'_m}(j'_m-1)D_1T^j(\overset{(i'_1)}{\delta(w)}\otimes\cdots\otimes\overset{(j'_m)}{\delta(u)}\otimes\cdots\otimes\overset{(i'_r)}{\delta(w)}\otimes\cdots\otimes u)$$

$$+\sum_{m=1}^{p-r}\sum_{j=p-j_m+1}^{p-1}(j'_m-1-p)D_1T^j\overset{(i'_1)}{\delta(w)}\otimes\cdots\otimes\overset{(j'_m)}{\delta(u)}\otimes\cdots\otimes\overset{(i'_r)}{\delta(w)}\otimes\cdots\otimes u)$$

$$=\sum_{m=1}^{p-r}\sum_{j=0}^{p-1}(j'_m-1)D_1\circ T^j(\overset{(i'_1)}{\delta(w)}\otimes\cdots\otimes\overset{(j'_m)}{\delta(u)}\otimes\cdots\otimes\overset{(i'_r)}{\delta(w)}\otimes\cdots\otimes u)$$

$$-\sum_{m=1}^{p-r}\sum_{j=p-j'_m+1}^{p-1}pD_1\circ T^j(\overset{(i'_1)}{\delta(w)}\otimes\cdots\otimes\overset{(j'_m)}{\delta(u)}\otimes\cdots\otimes\overset{(i'_r)}{\delta(w)}\otimes\cdots\otimes u)$$

$$=\sum_{m=1}^{p-r}(j_m'-1)[D_2\circ\delta(\overset{(i_1')}{\delta(w)}\otimes\cdots\otimes\overset{(j_m')}{\delta(u)}\otimes\cdots\otimes\overset{(i_r')}{\delta(w)}\otimes\cdots\otimes u)$$

$$-\delta\circ D_2(\overset{(i_1')}{\delta(w)}\otimes\cdots\otimes\overset{(j_m')}{\delta(u)}\otimes\cdots\otimes\overset{(i_r')}{\delta(w)}\otimes\cdots\otimes u)]$$

$$-\sum_{m=1}^{p-r}\sum_{j=p-j_m+1}^{p-1} pD_1\circ T^j(\overset{(i_1')}{\delta(w)}\otimes\cdots\otimes\overset{(j_m')}{\delta(u)}\otimes\cdots\otimes\overset{(i_r')}{\delta(w)}\otimes\cdots\otimes u)\,.$$

Since there exist the following equalities

$$\{G_{h+1}[pD_0(\overset{(i_1')}{\delta(w)}\otimes\cdots\otimes\overset{(i_r')}{\delta(w)}\otimes\cdots\otimes u]\}$$

$$=\{G_{h+1}[p\delta\circ D_0(w\otimes \overset{(i_1')}{u}\otimes\cdots\otimes\overset{(i_r')}{\delta(w)}\otimes\cdots\otimes u)+p\sum_{m=1}^{p-r}D_0(\overset{(i_1')}{w}\otimes\cdots\otimes\overset{(j_m')}{\delta(u)}\otimes$$

$$\cdots\otimes\overset{(i_r')}{\delta(w)}\otimes\cdots\otimes u)]\}=0\,,$$

we have

$$\text{(5)}\qquad \{G_{h+1}[\sum_{j=1}^{p-1}D_0\circ T^j(\overset{(i_1')}{\delta(w)}\otimes\cdots\otimes\overset{(i_r')}{\delta(w)}\otimes\cdots\otimes u)+\sum_{m=1}^{p-r}\sum_{j=0}^{p-j_m'}(j_m'+j)$$

$$D_1\circ T^j(\overset{(i_1')}{\delta(w)}\otimes\cdots\otimes\overset{(j_m')}{\delta(u)}\otimes\cdots\otimes\overset{(i_r')}{\delta(w)}\otimes\cdots\otimes u)-\sum_{m=1}^{p-r}\sum_{j=p-j_m'+1}^{p-1}$$

$$(j_m'+j-p)D_1\circ T^j(\overset{(i_1')}{\delta(w)}\otimes\cdots\otimes\overset{(j_m')}{\delta(u)}\otimes\cdots\otimes\overset{(i_r')}{\delta(w)}\otimes\cdots\otimes u)]\}=0\,.$$

It is easily verifies that (5) is a sufficient condition for our assertion. Q. E. D.

Next let n be odd. Then we have

$$\delta\circ D_0(u\otimes\cdots\otimes u)=\sum_{i=1}^{p}(-1)^{i-1}D_0(u\otimes\cdots\otimes\overset{(i)}{\delta(u)}\otimes\cdots\otimes u)\,,$$

and

$$D_1\circ\delta(u\otimes\cdots\otimes\overset{(i)}{\delta(u)}\otimes\cdots\otimes u)+\delta\circ D_1(u\otimes\cdots\otimes\overset{(i}{\delta(u)}\otimes\cdots\otimes u)$$

$$=-D_0(u\otimes\cdots\otimes\overset{(i+1)}{\delta(u)}\otimes\cdots\otimes u)-D_0(u\otimes\cdots\otimes\overset{(i)}{\delta(u)}\otimes\cdots\otimes u)\quad i\leqq p-1$$

$$=D_0(\delta(u)\otimes\cdots\otimes u)-D_0(u\otimes\cdots\otimes\delta(u))\qquad i=p\,.$$

Therefore we have

$$\delta(D_0(u\otimes\cdots\otimes u)+\sum_{k=1}^{p}k(-1)^{k-1}D_1(u\otimes\cdots\otimes\overset{(k)}{\delta(u)}\otimes\cdots\otimes u))$$

$$=pD_0(\delta(u)\otimes u\otimes\cdots\otimes u-\sum^{p}k(-1)^{k-1}D_1(\sum(-1)^{i-1}u\otimes\cdots\otimes\overset{(i)}{\delta(u)}\otimes$$

$$\cdots\otimes\overset{(k)}{\delta(u)}\otimes\cdots\otimes u+\sum_{k<i\leqq p}(-1)^{i}u\otimes\cdots\otimes\overset{(k)}{\delta(u)}\otimes\cdots\otimes\overset{(i)}{\delta(u)}\otimes\cdots\otimes u).$$

This implies that

$$G_{h+1}[D_0(u\otimes\cdots\otimes u)+\sum_{k=1}^{p}k(-1)^{k-1}D_1(u\otimes\cdots\otimes\overset{(k)}{\delta(u)}\otimes\cdots\otimes u)]$$

is a cocycle. We set

$$P_p^h(\alpha)=\{G_{h+1}[D_0(u\otimes\cdots\otimes u)+\sum_{k=1}^{p}k(-1)^{k-1}D_1(u\otimes\cdots\otimes\overset{(k)}{\delta(u)}\otimes\cdots\otimes u)]\}$$
$$\bmod \theta H^{pn}(X, Z_{p^{h-1}}).$$

This is also determined only by α, independently of D_i operators and of the choice of u', u. The proof is analogously performed as in the proof of the case where n is even.

REMARK 1.1. In a similar way we can also define an operation

$$'P_p^h:\ H^n(X, Z_{p^h})\rightarrow H^{pn}(X, Z_{p^{h+1}}).$$

REMARK 1.2. If n is odd, then it is easily verified that

$$\delta'_{h+1}\{G_{h+1}[D_0(u\otimes\cdots\otimes u)+\sum_{k=1}^{p}k(-1)^{k-1}D_1(u\otimes\cdots\otimes\overset{(k)}{\delta(u)}\otimes\cdots\otimes u)]\}=0.$$

2. Addition formula of the generalized Pontrjagin square.

In this section, we shall prove the following property of P_p^h:

(6) $$P_p^h(\alpha+\beta)=P_p^h(\alpha)+P_p^h(\beta)$$
$$+f'_h\left(\alpha^{p-1}\beta+\sum_{r=2}^{p-1}\frac{(p-1)\cdots(p-r+1)}{r!}\alpha^{p-r}\beta^r+\alpha\beta^{p-1}\right),\ \text{if } D(\alpha) \text{ is even.}$$

$$P_p^h(\alpha+\beta)=P_p^h(\alpha)+P_p^h(\beta),\ \text{if } D(\alpha) \text{ is odd.}$$

PROOF. of (6). Let u and v be cochains $\in C^n(X, Z)$ such that $\{G_1(u)\}=\alpha$, $\{G_1(v)\}=\beta$ and $\delta\circ G_h(u)=\delta\circ G_h(v)=0$. Case $p=2$. We have the following equalities

$$(u+v)\smile_0(u+v)+(u+v)\smile_1(\delta(u)+\delta(v))$$
$$=u\smile_0 u+u\smile_1\delta(u)+v\smile_0 v+v\smile_1\delta(v)+u\smile_0 v+v\smile_0 u$$

$$+u\smile_1\delta(v)+v\smile_1\delta(u)$$
$$=u\smile_0 u+u\smile_1\delta(u)+v\smile_0 v+v\smile_1\delta(v)-\delta(u\smile_1 v)+2u\smile_1\delta(v)$$
$$+[1+(-1)^n]v\smile_1\delta(u)+[1+(-1)^n]v\smile_0 u\,,$$

for

$$\delta(u\smile_1 v)=-u\smile_0 v+(-1)^n v\smile_0 u+\delta(u)\smile_1 v+(-1)^n u\smile_1\delta(v)\,.$$

Therefore

$$G_{h+1}[(u+v)\smile_0(u+v)+(u+v)\smile_1(\delta(u)+\delta(v))]$$
$$\begin{cases} =G_{h+1}(u\smile_0 u+u\smile_1\delta(u))+G_{h+1}(v\smile_0 v+v\smile_1\delta(v)) \\ \qquad\qquad +f'_h(v\smile_0 u) & \text{if } n \text{ is even,} \\ =G_{h+1}(u\smile_0 u+u\smile_1\delta(u))+G_{h+1}(v\smile_0 v+v\smile_1\delta(v)) & \text{if } n \text{ is odd.}\end{cases}$$

Case $p>2$. We give the proof only for the case n is even, as the case n is odd is treated analogously.

We have the following equalities

$$D_0((u+v)\otimes\cdots\otimes(u+v))+\sum_{k=1}^{p}kD_1((u+v)\otimes\cdots\otimes\overset{(k)}{(\delta(u)+\delta(v))}\otimes$$
$$\cdots\otimes(u+v))$$
$$=D_0(u\otimes\cdots\otimes u)+\sum_{k=1}^{p}kD_1(u\otimes\cdots\otimes\overset{(k)}{\delta(u)}\otimes\cdots\otimes u)+D_0(v\otimes\cdots\otimes v)$$
$$+\sum_{k=1}^{p}kD_1(v\otimes\cdots\otimes\overset{(k)}{\delta(v)}\otimes\cdots\otimes v)+\sum_{r=1}^{p-1}\sum_{i_1<\cdots<i_r}D_0(u\otimes\cdots\otimes\overset{(i_1)}{v}\otimes$$
$$\cdots\otimes\overset{(i_r)}{v}\otimes\cdots\otimes u)+\sum_{k=1}^{p}kD_1[\sum_{r=1}^{p-1}\sum_{\substack{i_1<\cdots<i_r\\ i_\mu\neq k}}u\otimes\cdots\otimes\overset{(i_1)}{v}\otimes\cdots\otimes\overset{(k)}{\delta(u)}\otimes$$
$$\cdots\otimes\overset{(i_r)}{v}\otimes\cdots\otimes u+\sum_{r=1}^{p-2}\sum_{\substack{i_1<\cdots<i_r\\ i_\mu\neq k}}u\otimes\cdots\otimes\overset{(i_1)}{v}\otimes\cdots\otimes\overset{(k)}{\delta(v)}\otimes\cdots\otimes\overset{(i_r)}{v}$$
$$\otimes\cdots\otimes u]+\sum_{k=1}^{p}kD_1(u\otimes\cdots\otimes\overset{(k)}{\delta(v)}\otimes\cdots\otimes u)\,.$$

Let $(i'_1,\cdots,i'_r)$ be any increasing subsequence of $(1,2,\cdots,p)$ with $i'_1=1$, and $(j'_1,\cdots,j'_{p-r})$ be the subsequence of $(1,2,\cdots,p)$ complementary to $(i'_1,\cdots,i'_r)$. Then the following equalities can be easily verified:

$$(7)\qquad \sum_{j=0}^{p-1}D_0\circ T^j(\overset{(i'_1)}{v}\otimes\cdots\otimes\overset{(i'_r)}{v}\otimes\cdots\otimes u)$$

$$=pD_0(\overset{(i'_1)}{v}\otimes u\otimes\cdots\otimes\overset{(i'_r)}{v}\otimes\cdots\otimes u)-\sum_{j=0}^{p-1}(j+1)D_0\circ\delta\circ T^j(\overset{(i'_1)}{v}\otimes u\otimes\cdots\otimes\overset{(i'_r)}{v}\otimes\cdots\otimes u)-\sum_{j=0}^{p-1}(j+1)\delta\circ D_1\circ T^j(\overset{(i'_1)}{v}\otimes\cdots\otimes\overset{(i'_r)}{v}\otimes\cdots\otimes u)\,,$$

$$\delta\circ T^j(\overset{(i'_1)}{v}\otimes\cdots\otimes\overset{(i'_r)}{v}\otimes\cdots\otimes u)$$

$$=T^j[\sum_{k=0}^{r}(\overset{(i'_1)}{v}\otimes\cdots\otimes\overset{(i'_k)}{\delta(v)}\otimes\cdots\otimes\overset{(i'_r)}{v}\otimes\cdots\otimes u)+\sum_{k=1}^{p-r}(\overset{(i'_1)}{v}\otimes\cdots\otimes\overset{(j'_k)}{\delta(u)}\cdots\otimes\overset{(i'_r)}{v}\otimes\cdots\otimes u)]\,.$$

In virtue of the similar computation as in (4), we have

$$-\sum_{j=0}^{p-1}(j+1)D_1T^j[\sum_{k=1}^{r}\overset{(i'_1)}{v}\otimes\cdots\otimes\overset{(i'_k)}{\delta(v)}\otimes\cdots\otimes\overset{(i'_r)}{v}\otimes\cdots\otimes u+\sum_{k=1}^{p-r}\overset{(i'_1)}{v}\otimes\cdots\otimes\overset{(j'_k)}{\delta(u)}\otimes\cdots\otimes\overset{(i'_r)}{v}\otimes\cdots\otimes u]+\sum_{k=1}^{r}[\sum_{j=0}^{p-i'_k}(i'_k+j)D_1\circ T^j(\overset{(i'_1)}{v}\otimes\cdots\otimes\overset{(i'_k)}{\delta(v)}\otimes\cdots\otimes\overset{(i'_r)}{v}\otimes\cdots\otimes u)+\sum_{j=p-i'_k+1}^{p-1}(i'_k+j-p)D_1\circ T^j(\overset{(i'_1)}{v}\otimes\cdots\otimes\overset{(i'_k)}{\delta(v)}\otimes\cdots\otimes\overset{(i'_r)}{v}\otimes\cdots\otimes u)]+\sum_{k=1}^{p-r}[\sum_{j=0}^{p-j'_k}(j'_k+j)D_1\circ T^j(\overset{(i'_1)}{v}\otimes\cdots\otimes\overset{(j'_k)}{\delta(u)}\otimes\cdots\otimes\overset{(i'_r)}{v}\otimes\cdots\otimes u)+\sum_{j=p-j'_k+1}^{p-1}(j'_k+j-p)D_1\circ T^j(\overset{(i'_1)}{v}\otimes\cdots\otimes\overset{(j'_k)}{\delta(u)}\otimes\cdots\otimes\overset{(i'_r)}{v}\otimes\cdots\otimes u)]$$

$$=\sum_{k=1}^{r}\sum_{j=0}^{p-1}(i'_k-1)D_1\circ T^j(\overset{(i'_1)}{v}\otimes\cdots\otimes\overset{(i'_k)}{\delta(v)}\otimes\cdots\otimes\overset{(i'_r)}{v}\otimes\cdots\otimes u)$$

$$-\sum_{k=1}^{r}\sum_{j=p-i'_k+1}^{p-1}pD_1\circ T^j(\overset{(i'_1)}{v}\otimes\cdots\otimes\overset{(i'_k)}{\delta(v)}\otimes\cdots\otimes\overset{(i'_r)}{v}\otimes\cdots\otimes u)$$

$$+\sum_{k=1}^{p-r}\sum_{j=0}^{p-1}(j'_k-1)D_1\circ T^j(\overset{(i'_1)}{v}\otimes\cdots\otimes\overset{(j'_k)}{\delta(j'_k)}\otimes\cdots\otimes\overset{(i'_r)}{v}\otimes\cdots\otimes u)$$

$$-\sum_{k=1}^{p-r}\sum_{j=p-j'_k+1}^{p-1}pD_1\circ T^j(\overset{(i'_1)}{v}\otimes\cdots\otimes\overset{(j'_k)}{\delta(u)}\otimes\cdots\otimes\overset{(i'_r)}{v}\otimes\cdots\otimes u)$$

$$=\sum_{k=1}^{r}(i'_k-1)[D_2\circ\delta(\overset{(i'_1)}{v}\otimes\cdots\otimes\overset{(i'_k)}{\delta(v)}\otimes\cdots\otimes\overset{(i'_r)}{v}\otimes\cdots\otimes u)-\delta\circ D_2(\overset{(i'_1)}{v}\otimes$$

$$\cdots\otimes\overset{(i'_k)}{\delta(v)}\otimes\cdots\otimes\overset{(i'_r)}{v}\otimes\cdots\otimes u)]-\sum_{k=1}^{r}\sum_{j=p-i'_k+1}^{p-1}pD_1\circ T^j(\overset{(i'_1)}{v}\otimes\cdots\otimes\overset{(i'_k)}{\delta(v)}\otimes$$

$$\cdots\otimes\overset{(i'_r)}{v}\otimes\cdots\otimes u)+\sum_{k=1}^{p-r}(j'_k-1)[D_2\circ\delta(\overset{(i'_1)}{v}\otimes\cdots\otimes\overset{(j'_k)}{\delta(u)}\otimes\cdots\otimes\overset{(i'_r)}{v}\otimes\cdots u)$$

$$-\delta\circ D_2(\overset{(i'_1)}{v}\otimes\cdots\otimes\overset{(j'_k)}{\delta(u)}\otimes\cdots\otimes\overset{(i'_r)}{v}\otimes\cdots\otimes u)-\sum_{k=1}^{p-r}\sum_{j=p-j'_k+1}^{p-1}pD_1\circ T^j(\overset{(i'_1)}{v}\otimes$$

$$\cdots\otimes\overset{(j'_k)}{\delta(u)}\otimes\cdots\otimes\overset{(i'_r)}{v}\otimes\cdots\otimes u).$$

Therefore we have

(8) $$\{G_{h+1}[-\sum_{j=0}^{p-1}(j+1)D_1\circ T^j[\sum_{k=1}^{r}\overset{(i'_1)}{v}\otimes\cdots\otimes\overset{(j'_k)}{\delta(v)}\otimes\cdots\otimes\overset{(i'_r)}{v}\otimes\cdots\otimes u$$

$$+\sum_{k=1}^{p-r}\overset{(i'_1)}{v}\otimes\cdots\otimes\overset{(j'_k)}{\delta(u)}\otimes\cdots\otimes\overset{(i'_r)}{v}\otimes\cdots\otimes u+\sum_{k=1}^{r}[\sum_{j=0}^{p-i'_k}(i'_k+j)D_1\circ T^j(\overset{(i'_1)}{v}\otimes$$

$$\cdots\otimes\overset{(i'_k)}{\delta(v)}\otimes\cdots\otimes\overset{(i'_r)}{v}\otimes\cdots\otimes u)+\sum_{j=p-i'_k+1}^{p-1}(i'_k+j-p)D_1\circ T^j(\overset{(i'_1)}{v}\otimes\cdots\otimes\overset{(i'_k)}{\delta(v)}\otimes$$

$$\cdots\otimes\overset{(i'_r)}{v}\otimes\cdots\otimes u)]+\sum_{k=1}^{p-r}[\sum_{j=0}^{p-j'_k}(i'_k+j)D_1\circ T^j(\overset{(i'_1)}{v}\otimes\cdots\otimes\overset{(j'_k)}{\delta(u)}\otimes\cdots\otimes\overset{(i'_r)}{v}\otimes$$

$$\cdots\otimes u)+\sum_{j=p-j'_k+1}^{p-1}(j'_k+j-p)D_1\circ T^j(\overset{(i'_1)}{v}\otimes\cdots\otimes\overset{(j'_k)}{\delta(u)}\otimes\cdots\otimes\overset{(i'_r)}{v}\otimes\cdots\otimes u]]\}=0.$$

It is easily verified that (8) is a sufficient condition for our assertion.

REMARK 2.1. For the operation $`P^h_p$, we have analogously

$$`P^h_p(\alpha+\beta)=`P^h_p(\alpha)+`P^h_p(\beta)+f_h(\alpha^{p-1}\beta+\sum_{r=1}^{p-1}\frac{(p-1)\cdots(p-r+1)}{r!}$$

$$\alpha^{p-r}\beta^r+\alpha\beta^{p-1}).$$

3. Auxiliary operations.

Consider the exact sequence

(9) $\cdots \to H^{pn-1}(X, Z_{p^h}) \to H^{pn}(X, Z_p) \to H^{pn}(X, Z_{p^{h+1}}) \to H^{pn}(X, Z_{p^h}) \to \cdots$,

induced by the exact sequence

$$0 \longrightarrow Z_p \xrightarrow{f'_h} Z_{p^{h+1}} \xrightarrow{g'_h} Z_{p^h} \longrightarrow 0 .$$

Let α be an element of $H^n(X, Z_{p^h})$ such that $\alpha^p = 0$. Then the exactness of (9) implies that there exists an element $\beta \in H^{pn}(X, Z_p)$ such that $f'_h(\beta) = {}^\backprime P^h_p(\alpha)$. Obviously β is uniquely determined mod Im δ'_h, and we now define

$$(1/_{p^h}{}^\backprime P^h_p)\,(\alpha) = \beta \qquad \text{mod } \delta'_h H^{pn-1}(X, Z_{p^h}) .$$

Let α be an element of $H^n(X, Z_p)$ such that $\delta_{h-1}(\alpha) = 0$ and $\alpha^p = 0$, then by the exactness of the following sequence

$$\cdots \to H^{pn-1}(X, Z_p) \to H^{pn}(X, Z_{p^h}) \to H^{pn}(X, Z_{h+1}) \to H^{pn}(X, Z_p) \to \cdots$$

there exists an element γ such that $f_h(\gamma) = P^h_p(\alpha)$.

We set

$$(1/_p P^h_p)\,(\alpha) = \gamma \ \text{ mod } \delta_h H^{pn-1}(X, Z_p) + f_{h-1} H^{pn}(X, Z_{p^{h-1}}) .$$

EXAMPLE 3.1. Let S^{n+1} $(n \geqq 1)$ be an odd-dimensional sphere. Then Serre proved that $H^*(\Omega(S^{n+1}), Z)$ has a base $(\alpha_1, \alpha_2, \cdots)$ with $D(\alpha_i) =$ in $i = 1, 2, \cdots$ such that $\alpha_i \alpha_j = \binom{i+1}{i} \alpha_{i+j}$. This shows $H^*(\Omega(S^{n+1}), Z_p) = Q_p[\beta_0, \beta_1, \beta_2, \cdots]$, $D(\beta_i) = p^i n$ $i = 0, 1, 2, \cdots$, where $Q_p[\beta_0, \beta_1, \beta_2, \cdots]$ means the factor algebra of the polynomial algebra generated by the elements $\beta_0, \beta_1, \beta_2, \cdots$, by the ideal generated by the elements $\beta_0^p, \beta_1^p, \beta_2^p, \cdots$. Then it is easily verified that

$$\beta_{i+1} = 1/_p {}^\backprime P^1(-\beta_i)\ (= 1/_p P^1_p(-\beta_i)) \quad i = 0, 1, 2, \cdots .$$

REMARK 3.2. We have the following relations ($D(\alpha)$ even).

If $p > 2$, $\quad \delta'_h \circ (1/_p P^h_p)\,(\alpha) = \alpha^{p-1} \Delta^h_p(\alpha)$

$$\text{mod } \delta'_{h-1} H^{pn}(X, Z_{p^{h-1}}) .$$

If $p = 2$ and $h = 1$, $\delta'_1 \circ (1/_2 P^1_2)\,(\alpha) = \alpha \Delta^1_2(\alpha) + Sq^n \Delta^1_2(\alpha)$,

if $p = 2$ and $h > 1$, $\delta'_h \circ (1/_2 P^h_2)\,(\alpha) = \alpha \Delta^h_2(\alpha)$

$$\text{mod } \delta'_{h-1} H^{2n}(X, Z_{2^{h-1}}) .$$

Musashi Institute of Technology
and University of Tokyo.

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