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A note on close-to-convex functions.

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I. Introduction

G. Szegö [1] proved the following theorem on the partial sums of the normalized schlicht functions in the unit circle;

Let the function

$$f(z)=z+\sum_{\nu=2}^{\infty}a_{\nu}z^{\nu}$$

be analytic and schlicht in $|z|<1$. *Then any one of the partial sums*

$$z+\sum_{\nu=2}^{n}a_{\nu}z^{\nu} \qquad (n=2, 3,\cdots)$$

is also schlicht in $|z|<\frac{1}{4}$, *and the constant* $\frac{1}{4}$ *can not be replaced by any greater one.*

Szegö [1] proved also that in this theorem the word 'schlicht' may be replaced by 'star-shaped with respect to the origin' or by 'convex', both in the hypothesis and in the conclusion in corresponding manner.

In this note, we shall prove a similar theorem for the class of close-to-convex functions defined by W. Kaplan [2].

We call an analytic function $f(z)$ close-to-convex for $|z|<R$, if there exists a function $\varphi(z)$, convex and schlicht for $|z|<R$, such that $f'(z)/\varphi'(z)$ has positive real part for $|z|<R$. This function $\varphi(z)$ will be called an *associate* function to the close-to-convex function $f(z)$, and we shall call $f(z)$ close-to-convex *with respect* to $\varphi(z)$, when it is needed to indicate an associate function. Close-to-convex functions in the unit circle will be simply called close-to-convex functions.

Thus close-to-convex functions are clearly schlicht for $|z|<1$, and the class of these functions includes the classes of star-shaped functions and convex functions for $|z|<1$, ([2]). We aim at proving the following theorem:

Let the function

$$f(z)=z+\sum_{\nu=2}^{\infty}a_{\nu}z^{\nu}$$

be close-to-convex for $|z|<1$ *with respect to the function*

$$\varphi(z)=z+\sum_{\nu=2}^{\infty}b_{\nu}z^{\nu}.$$

Then the n-th partial sum

$$s_{n}(z)=z+\sum_{\nu=2}^{n}a_{\nu}z^{\nu}$$

of $f(z)$ *is also close-to-convex for* $|z|<\frac{1}{4}$ *with respect to the partial sum*

$$\sigma_{n}(z)=z+\sum_{\nu=2}^{n}b_{\nu}z^{\nu}$$

of $\varphi(z)$. *The constant* $\frac{1}{4}$ *can not be replaced by any greater one.*

II. Lemmas

Following lemmas are needed in the proof of our theorem:

1°. *Let* $f(z)=z+\sum_{\nu=2}^{\infty}a_{\nu}z^{\nu}$ *be close-to-convex, then the coefficients* a_n *satisfy the inequalities*

$$|a_{n}|\leqq n \qquad (n=2, 3, \cdots)$$

(Reade [4]).

2°. *Let* $f(z)=z+\sum_{\nu=2}^{\infty}a_{\nu}z^{\nu}$ *be close-to-convex with respect to* $\varphi(z)=z+\sum_{\nu=2}^{\infty}b_{\nu}z^{\nu}$ *then*

$$\frac{1-|z|}{1+|z|}\leqq\Re\frac{f'(z)}{\varphi'(z)}\leqq\frac{1+|z|}{1-|z|},$$

$$\frac{1-|z|}{1+|z|} \leqq \frac{f'(z)}{\varphi'(z)} \leqq \frac{1+|z|}{1-|z|}$$

for $|z|<1$.

PROOF. Let $F(z)$ be analytic and have positive real part for $|z|<1$. If $F(0)$ be real, then we have

$$\frac{1-|z|}{1+|z|} F(0) \leqq \Re F(z) \leqq \frac{1+|z|}{1-|z|} F(0),$$

$$\frac{1-|z|}{1+|z|} F(0) \leqq |F(z)| \leqq \frac{1+|z|}{1-|z|} F(0)$$

for $|z|<1$ (Pólya-Szegö [5], p. 140). Our conclusion follows in putting $F(z) \equiv f'(z)/\varphi'(z)$.

3°. *Let* $\varphi(z)=z+\sum_{\nu=2}^{\infty} b_\nu z^\nu$ *be schlicht and convex for* $|z|<1$, *then, as is well known,* $|b_n| \leqq 1$ $(n=2, 3, \cdots)$ *and*

$$\frac{1}{(1+|z|)^2} \leqq |\varphi'(z)| \leqq \frac{1}{(1-|z|)^2}$$

for $|z|<1$. (K. Löwner [7]).

III. Proof of the theorem

1°. It is easy to show that $\frac{1}{4}$ is the best constant. In fact, the function $z/(1-z)^2=z+\sum_{\nu=2}^{\infty} \nu z^\nu$ is analytic and star-shaped with respect to the origin for $|z|<1$ and so close-to-convex, but the derivative of the second partial sum $s_2(z)$ of this function has a zero at the point $z=-\frac{1}{4}$, and hence $s_2(z)$ is not schlicht, a fortiori not close-to-convex, for $|z|<\rho$, if $\rho>\frac{1}{4}$.

2°. As for the first half of the theorem, it is sufficient to prove $\Re\{s_n'(z)/\sigma_n'(z)\}>0$ $(n=2, 3, \cdots)$ for $|z|<\frac{1}{4}$, as $\sigma_n(z)$ is convex for $|z|<\frac{1}{4}$ by a result of Szegö.

If we put

$$\varphi(z)=\sigma_n(z)+\rho_n(z) \qquad (n=2, 3, \cdots),$$

then $\rho_n(z)=\sum_{\nu=n+1}^{\infty} b_\nu z^\nu$ and

$$|\sigma_n'(z)|\geqq|\varphi'(z)|-|\rho_n'(z)| \qquad (n=2, 3, \cdots).$$

By the lemma 3°, we have for $|z|<1$,

$$|\varphi'(z)|\geqq 1/(1+|z|)^2,$$

and

$$|\rho_n'(z)|\leqq\sum_{\nu=n+1}^{\infty}\nu|b_\nu||z|^{\nu-1}$$

$$\leqq\sum_{\nu=n+1}^{\infty}\nu|z|^{\nu-1}.$$

Hence we have for $|z|=\frac{1}{4}$

$$|\sigma_n'(z)|\geqq\frac{1}{\left(1+\frac{1}{4}\right)^2}-\left(\frac{1}{4}\right)^n\times\frac{(n+1)-n\times\frac{1}{4}}{\left(1-\frac{1}{4}\right)^2}$$

$$=\frac{16}{25}-\frac{3n+4}{9\times 4^{n-1}}$$

$$\geqq\frac{16}{25}-\frac{5}{18}>0$$

for $n\geqq 2$. Therefore, the function

$$\Re\frac{s_n'(z)}{\sigma_n'(z)}=\Re\frac{1+2a_2z+\cdots+na_nz^{n-1}}{1+2b_2z+\cdots+nb_nz^{n-1}}, \qquad n\geqq 2$$

is harmonic for $|z|\leqq\frac{1}{4}$.

If we put

$$f(z)=s_n(z)+r_n(z) \qquad (n=2,3,\cdots),$$

then we have $r_n(z)=\sum_{\nu=n+1}^{\infty} a_\nu z^\nu$ and

$$\Re\frac{s'_n(z)}{\sigma'_n(z)}=\Re\frac{f'(z)-r'_n(z)}{\varphi'(z)-\rho'_n(z)}$$

$$=\Re\frac{f'(z)}{\varphi'(z)}+\Re\frac{\rho'_n(z)\dfrac{f'(z)}{\varphi'(z)}-r'_n(z)}{\varphi'(z)-\rho'_n(z)}$$

$$\geqq\Re\frac{f'(z)}{\varphi'(z)}-\frac{|\rho'_n(z)|\cdot\left|\dfrac{f'(z)}{\varphi'(z)}\right|+|r'_n(z)|}{|\varphi'(z)|-|\rho'_n(z)|}.$$

By the lemma 2°, we have

$$\Re\frac{f'(z)}{\varphi'(z)}\geqq\frac{1-|z|}{1+|z|} \qquad (|z|<1)$$

$$=\frac{3}{5} \qquad \left(|z|=\frac{1}{4}\right).$$

On the other hand, by lemmas 1°, 2° and 3° we have for $|z|<1$,

$$|\rho'_n(z)|\leqq|z|^n\times\frac{(n+1)-n|z|}{(1-|z|)^2},$$

$$\left|\frac{f'(z)}{\varphi'(z)}\right|\leqq\frac{1+|z|}{1-|z|},$$

and

$$|r'_n(z)|\leqq\sum_{\nu=n+1}^{\infty}\nu|a_\nu||z|^{\nu-1}$$

$$\leqq\sum_{\nu=n+1}^{\infty}\nu^2|z|^{\nu-1}.$$

Therefore we have for $|z|=\dfrac{1}{4}$,

$$\frac{|\rho_n'(z)|\cdot\left|\frac{f'(z)}{\varphi'(z)}\right|+|r_n'(z)|}{|\varphi'(z)|-|\rho_n'(z)|}\leqq\frac{\frac{3n+4}{9\times 4^{n-1}}\times\frac{5}{3}+\frac{9n^2+24n+20}{27\times 4^{n-1}}}{\frac{16}{25}-\frac{3n+4}{9\times 4^{n-1}}}.$$

Hence

$$\frac{|\rho_4'(z)|\cdot\left|\frac{f'(z)}{\varphi'(z)}\right|+|r_4'(z)|}{|\varphi'(z)|-|\rho_4'(z)|}\leqq\frac{\frac{1}{36}\times\frac{5}{3}+\frac{65}{432}}{\frac{16}{25}-\frac{1}{36}}$$

$$=\frac{2125}{6612}<\frac{1}{3}<\frac{3}{5}\qquad\text{for } |z|=\frac{1}{4}.$$

From this we can conclude

$$\Re\frac{s_4'(z)}{\sigma_4'(z)}>0$$

for $|z|=\frac{1}{4}$. By the maximum principle for harmonic functions, this inequality holds also in $|z|\leqq\frac{1}{4}$. Moreover, the inequalities for the case $n\geqq 4$ follow clearly from this inequality. Thus we have,

$$\Re\frac{s_n'(z)}{\sigma_n'(z)}>0$$

for $|z|<\frac{1}{4}$, $n\geqq 4$, that is, the theorem is true for the case $n\geqq 4$.

3°. For the case $n=3$, since the function

$$\Re\frac{s_3'(z)}{\sigma_3'(z)}=\Re\frac{1+2a_2z+3a_3z^2}{1+2b_2z+3b_3z^2}$$

is harmonic for $|z|\leqq\frac{1}{4}$, we have only to prove

(1) $$\Re\{s_3'(z)/\sigma_3'(z)\}>0\qquad\text{for } |z|=\frac{1}{4}.$$

By considering $\bar{\varepsilon}f(\varepsilon z)$ in place of $f(z)$ with a suitable ε, $|\varepsilon|=1$, the proof of (1) is reduced to that of (1) with $z=\frac{1}{4}$, i.e.

$$\Re\left\{\left(1+\frac{a_2}{2}+\frac{3a_3}{16}\right)\Big/\left(1+\frac{b_2}{2}+\frac{3b_3}{16}\right)\right\}>0.$$

As $f(z)$ is close-to-convex, the function

$$(2)\qquad \frac{f'(z)}{\varphi'(z)}=\frac{1+2a_2z+3a_3z^2+\cdots}{1+2b_2z+3b_3z^2+\cdots}=1+c_1z+c_2z^2+\cdots$$

is analytic for $|z|<1$, and has positive real part there, hence by Carathéodory-Toeplitz's theorem (Bieberbach [6]),

$$(3)\qquad |c_1|\leqq 2,\quad |2c_2-c_1^2|\leqq 4-|c_1|^2.$$

While, from (2), $2a_2=2b_2+c_1$, $3a_3=3b_3+2b_2c_1+c_2$, so

$$(4)\qquad a_2=b_2+\frac{c_1}{2},\quad a_3=b_3+\frac{2b_2c_1}{3}+\frac{c_2}{3}.$$

On the other hand, as $\varphi(z)$ is schlicht and convex for $|z|<1$, the function

$$(5)\qquad 1+z\frac{\varphi''(z)}{\varphi'(z)}=\frac{1+4b_2z+9b_3z^2+\cdots}{1+2b_2z+3b_3z^2+\cdots}=1+d_1z+d_2z^2+\cdots$$

is analytic for $|z|<1$ and has positive real part there. Hence again by Carathéodory-Toeplitz's theorem, we have

$$(6)\qquad |d_1|\leqq 2,\quad |2d_2-d_1^2|\leqq 4-|d_1|^2.$$

Since $4b_2=2b_2+d_1$, $9b_3=3b_3+2b_2d_1+d_2$ by (6), we have $d_1=2b_2$ and so $b_3=2b_2^2/3+d_2/6$.

Moreover, by the second inequality of (6), we can put $2d_2-d_1^2=\varepsilon(4-|d_1|^2)$ $(|\varepsilon|\leqq 1)$, so we have

$$(7)\qquad b_3=b_2^2+\frac{\varepsilon(1-|b_2|^2)}{3}.$$

Hence by the second formula of (4), we obtain

$$(8)\qquad a_3=b_2^2+\frac{2b_2c_1}{3}+\frac{c_2}{3}+\frac{\varepsilon(1-|b_2|^2)}{3}.$$

After all, by using (4), (7) and (8), it remains only to prove that

$$(9)\qquad \Re\frac{1+\dfrac{b_2}{2}+\dfrac{c_1}{4}+\dfrac{3b_2^2}{16}+\dfrac{b_2c_1}{8}+\dfrac{c_2}{16}+\dfrac{\varepsilon(1-|b_2|^2)}{16}}{1+\dfrac{b_2}{2}+\dfrac{3b_2^2}{16}+\dfrac{\varepsilon(1-|b_2|^2)}{16}}>0 .$$

This fraction, regarded as a function of ε, is analytic for $|\varepsilon|\leqq 1$, because

$$\left|\frac{b_2}{2}+\frac{3b_2^2}{16}+\frac{\varepsilon(1-|b_2|^2)}{16}\right|\leqq\frac{|b_2|}{2}+\frac{3|b_2|^2}{16}+\frac{1-|b_2|^2}{16}$$

$$\leqq\frac{1}{2}+\frac{1}{16}+\frac{1}{8}<1$$

Hence the proof of (9) is reduced to that of the following inequality for $|\varepsilon|=1$;

$$(10)\qquad \left[\left\{1+\frac{b_2}{2}+\frac{c_1}{4}+\frac{3b_2^2}{16}+\frac{b_2c_1}{8}+\frac{c_2}{16}+\frac{\varepsilon(1-|b_2|^2)}{16}\right\}\right.$$
$$\left.\times\left\{1+\frac{\bar{b}_2}{2}+\frac{3\bar{b}_2^2}{16}+\frac{\bar{\varepsilon}(1-|b_2|^2)}{16}\right\}\right]>0 .$$

Here we put

$$(11)\qquad \begin{cases} u=1+\dfrac{b_2}{2}+\dfrac{c_1}{4}+\dfrac{3b_2^2}{16}+\dfrac{b_2c_1}{8}+\dfrac{c_2}{16}, \\[2ex] v=1+\dfrac{b_2}{2}+\dfrac{3b_2^2}{16}, \end{cases}$$

then the left hand side of (10) has the form

$$\Re\left[\left\{u+\frac{\varepsilon(1-|b_2|^2)}{16}\right\}\left\{\bar{v}+\frac{\bar{\varepsilon}(1-|b_2|^2)}{16}\right\}\right]$$

$$=\Re u\bar{v}+\frac{(1-|b_2|^2)^2}{16^2}+\frac{1-|b_2|^2}{16}\Re(u+v)\bar{\varepsilon}$$

$$\geqq\Re u\bar{v}+\frac{(1-|b_2|^2)^2}{16^2}-\frac{1-|b_2|^2}{16}|u+v|$$

$$= \frac{|u+v|^2}{4} - \frac{|u-v|^2}{4} + \frac{(1-|b_2|^2)^2}{16^2} - \frac{1-|b_2|^2}{16}|u+v|$$

$$= \left(\frac{|u+v|}{2} + \frac{|u-v|}{2} - \frac{1-|b_2|^2}{16}\right)\left(\frac{|u+v|}{2} - \frac{|u-v|}{2} - \frac{1-|b_2|^2}{16}\right).$$

Noting that

$$\left(\frac{|u+v|}{2} + \frac{|u-v|}{2} - \frac{1-|b_2|^2}{16}\right) - \left(\frac{|u+v|}{2} - \frac{|u-v|}{2} - \frac{1-|b_2|^2}{16}\right)$$

$$= |u-v| \geqq 0\,,$$

we have only to prove that

$$\frac{|u+v|}{2} - \frac{|u-v|}{2} - \frac{1-|b_2|^2}{16} > 0\,. \tag{12}$$

On the other hand we have from (11),

$$|u+v| = 2 + b_2 + \frac{3b_2^2}{8} + \frac{c_1}{4} + \frac{b_2 c_1}{8} + \frac{c_2}{16}$$

$$\geqq 2 + b_2 + \frac{3b_2^2}{8} - \frac{|2c_1 + b_2 c_1|}{8} - \frac{c_2}{16}$$

$$= 2 + b_2 + \frac{3b_2^2}{8} - \frac{|c_1||2+b_2|}{8} - \frac{2c_2 - c_1^2}{32} + \frac{c_1^2}{32}$$

$$\geqq 2 + b_2 + \frac{3b_2^2}{8} - \frac{|2+b_2|}{4} - \frac{4-|c_1|^2}{32} - \frac{|c_1|^2}{32}$$

$$= 2 + b_2 + \frac{3b_2^2}{8} - \frac{|2+b_2|}{4} - \frac{1}{8}\,,$$

and

$$|u-v| = \frac{c_1}{4} + \frac{b_2 c_1}{8} + \frac{c_2}{16}$$

$$\leqq \frac{|2c_1 + b_2 c_1|}{8} + \frac{c_2}{16}$$

$$\leqq \frac{|2+b_2|}{4} + \frac{1}{8}\,,$$

and so it suffices to prove

$$\left|2+b_2+\frac{3b_2^2}{8}\right|-\frac{|2+b_2|}{4}-\frac{1}{8}-\frac{|2+b_2|}{4}-\frac{1}{8}-\frac{1-|b_2|^2}{8}>0,$$

that is,

$$\left|4+2b_2+\frac{3b_2^2}{4}\right|-|2+b_2|-\frac{3-|b_2|^2}{4}>0, \qquad |b_2|\leqq 1. \tag{13}$$

Now we put $2+b_2=re^{i\psi}$. Then, we have $1\leqq r\leqq 3$, and for arbitrary fixed r $(1\leqq r\leqq 3)$, ψ satisfies the inequality

$$|\psi|\leqq\psi_0(r), \tag{14}$$

where $\psi_0(r)$ is determined by the equation $|-2+re^{i\psi}|=1$ $\left(0<\psi<\frac{\pi}{2}\right)$, that is, the point $-2+re^{i\psi_0(r)}$ lies on the unit circle, and

$$\cos\psi_0(r)=(3+r^2)/4r. \tag{15}$$

Then we have

$$\left|4+2b_2+\frac{3b_2^2}{4}\right|-|2+b_2|-\frac{3-|b_2|^2}{4}$$

$$=\left|3-re^{i\psi}+\frac{3}{4}r_2e^{2i\psi}\right|-\left(\frac{3}{4}+Q\right),$$

where $Q=r-\frac{|-2+re^{i\psi}|^2}{4}=-1+r\,(1+\cos\psi)-\frac{r^2}{4}$, and by (14) and (15),

$$Q\geqq -1-r+\frac{3+r^2}{4}-\frac{r^2}{4}=-\frac{1}{4}+r.$$

Here we put

$$\Phi(r,Q)\equiv\left|3-re^{i\psi}+\frac{3}{4}r^2e^{2i\psi}\right|^2-\left(\frac{3}{4}+Q\right)^2, \tag{16}$$

where $\cos\psi=\left(Q+1-r+\frac{r^2}{4}\right)\Big/r$. Then the function

$$\Phi(r,Q)=9-\frac{7r^2}{2}+\frac{9r^4}{16}-\frac{3r(4+r^2)}{2}\cos\psi$$

$$+9r^2\cos^2\psi-\left(\frac{3}{4}+Q\right)^2$$

is monotone increasing with Q. In fact, we have

$$\frac{\partial\Phi}{\partial Q}=-\frac{3r(4+r^2)}{2}\times\frac{1}{r}+18r^2\times\frac{Q+1-r+\dfrac{r^2}{4}}{r}\times\frac{1}{r}-2\left(\frac{3}{4}+Q\right)$$

$$=21/2-18r+3r^2+16Q\,,$$

$$\frac{\partial^2\Phi}{\partial Q^2}=16>0\,,$$

and

$$\left.\frac{\partial\Phi}{\partial Q}\right|_{Q=-\frac{1}{4}+r}=21/2-18r+3r^2+16\left(-\frac{1}{4}+r\right)$$

$$=3\left(r-\frac{1}{3}\right)^2+\frac{37}{6}>0\,.$$

Hence $\Phi(r, Q)$ is a monotone increasing function of Q in $Q_1(r)\geqq Q\geqq -\frac{1}{4}+r$ for fixed r $(1\leqq r\leqq 3)$, and so attains its minimum at $Q=-\frac{1}{4}+r$. This condition $Q=-\frac{1}{4}+r$ means $\cos\psi=(3+r^2)/4r=\cos\psi_0(r)$, by (14) and (15) or, in other words, $|b_2|=1$. Hence, if we put $b_2=e^{i\theta}$ (θ: real), we have to prove, instead of (13),

$$\left|4+2e^{i\theta}+\frac{3e^{2i\theta}}{4}\right|-|2+e^{i\theta}|-\frac{1}{2}>0\,,$$

that is,

$$\left|4+2e^{i\theta}+\frac{3e^{2i\theta}}{4}\right|^2-\left(|2+e^{i\theta}|+\frac{1}{2}\right)^2>0\,. \tag{17}$$

The left hand side of (17) is

$$\left|4e^{-i\theta}+2+\frac{3e^{i\theta}}{4}\right|^2-\left(|2+e^{i\theta}|+\frac{1}{2}\right)^2$$

$$=\left(\frac{19}{4}\cos\theta+2\right)^2+\left(\frac{13}{4}\sin\theta\right)^2-\left(\frac{21}{4}+4\cos\theta+\sqrt{5+4\cos\theta}\right)$$

$$=12\cos^2\theta+15\cos\theta+\frac{149}{16}-\sqrt{5+4\cos\theta}\,,$$

while $12\cos^2\theta+15\cos\theta+\frac{149}{16}=12\left(\cos\theta+\frac{5}{8}\right)^2+\frac{37}{8}\geqq\frac{37}{8}$ and $\sqrt{5+4\cos\theta}\leqq 3<\frac{37}{8}$, hence the inequality (17) holds. Thus, our theorem is proved for the case $n=3$.

4°. Finally we consider the case $n=2$. In this case, by using (4), it is sufficient to prove that

$$\Re\frac{s_2'(z)}{\sigma_2'(z)}=\Re\frac{1+2a_2z}{1+2b_2z}$$

$$=\Re\frac{1+(2b_2+c_1)z}{1+2b_2z}>0 \qquad \text{for } |z|<\frac{1}{4}. \tag{18}$$

Since

$$\Re\frac{1+(2b_2+c_1)z}{1+2b_2z}=\frac{1}{|1+2b_2z|^2}\times\Re[\{1+(2b_2+c_1)z\}\times\{1+2\bar{b}_2\bar{z}\}]\,,$$

it is sufficient to prove that

$$\Re[\{1+(2b_2+c_1)z\}\times\{1+2\bar{b}_2\bar{z}\}]>0 \qquad \text{for } |z|<\frac{1}{4}. \tag{19}$$

Now we put

$$\begin{cases} U=1+(2b_2+c_1)z\,, \\ V=1+2b_2z\,, \end{cases} \tag{20}$$

then the left hand side of (19) has the form

$$\Re U\bar{V}=\frac{|U+V|^2}{4}-\frac{|U-V|^2}{4}.$$

Now by (20), we have

$$\begin{aligned}|U+V|&=|2+4b_2z+c_1z|\\ &\geqq 2-4|b_2|\,|z|-|c_1|\,|z|\\ &>2-4\cdot1\cdot\frac{1}{4}-2\cdot\frac{1}{4}=\frac{1}{2}\end{aligned}$$

and

$$|U-V|=|c_1 z|<2\cdot\frac{1}{4}=\frac{1}{2}.$$

Hence $|U+V|>|U-V|$ and therefore the inequality (19) holds, that is, for the case $n=2$, our theorem is proved.

Thus the proof of our theorem is completed.

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