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Note on an absolute neighborhood extensor for metric spaces.

By Yukihiro KODAMA

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1. Introduction

Recently, K. Morita [4] has introduced the following idea. Let X be a topological space and $\{A_\alpha\}$ a closed covering of X. Then X is said to *have the weak topology with respect to* $\{A_\alpha\}$, if the union of any subcollection $\{A_\beta\}$ of $\{A_\alpha\}$ is closed in X and any subset of $\bigcup_\beta A_\beta$ whose intersection with each A_β is closed relative to the subspace topology of A_β is necessarily closed in the subspace $\bigcup_\beta A_\beta$.

E. Michael [3] has introduced the following notion. A topological space X is called an *absolute extensor* (resp. *absolute neighborhood extensor*) *for metric spaces* if, whenever Y is a metric space and B is a closed subset of Y, then any continuous mapping from B into X can be extended to a continuous mapping from Y (resp. some neighborhood of B in Y) into X. A topological space X is called an *absolute retract* (resp. *absolute neighborhood retract*) *for metric spaces* if, whenever X is a closed subset of a metric space Y, there exists a continuous mapping from Y (resp. some neighborhood of Y in X) onto X which keeps X pointwise fixed. We shall use the following abbreviations as Michael [3]:

AE = absolute extensor.
ANE = absolute neighborhood extensor.
AR = absolute retract.
ANR = absolute neighborhood retract.

The purpose of this paper is to establish the following theorem.

THEOREM. *Let X be a topological space having the weak topology with respect to a closed covering $\{A_\alpha\}$. We assume that, for each finite subcollection $\{A_{\alpha_1}, A_{\alpha_2}, \cdots, A_{\alpha_n}\}$ of $\{A_\alpha\}$ with non-void intersection,* $\bigcap_{i=1}^{n} A_{\alpha_i}$

is an ANE for metric spaces. Then X is an ANE for metric spaces.

The following theorems proved by K. Borsuk [1, p. 226] and O. Hanner [2, 25.1] are consequencnces of the above theorem.

COROLLARY 1. *If A_1 and A_2 are closed subsets of a metric space X such that $A_1 \cup A_2 = X$ and A_1, A_2 and $A_1 \cap A_2$ are ANR for metric spaces, then X is an ANR for metric spaces.*

COROLLARY 2. *Any simplicial complex with the weak topology is an ANE for metric spaces.*

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2. Lemmas

LEMMA 1. *Let X be a topological space having the weak topology with respect to a closed covering $\{A_\alpha \mid \alpha \in \Lambda\}$ and Y a metric space. Let f be a continuous mapping of Y into X. Put $Y_\alpha = f^{-1}(A_\alpha)$, $\alpha \in \Lambda$. Then there exists a closed covering $\{B_\alpha \mid \alpha \in \Lambda\}$ of Y which satisfies the following conditions:*

i) $B_\alpha \subset Y_\alpha$, $\alpha \in \Lambda$.

ii) *$\{B_\alpha \mid \alpha \in \Lambda\}$ is locally finite.*

PROOF. We assume that the set Λ of indices α consists of all ordinals α less than a fixed ordinal η. Put $B_\alpha = \overline{Y_\alpha - \bigcup_{\beta<\alpha} Y_\beta}$, $\alpha < \eta$. Then $\{B_\alpha \mid \alpha < \eta\}$ is a closed covering of Y and we have $Y_\alpha \supset B_\alpha$, $\alpha < \eta$.

Now we shall show that $\{B_\alpha \mid \alpha < \eta\}$ is locally finite. Put, for each $\tau < \eta$,

$$P_\tau = \{B_\alpha \mid \alpha \leqq \tau\}, \quad Q_\tau = \{B_\alpha \mid \alpha < \tau\}.$$

We assume that for each θ less than τ $(<\eta)$ P_θ is locally finite. Since $\bigcup_{\alpha<\tau} A_\alpha$ is closed in X by the definition of the weak topology and $Y_\alpha = f^{-1}(A_\alpha)$, $\alpha < \tau$, $\bigcup_{\alpha<\tau} Y_\alpha$ is closed. Moreover, since $\bigcup_{\alpha<\tau} Y_\alpha = \bigcup B_\alpha$, $\bigcup_{\alpha<\tau} B_\alpha$ is closed in Y. Therefore, to prove that Q_τ is locally finite, it is sufficient to prove that any point p of $\bigcup_{\alpha<\tau} B_\alpha$ has some neighborhood which meets only a finite number of elements of Q_τ. Suppose that every neighborhood of a point p meets infinite elements of Q_τ. We assume p belongs to B_β for some $\beta<\tau$. Then, since Y is a metric space, we can find the following sequence of points $\{p_k\}$ of Y:

$$p_k \to p \ (k\to\infty),$$

$$p_k \in B_{\beta_k}, \ \beta<\beta_k<\beta_{k+1}<\tau, \quad k=1,2,\cdots.$$

Since $p_k\in B_{\beta_k}=\overline{Y_{\beta_k}-\bigcup_{\gamma<\beta_k} Y_\gamma}$, $k=1,2,\cdots$, we can find the following sequence $\{p^j_k \mid j=1,2,\cdots\}$ of points of $Y_{\beta_k}-\bigcup_{\gamma\in\beta_k} Y_\gamma$:

$$p^j_k \to p_k \ (j\to\infty).$$

For each $k=1,2,\cdots$, we can select j_k such that

$$p^{j_k}_k \to p \quad (k\to\infty).$$

Since $p\in B_\beta$ and f is continuous, we have $f(p)\in A_\beta$. On the other hand, since $p^{j_k}_k\in Y_{\beta_k}-\bigcup_{\gamma<\beta_k} Y_\gamma$ and $Y_\alpha=f^{-1}(A_\alpha)$, we have $f(p^{j_k}_k)\in A_{\beta_k}-\bigcup_{\gamma<\beta_k} A_\gamma$. Therefore we have

$$(*) \qquad f(p^{j_k}_k)\neq f(p^{j_l}_l), \ k\neq l; \ f(p^{j_k}_k)\neq f(p), \ k=1,2,\cdots.$$

Put $A=\bigcup A_{\beta_i}$ and $B=\{f(p^{j_k}_k)\mid k=1,2,\cdots\}$. Since $A_{\beta_k}\cap B\subset \bigcup_{i=1}^{k} f(p^{j_i}_i)$, B is closed in A which is closed in X by the definition of weak topology. But $f(p)\notin B$ by (*). This contradicts the fact that f is continuous. Thus Q_τ is locally finite. Since $P_\tau=\{Q_\tau; B_\tau\}$, P_τ is locally finite. This completes the proof of Lemma 1.

LEMMA 2. *Let Y be a metric space, B a closed subset of Y and $\{B_\alpha \mid \alpha\in\Lambda\}$ a locally finite closed covering of B. Then there exists a closed neighborhood F of* B *in Y and a locally finite closed covering $\{F_\alpha\mid\alpha\in\Lambda\}$ of F which satisfies the following conditions:*

i) $F_\alpha\cap B=B_\alpha$, $\alpha\in\Lambda$.

ii) *$\{F_\alpha\mid\alpha\in\Lambda\}$ is similar to $\{B_\alpha\mid\alpha\in\Lambda\}$.*

PROOF. Since B is fully normal and $\{B_\alpha\mid\alpha\in\Lambda\}$ is locally finite closed covering, by K. Morita [4, 1.3] there exists a locally finite covering $\{S_\alpha \mid \alpha \in \Lambda\}$ of B as follows:

i) $S_\alpha \supset B_\alpha, \quad \alpha \in \Lambda$.

ii) S_α is open relative to B.

iii) $\{S_\alpha \mid \alpha \in \Lambda\}$ is similar to $\{B_\alpha \mid \alpha \in \Lambda\}$.

Since $S_\alpha, \alpha \in \Lambda$, is F_σ as an open set of the metric space B, by K. Morita [6, Lemma 1], we can find a locally finite system $\{H_\alpha \mid \alpha \in \Lambda\}$ of open sets in Y as follows:

i) $H_\alpha \cap B = S_\alpha, \quad \alpha \in \Lambda$.

ii) $\{H_\alpha \mid \alpha \in \Lambda\}$ is similar to $\{B_\alpha \mid \alpha \in \Lambda\}$.

Take, for each α, an open set V_α of Y such that

$$H_\alpha \supset \bar{V}_\alpha \supset V_\alpha \supset B_\alpha .$$

Then $\{\bar{V}_\alpha \mid \alpha \in \Lambda\}$ is a locally finite system of closed sets of Y and is similar to $\{B_\alpha \mid \alpha \in \Lambda\}$. Put $F = \bigcup \{\bar{V}_\alpha \mid \alpha \in \Lambda\}$. Then F is a closed neighborhood of B in Y. By transfinite induction we shall prove the existence of a locally finite closed covering of F satisfying the conditions of Lemma 2. We can assume the set Λ of indices α consists of all ordinals α less than a fixed ordinal η. Suppose that for each α less than $\tau (< \eta)$ there exists a locally finite closed covering $P_\alpha = \{F_\beta, \beta \leqq \alpha ; \bar{V}_\gamma, \alpha < \gamma\}$ of F which satisfies the following conditions:

i)$_\alpha$ $F_\beta \cap B = B_\beta, \ F_\beta \subset \bar{V}_\beta, \quad \beta \leqq \alpha$.

ii)$_\alpha$ If a point p of B belongs to only $B_{\alpha_i}, \alpha_i \leqq \alpha, \ i = 1, 2, \cdots, n$, then $p \in$ ***Interior*** $(F_{\alpha_1} \cup F_{\alpha_2} \cup \cdots \cup F_{\alpha_n})$. Put $Q_\tau = \{F_\beta, \beta < \tau ; \bar{V}_\gamma, \tau \leqq \gamma\}$. Obviously Q_τ is a locally finite closed covering of F and we have

i)$_*$ $F_\beta \cap B = B_\beta, \ F_\beta \cap \bar{V}_\beta, \ \beta < \tau$.

ii)$_*$ If a point p belongs to only $B_{\alpha_i}, \alpha_i < \tau, i = 1, 2, \cdots, n$, then $p \in$ ***Interior*** $(F_{\alpha_1} \cup F_{\alpha_2} \cup \cdots \cup F_{\alpha_n})$.

We divide $\bar{V}_\tau \cap B - B_\tau$ into two disjoint subsets S_1, S_2 as follows. If a point p of $\bar{V}_\tau \cap B - B_\tau$ belongs to only $B_{\alpha_i}, \alpha_i < \tau, i = 1, \cdots, n$, then $p \in S_1$. Put $S_2 = \bar{V}_\tau \cap B - B_\tau - S_1$. Take $p \in S_1$. By the assumption we can find

$B_{\alpha_i}, i=1,\cdots,n,$ such that p belongs to only $B_{\alpha_i}, \alpha_i<\tau, i=1,\cdots,n.$ We have by ii)$_*$ $p\in Interior\ (F_{\alpha_1}\cup\cdots\cup F_{\alpha_n})$. Therefore we can find an open neighborhood $L_1(p)$ of p as follows:

$$L_1(p)\subset Interior\ (F_{\alpha_1}\cup F_{\alpha_2}\cup\cdots\cup F_{\alpha_n})$$

and

$$\overline{L_1(p)}\cap B_\tau=\phi\,.$$

Take $p'\in S_2$. By the assumption there exists some $V_\gamma, \gamma>\tau,$ containing p'. Therefore we can find an open spherical neighborhood $L_2(p')$ as follows:

$$\overline{L_2(p')}\subset V_\gamma$$

and

$$\text{the radius of } L_2(p')\leqq\frac{1}{2}\rho(p', B_\tau)\,,$$

where ρ is a metric function in Y. Put $F_\tau=\overline{V}_\tau-\bigcup_{p\in S_1}L_1(p)\bigcup_{p'\in S_2}L_2(p')$. By the construction we have $F_\tau\subset\overline{V}_\tau$ and $F_\tau\cap B=B_\tau$, i. e. i)$_\tau$ holds. Put $P_\tau=\{F_\beta, \beta\leqq\tau\,;\ \overline{V}_\gamma, \tau<\gamma\}$. Obviously P_τ is locally finite as a refinement of Q_τ. Moreover P_τ is a closed covering of F since Q_τ is a closed covering of F and

$$\overline{V}_\tau-F_\tau\subset\bigcup_{p\in S_1}L_1(p)\bigcup_{p'\in S_2}L_2(p')\subset\bigcup\{F_\beta, \beta<\tau\}\bigcup\{\overline{V}_\gamma, \tau<\gamma\}\,.$$

Next, we shall prove that P_τ satisfies the condition ii)$_\tau$. For this purpose, it is sufficient to prove that if a point q of B belongs to only $B_{\alpha_1}, B_{\alpha_2},\cdots, B_{\alpha_n}, B_\tau, \alpha_i<\tau, i=1, 2,\cdots, n,$ then

$$q\in Interior\ (F_{\alpha_1}\cup F_{\alpha_2}\cup\cdots\cup F_{\alpha_n}\cup F_\tau)\,,$$

since P_τ satisfies the condition ii)$_\alpha$ for each $\alpha<\tau$. Suppose $q\notin Interior$ $(F_{\alpha_1}\cup F_{\alpha_2}\cup\cdots\cup F_{\alpha_n}\cup F_\tau)$. Since P_τ is a locally finite closed covering of F and the condition i)$_\tau$ holds, q does not belong to $Interior \bigcup\{F_\beta,$ $\beta\leqq\tau\}$. Since V_τ is an open set containing q, we have

$$q\in\overline{V_\gamma-\bigcup\{F_\beta, \beta\leqq\tau\}}\subset\overline{\bigcup_{p\in S_1}L_1(p)\bigcup_{p'\in S_2}L_2(p')-\bigcup\{F_\beta, \beta<\tau\}}\,.$$

Moreover, since $\bigcup_{p\in S_1} L_1(p)\subset\bigcup\{F_\beta, \beta<\tau\}$, we have

$$q\in\overline{\bigcup_{p'\in S_2} L_2(p')-\bigcup\{F_\beta, \beta<\tau\}}\,.$$

Therefore we can find the following sequence of points of $\bigcup_{p'\in S_2} L_2(p')-\bigcup\{F_\beta, \beta<\tau\}$:

$$q_i\to q\ (i\to\infty)\,;\ q_i\in L_2(p_i), p_i\in S_2,\ i=1, 2,\cdots.$$

By the construction of $L_2(p_i), p_i\in S_2$, and $q\in B_\tau$ we have the following inequality:

$$\text{(1)}\qquad \rho(p_i, q_i)<\frac{1}{2}\rho(p_i, B_\tau)\leqq\frac{1}{2}\rho(p_i, q)\,.$$

Moreover,

$$\text{(2)}\qquad \rho(p_i, q)\leqq\rho(p_i, q_i)+\rho(q_i, q)\,.$$

Therefore, we have by (1) and (2)

$$(*)\qquad \frac{1}{2}\rho(p_i, q)<\rho(q_i, q)\,.$$

On the other hand, since $\{B_\alpha\,|\,\alpha<\eta\}$ is the locally finite closed covering of B and the point q belongs to only $B_{\alpha_1},\cdots, B_{\alpha_n}, B_\tau$, we have

$$q\in Interior_B\,(B_{\alpha_1}\cup\cdots\cup B_{\alpha_n}\cup B_\tau)\,,$$

where $Interior_B$ means the set of interior points relative to B. By the construction of S_2, $S_2\cup Interior_B\,(B_{\alpha_1}\cup\cdots\cup B_{\alpha_n}\cup B_\tau)=\phi$, i. e. $\rho(q, S_2)>0$. Therefore, by (*) and $p_i\in S_2, i=1, 2,\cdots$, we have

$$0<\frac{1}{2}\rho(q, S_2)<\rho(q_i, q)\,,\quad i=1, 2,\cdots.$$

This contradicts the fact that $q_i\to q\ (i\to\infty)$. Therefore $q\in$ *Interior* $(F_{\alpha_1}\cup\cdots\cup F_{\alpha_n}\cup F_\tau)$. We have proved that P_τ satisfies the conditions i)$_\tau$ and ii)$_\tau$. Put $\mathfrak{F}=\{F_\beta, \beta<\eta\}$. Then it is obvious that $\mathfrak{F}$ is a locally finite closed covering of F which we require.

LEMMA 3. *Let Y be a topological space, B a closed subset of Y*

and F a closed neighborhood of B in Y. Moreover let $\{F_\alpha \mid \alpha \in \Lambda\}$ be a locally finite closed covering of F. Suppose that for each α there is a closed neighborhood C_α of $F_\alpha \cap B$ in F_α. Then

$$C = \bigcup \{C_\alpha \mid \alpha \in \Lambda\}$$

is a closed neighborhood of B in Y.

PROOF. Since this theorem is a trivial modification of [2, 20.2], we omit the proof.

LEMMA 4. *Let Q be a class of topological spaces. Let X be a topological space and $\{A_i \mid i = 1, \cdots, n\}$ a closed covering of X. If $\bigcup_{j=1}^{p} A_{i_j} \neq \phi$, $i_j \in (1, \cdots, n)$, $j = 1, \cdots, p$,* let *$\bigcup_{j=1}^{p} A_{i_j}$ be an ANE for Q-spaces. Moreover let Y be a Q-space, B a closed subset of Y and $\{Y_i \mid i = 1, \cdots, n\}$ a closed covering of Y. Put $B_i = B \cap Y_i$, $i = 1, \cdots, n$. Let f be a continuous mapping of B into X such that $f(B_i) \subset A_i$, $i = 1, \cdots, n$. Let Q-spaces be normal. Then there exist a closed neighborhood F of B in Y and an extention h of f such that $h: F \to X$ and $h(F \cup Y_i) \subset A_i$, $i = 1, 2, \cdots, n$.*

PROOF. Put $H = \bigcup \{\bigcap_{j=1}^{p} Y_{i_j} \mid \bigcap_{j=1}^{p} Y_{i_j} \cap B = \phi, i_1, \cdots, i_p \in (1, \cdots, n)\}$. Since $H \cap B = \phi$ and Y is normal, we can find a closed neighborhood D of B in Y such that $D \cap H = \phi$. Put $D_i = D \cap Y_i$, $i = 1, \cdots, n$. Then $\{D_i\}$ is similar to $\{B_i\}$. Denote by K the nerve of $\{D_i\}$. A simplex of K is denoted by $(i_0, \cdots, i_p)$, $i_0, \cdots, i_p \in (1, \cdots, n)$. For each simplex $s = (i_0, \cdots, i_p)$ of K put $|s| = \bigcap_{j=1}^{p} D_{i_j}$. Give a simple order to the simplexes of K as follows; at first, give same dimensional simplexes a suitable order; next, if $\dim s > \dim s'$, we define s is less than s', i. e. $s < s'$. Assume that for each simplex $s <$ a fixed $\bar{s}$ the following mapping f_s and a closed set $M(s)$ are constructed:

i)$_s$ $M(s)$ is a closed neighborhood of $M(s) \cap B$ in $|s|$.

ii)$_s$ f_s is a continuous mapping of $M(s)$ into $\bigcap A_{i_j}$, where $s = (i_0, \cdots, i_p)$, such that $f_s \mid B \cap M(s) = f \mid B \cap M(s)$.

iii)$_s$ Let $s_1 = (i_0, \cdots, i_p)$ and $s_2 = (j_0, \cdots, j_q)$ be two simplexes such that $s_1 \leqq s_2 \leqq s$ and s_1, s_2 spans a simplex $s_3 = (h_0, \cdots, h_r)$ of K, where $h_0 = i_0, \cdots, h_{r-q} = i_{r-q} = j_0, \cdots, h_p = i_p = j_{p+q-v}, \cdots, h_r = j_q$. Then we have

$$(M(s_1)\cap|s_3|)\cup(M(s_2)\cap|s_3|)\subset M(s_3)$$

and

$$f_{s_1}|M(s_1)\cap M(s_2)=f_{s_2}|M(s_1)\cap M(s_2)\,.$$

We shall construct a closed neighborhood $M(\bar{s})$ of $|\bar{s}|\cap B$ in $|\bar{s}|$ and a mapping $f_{\bar{s}}$ satisfying i)$_{\bar{s}}$, ii)$_{\bar{s}}$ and iii)$_{\bar{s}}$.

Let $\bar{s}=(k_0,\cdots,k_r)$, $k_j\in(1,\cdots,n)$, $j=0,\cdots,r$. At first, let $\bar{s}$ be a principal simplex. Then since $|\bar{s}|\cap(\bigcup\{|s|,s<\bar{s}\})=\phi$ and $\bigcap_{j=0}^{r}A_{k_j}$ is an ANE for Q-spaces, there exist a closed neighborhood $M(\bar{s})$ of $|\bar{s}|\cap B$ in $|\bar{s}|$ and an extention $f_{\bar{s}}$ of $f||\bar{s}|\cap B$ over $M(\bar{s})$ such that $f_{\bar{s}}(M(\bar{s}))\subset\bigcap_{j=0}^{r}A_{k_j}$. It is obvious that the conditions i)$_{\bar{s}}$, ii)$_{\bar{s}}$ and iii)$_{\bar{s}}$ are satisfied. Next, let $\bar{s}$ be a face of s_i^{r+1}, $i=1,\cdots,m$. Then since $M(s_i^{r+1})$ is a closed neighborhood of $|s_i^{r+1}|\cap B$ in $|s_i^{r+1}|$ and $|s_i^{r+1}|\subset|\bar{s}|$, $i=1,\cdots,m$, we have $(\bigcup_{i=0}^{m}\overline{|s_i^{r+1}|-M(s_i^{r+1})})\cap|\bar{s}|\cap B=\phi$. Since $|\bar{s}|$ is a normal space there exists a closed neighborhood N of $|\bar{s}|\cap B$ in $|\bar{s}|$ such that $N\cap(\bigcup_{i=1}^{m}\overline{|s_i^{r+1}|-M(s_i^{r+1})})=\phi$. Define $g:\bigcup_{i=1}^{m}M(s_i^{r+1})\cup(|\bar{s}|\cap B)\to\bigcap_{j=0}^{r}A_{k_j}$ as follows:

$$g|M(s_i^{r+1})=f_{s_i^{r+1}},\ i=1,\cdots,m,\ g||\bar{s}|\cap B=f\,.$$

Then g is a single-valued continuous mapping by the assumption of induction. Since $\bigcap_{j=0}^{r}A_{k_j}$ is an ANE for Q-space there exist a closed neighborhood $M(\bar{s})$ of $\bigcup_{i=1}^{m}M(s_i^{r+1})\cup(|\bar{s}|\cap B)$ in $\bigcup_{i=1}^{m}M(s_i^{r+1})\cup N$ and an extention $f_{\bar{s}}$ of g over $M(\bar{s})$. Since $\bigcup_{i=1}^{m}M(s_i^{r+1})\cup N$ is a closed neighborhood of $|\bar{s}|\cap B$ in $|\bar{s}|$, it is obvious that the conditions i)$_{\bar{s}}$, ii)$_{\bar{s}}$ and iii)$_{\bar{s}}$ are satisfied. Therefore we can construct $M(s)$ and f_s satisfying i)$_s$, ii)$_s$ and iii)$_s$ for each s of K. If we put $F=\bigcup\{M(s),s\in K\}$, F is a closed neighborhood of B in Y by Lemma 3. Define $h:F\to X$ by $h|M(s)=f_s$. Since condition iii)$_s$ is satisfied for each s of K, F and h are respectively the closed neighborhood and the extention which we require.

3. The proof of Theorem

Let Y be a metric space, B a closed subset of Y and f a continuous mapping from B into X. We shall show that there exist a closed neighborhood G of B in Y and an extention h of f such that $h|B=f$ and $h: G\to Y$.

Put $C_\alpha=f^{-1}(A_\alpha)$ and $B_\alpha=\overline{C_\alpha-\bigcup_{\beta<\alpha}C_\beta}$ for each $\alpha<\eta$. Then $\mathfrak{F}_1=\{B_\alpha|\alpha<\eta\}$ is a locally finite closed covering of B by Lemma 1. By the application of Lemma 2 we can find a closed neighborhood F of B in Y and a locally finite closed covering $\mathfrak{F}_2=\{F_\alpha|\alpha<\eta\}$ such that $\mathfrak{F}_2|B=\mathfrak{F}_1$ and $\mathfrak{F}_2$ is similar to $\mathfrak{F}_1$. Since $\mathfrak{F}_2$ is a locally finite open covering of F there exists a locally finite open covering $\{V_\pi\}$ of F each closure of which meets only finite number of elements of $\mathfrak{F}_2$. Put $\mathfrak{V}=\{\overline{V}_\pi|V_\pi\cap B\neq\phi\}$. Then $\mathfrak{V}$ is a locally finite closed covering of B and $\bigcup\{\overline{V}_\pi|V_\pi\in\mathfrak{V}\}$ is a closed neighborhood of B in Y. We assume that the set of indices π consists of all ordinals π less than a fixed ordinal δ and put, for each $\theta<\delta$,

$$Q_\theta=\bigcup\{\overline{V}_\pi|\pi<\theta\}\,,\quad P_\theta=\bigcup\{\overline{V}_\pi|\pi\leqq\theta\}\,.$$

Let $\mu<\delta$. Assume for each $\theta<\mu$ the following closed set N_θ and continuous mapping f_θ are constructed:

i)$_\theta$ N_θ is a closed neighborhood of $P_\theta\cap B$ in P_θ.

ii)$_\theta$ f_θ is a continuous mapping of $N_\theta\cup B$ into X.

iii)$_\theta$ $f_\theta|B=f$.

iv)$_\theta$ If $\nu<\theta$ we have $N_\nu\subset N_\theta$ and $f_\theta|N_\nu=f_\nu$.

v)$_\theta$ For each $\alpha<\delta$ we have

$$f_\theta(N_\theta\cap F_\alpha)\subset A_\alpha\,.$$

Put $M=\bigcup\{N_\theta|\theta<\mu\}\cup B$. Define $g: M\to X$ by $g|N_\theta\cup B=f_\theta$. By iv)$_\theta$ and the local finiteness of $\mathfrak{V}$, g is single-valued and continuous. Let $F_{\alpha_1}, F_{\alpha_2},\cdots, F_{\alpha_n}$ be all elements of $\mathfrak{F}_2$ which meet $\overline{V}_\mu$. Now we apply Lemma 4 to $\overline{V}_\mu$, $\overline{V}_\mu\cap M$, $\{\overline{V}_\mu\cap F_{\alpha_i}, i=1,\cdots,n\}$ and $A_{\alpha_1}\cup\cdots\cup A_{\alpha_n}$. We can find a closed neighborhood M_μ of $\overline{V}_\mu\cap M$ in $\overline{V}_\mu$ and a continuous mapping $h: M_\mu\to A_{\alpha_1}\cup\cdots\cup A_{\alpha_n}$ such that $h|\overline{V}_\mu\cap M=g|\overline{V}_\mu\cap M$ and

$h(M_\mu \cap F_{\alpha_i}) \subset A_{\alpha_i}$, $i=1,\cdots,n$. Put $N_\mu = \bigcup\{N_\theta \mid \theta < \mu\} \cup M_\mu$. Define f_μ: $N_\mu \cup B \to X$ by $f_\mu \mid M = g$ and $f_\mu \mid M_\mu = h$. It is obvious that the conditions i)$_\mu$, ii)$_\mu$, iii)$_\mu$ iv)$_\mu$ and v)$_\mu$ are satisfied. Put $G = \bigcup\{N_\theta \mid \theta < \delta\}$. Define $h: G \to X$ by $h \mid N_\theta \cup B = f_\theta$, $\theta < \delta$. Then G is the closed neighborhood of B in Y by Lemma 3 and h is an extention of f over G by the construction. This completes the proof of the theorem.

REMARK. The above theorem cannot be strengthend by replacing "$\bigcup_{j=1}^{n} A_{\alpha_j}$ is an ANE for metric spaces for each finite collection $\{A_{\alpha_1}, A_{\alpha_2}, \cdots, A_{\alpha_n}\}$ with non-void intersection" by "A_α is an ANE for metric spaces for each α", as is shown by the following simple example.

Let S_i be the circumference in the xy-plane with $\left(\frac{1}{i}, 0\right)$, $\left(\frac{1}{i+1}, 0\right)$ as the end points of diameter, $i=1,2,\cdots$. Put $X=(0,0) \cup (\bigcup_{i=0}^{\infty} S_i)$, $A_1 = \{(x,y) \mid (x,y) \in X \text{ and } y \geqq 0\}$ and $A_2 = \{(x,y) \mid (x,y) \in X \text{ and } y \leqq 0\}$. Then it is easily shown that X is not ANE for metric spaces, though A_1 and A_2 are AE for metric spaces.

Next, the above theorem cannot be strengthend by replacing "ANE for metric spaces" by "ANE for compact Hausdorff spaces". This is shown by an example of O, Hanner [2, 23.4].

Tokyo University of Education

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