Journal of the Mathematical Society of Japan Vol. 8, No. 3, July, 1956

The linear equivalence theory of cycles and cycles of dimension zero on abelian varieties.

By Atuhiro HIRAI

(Received, March 10, 1955)

The theory of linear equivalence relation in the algebraic geometry has been hitherto developed principally for divisors on varieties. In the present paper an attempt is made to generalize this theory to the case of cycles of arbitrary dimensions. In § 1, we shall define such equivalence and show that all properties announced by A. Weil in his book "Foundations of Algebraic Geometry" as necessary properties of such equivalence are satisfied.

Linear equivalence of two cycles X, Y will be denoted by $X\sim Y$. Functions $f(X)$ (generally with rational integral values) of cycles with the property: '$X\sim Y$ implies $f(X)=f(Y)$' will be called *linear invariant*. For instance, the ranks of complete linear systems or the indices of speciality of divisors on a curves are linear invariant.

In § 2, we shall deal with cycles $\mathfrak{a}$ of dimension zero on a product variety of complete non-singular curves Γ_i, $1\leqq i\leqq n$, and introduce a linear invariant $l(\mathfrak{a})$ in generalization of the rank of complete linear system in case $n=1$.

In § 3, we consider cycles $\mathfrak{a}$ of dimension zero on an abelian variety. Using the result of § 2, we define a linear invariant $l(\mathfrak{a})$ and $\beta_i(\mathfrak{a})$, $1\leqq i\leqq n$. We define further the index d and the pseudogenus g of an abelian variety. $l(\mathfrak{a})$ can be written in the form:

$$l(\mathfrak{a})=d^n(\deg \mathfrak{a})^n+d^{n-1}\beta_1(\mathfrak{a})(\deg \mathfrak{a})^{n-1}+\cdots+d^{n-i}\beta_i(\mathfrak{a})(\deg \mathfrak{a})^{n-i}+\cdots$$

$+\cdots+\beta_n(\mathfrak{a})$, and if $\deg \mathfrak{a}$ is sufficiently large, $\beta_i(\mathfrak{a})$ becomes a constant $(-1)^i\binom{n}{i}(g-1)^i$.

In § 4, we prove the birational invariance of these $l(\mathfrak{a})$, $\beta_i(\mathfrak{a})$, g, and d.

As to the notations and terminologies, we follow the usage in Weil [1], [2], [3].

The author wishes to express his thanks to Professor S. Iyanaga, and Mr. Y. Taniyama for their valuable remarks on this paper.

§ I. Linear equivalence relation

We shall first define a linear equivalence relation between cycles of arbitrary dimensions of a variety U.

DEFINITION. Let X be a cycle of dimension r on a variety U of dimension $n, r+1 \leqq n$. We call X linearly equivalent to zero, and write $X \sim 0$ on U if X can be written in the form $\sum_{i=1}^{m} a_i A_i \cdot (\varphi_i)$, where φ_i are m functions on U, A_i are m subvarieties of U, of dimension $r+1$, and a_i are m rational integers. If X, Y are two cycles of dimension r on U and $X-Y \sim 0$, then we write $X \sim Y$ and call X, Y linearly equivalent to each other on U.

In order to deduce some properties of cycles, which are linearly equivalent to zero, we need the following lemmas.

LEMMA 1. *Let V be a subvariety of a variety U, and φ be a function on V. Then there exists a function ψ, which induces φ on V.*

PROOF. Let k be a common field of definition of U, V, and φ; let M, N be generic points of U, V respectively over k, then $\varphi(N)$ is an element of $k(N)$, and it can be written in the form $\frac{f(N_\alpha)}{g(N_\alpha)}$, where N_α is a representative of N, and $f(X)$, $g(X)$ are polynomials with coefficients in k. N_α is a generic point of a representative V_α of V. Let U_α be the representative of U and M_α the representative of M. Then M_α is a generic point of U_α and $z = \frac{f(M_\alpha)}{g(M_\alpha)}$ is an element of $k(M_\alpha)$: because the denominator of z is not zero. Therefore there exists the function ψ on U, defined over k, such that $\psi(M) = z$, and obviously this function ψ induces φ on V.

LEMMA 2. *Let V be a simple subvariety of U, of dimension r. We suppose that every simple subvariety of V is simple on U, and every simple subvariety of U, contained in V is simple on V. We suppose further more that every simple subvariety of U of dimension $r-1$ contained in V, has a representative in the fixed representative U_α of U. Let φ be a function on U, defined along V. Then there exists* *a function $\bar{\varphi}$ on U satisfying the following conditions:* $(\varphi)\cdot V=(\bar{\varphi})\cdot V$ *and* $(\bar{\varphi})_0\cdot V, (\bar{\varphi})_\infty\cdot V$ *have no common component.*

PROOF. Let k be a common field of definition of φ, U, and V, which is algebraically closed. Let M, N be generic points of U, V respectively, and ψ be the function induced by φ on V. As $\psi(N)=\varphi(N)$ is an element of $k(N)$, $\psi(N)$ can be written in the form $\frac{f(N_\alpha)}{g(N_\alpha)}$, where N_α is the representative of N in U_α, and $f(X), g(X)$ are polynomials with coefficients in k. The function ψ is defined along every simple subvariety of V, of dimension $r-1$, (cf. Weil [1] VIII prop. 5) and from the assumption, every such a subvariety of V and so its generic point has a representative in U_α and accordingly $f(X), g(X)$ do not take zero at the same time at represenstives of generic points of every simple subvariety of V, of dimension $r-1$ in U_α. If we put $z=\frac{f(M_\alpha)}{g(M_\alpha)}$, where M_α is the representative of M in U_α, then z is an element of $k(M)$, and it defines a function $\bar{\varphi}$ on U over k. This function $\bar{\varphi}$ satisfies the required conditions. Namely, with respect to the first assertion, it is obvious from the definition of $\bar{\varphi}$ and Weil [1] VIII th. 4. As for the second assertion, we shall show the absurdity of the assertion that $(\bar{\varphi})_0\cdot V$ and $(\bar{\varphi})_\infty\cdot V$ have a common component C. Under this assumption, it would follow from Weil [1] VIII th. 1, the function $\bar{\varphi}$ is not defined along C.

Let P be a generic point of C over k; then it has the representative P_α in U_α, and we have $f(P_\alpha)=0, g(P_\alpha)=0$. But it is impossible since C is also simple on V. This prove our lemma.

LEMMA 3. *Let U be a non-singular variety, V a complete non-singular variety, and A a subvariety of dimension r, of* $U\times V$ *whose projection A′ on U is itself non-singular.*

Let φ *be a function on* $U\times V$, *which induces a function, other than constant zero, on A. If the projection of A on U does not have the dimension* $r-1$, *then there exists a function* ψ *on U, such that* $pr_U A\cdot(\varphi)=A'\cdot(\psi)$.

PROOF. The function θ on $A'\times V$ induced by φ is obviously not a constant zero. As $A'\times V$, and $U\times V$ are non-singular, we have, by virtue of Weil [1] VII th. 18 cor., $A\cdot(\varphi)=(A\cdot(\theta))_{A'\times V}$ where $(\cdot)_{A'\times V}$ means the intersection-product on $A'\times V$ and therefore we have

$pr_U A\cdot(\varphi)=pr_{A'}\{A\cdot(\theta)\}_{A'\times V}$.

If $[A:A']=0$, then from our assumption follows that the dimension of A' is strictly smaller than that of $A\cdot(\varphi)$, thus we have $pr_{A'} A\cdot(\varphi)=0$.

This means that $pr_U A\cdot(\varphi)$ can be considered as the intersection of A' and a divisor of a constant function on U.

If $[A:A']$ is not zero, then there exists a function η such that $pr_{A'} A\cdot(\theta)=(\eta)$, by virture of Weil [1] VIII th. 7, as A is simple on $A'\times V$. From lemma 1 and Weil [1] VIII th. 4, there exists a function ψ on U, such that $(\eta)=A'\cdot(\psi)$.

Therefore we have $pr_U A\cdot(\varphi)=A'\cdot(\psi)$.

PROPOSITION 1. *Let U and V be two varieties, and X a cycle of U, which is linearly equivalent to zero on U. Then $X\times V$ is linearly equivalent to zero on $U\times V$.*

PROOF. By linearity of the product of varieties, we may assume that the cycle X is of the form $(\varphi)\cdot A$, where φ is a function on U, and A is a subvariety of U. Let k be a common field of definition of U, V, A, and φ; let $M\times N$ be a generic point of $U\times V$ over k, where M, N are the generic points of U, V respectively. Then there exists a function ψ on $U\times V$, defined over k, such that $\psi(M\times N)=\varphi(M)$, and $(\psi)=(\varphi)\times V$, by Weil [1] VIII th. 1 cor. 1. As $(\varphi)\cdot A$ is defined, $((\varphi)\times V)\cdot(A\times V)$ is also defined. Therefore we have $((\varphi)\cdot A)\times V=((\varphi)\times V)\cdot(A\times V)=(\psi)\cdot(A\times V)$.

PROPOSITION 2. *Let U be a non-singular variety, and V a subvariety of U, which is itself non-singular. Let X be a cycle of U, which is ~ 0 on U. If there exist m functions φ_i, m subvarieties A_i, and m integers a_i, such that $\sum_{i=1}^{m} a_i A_i\cdot(\varphi_i)=X$, and $A_i\cdot V$, $(A_i\cdot(\varphi_i)_0)\cdot V$, $(A_i\cdot(\varphi_i)_\infty)\cdot V$ are defined for every i, then the cycle $X\cdot V\sim 0$ on V.*

Proof. As $A_i\cdot V$, $((\varphi_i)_0\cdot A_i)\cdot V$, $((\varphi_i)_\infty\cdot A_i)\cdot V$ are all defined, we have $X\cdot V=(\sum_{i=1}^{m} a_i A_i\cdot(\varphi_i))\cdot V=(\sum_{i=1}^{m} a_i A_i\cdot(\varphi_i)_0-\sum_{i=1}^{m} a_i A_i\cdot(\varphi_i)_\infty)\cdot V=\sum_{i=1}^{m} a_i(A_i\cdot(\varphi_i)_0)\cdot V-\sum_{i=1}^{m} a_i(A_i\cdot(\varphi_i)_\infty)\cdot V=\sum_{i=1}^{m} a_i A_i\cdot((\varphi_i)_0\cdot V)-\sum_{i=1}^{m} a_i A_i\cdot((\varphi_i)_\infty\cdot V)=\sum_{i=1}^{m} a_i A_i\cdot((\varphi_i)_0\cdot V-(\varphi_i)_\infty\cdot V)=\sum_{i=1}^{m} a_i A_i((\varphi_i)\cdot V)$, by Weil [1] VII th. 10 and its cor. Let ψ_i be a function induced by φ_i on V, then we have

$X\cdot V=\sum_{i=1}^{m} a_i A_i\cdot(\psi_i)$. From the assumption that U and V are both non-singular, we have by virtue of Weil VII th. 18 cor., $X\cdot V=\sum_{i=1}^{m} a_i A_i\cdot(\psi_i)=\sum_{i=1}^{m} a_i((A_i\cdot V)\cdot(\psi_i))_V$, where the right side means the intersection-product of $(A\cdot V)$ and (ψ_i) on V. This proves our proposition.

To show a corollary of this proposition, we introduce the following definition.

DEFINITION. Let X be a cycle of dimension r, ~ 0 on a variety U of dimension $n, r+1\leqq n$, so that $X=\sum_{i=1}^{m} a_i A_i\cdot(\varphi_i)$, where a_i are m rational integers, φ_i are m functions on U, and A_i are m subvarieties of dimension $r+1$. If A_i are all non-singular varieties, then X will be called linearly equivalent to zero on U in strong sense, and we shall write $X\approx 0$ on U, $X\approx Y$ on U will mean $X-Y\approx 0$ on U.

COROLLARY. *Let U be a non-singular variety, V a subvariety of U, which is itself non-singular, and X a cycle of dimension r, ≈ 0 on U, so that $X=\sum_{i=1}^{m} a_i A_i\cdot(\varphi_i)$ and A_i are non-singular. Suppose furthermore that every subvariety of U of dimension r contained in some A_i has a representative in a fixed representative U_α of U. If $A_i\cdot V$, $(A_i\cdot(\varphi_i))\cdot V$ are defined for every i, then $X\cdot V\sim 0$ on V.*

PROOF. From lemma 2, there exist m functions $\bar{\varphi}_i$ on U, such that $A_i\cdot(\bar{\varphi}_i)=A_i\cdot(\varphi_i)$, and $A_i\cdot(\bar{\varphi}_i)_0$, $A_i\cdot(\bar{\varphi}_i)_\infty$ have no common component.

Therefore the fact that $(A_i\cdot(\bar{\varphi}))\cdot V$ is defined implies that $(A_i\cdot(\bar{\varphi}_i)_0)\cdot V$, $(A_i\cdot(\bar{\varphi}_i)_\infty)\cdot V$ are both defined for every i, and our corollary follows from proposition 2.

PROPOSITION 3. *Let U be a variety, and Y a cycle on U. Let X be a cycle of U, which is ~ 0 on U, so that it can be written in the form $\sum_{i=1}^{m} a_i A_i\cdot(\varphi_i)$, in such a way that $A_i\cdot Y$, $(A_i\cdot(\varphi_i)_0)\cdot Y$, and $(A_i\cdot(\varphi_i)_\infty)\cdot Y$ are defined. Then $X\cdot Y\sim 0$ on U.*

This is an immediate consequence of Weil II th. 10, and its cor.

COROLLARY. *Let U be a non singular variety and Y a cycle on U. Let X be a cycle of dimension r on U, which is ≈ 0 on U so that it can be written in the form $\sum_{i=1}^{m} a_i A_i\cdot(\varphi_i)$, where A_i are themselves* *non-singular. Suppose that every subvariety of dimension r of U, which is contained in some A_i, has a representative in a fixed representative U_α of U. Then, if $A_i\cdot Y$ and $(A_i\cdot(\varphi_i))\cdot Y$ are defined for every i, $X\cdot Y\sim 0$ on U.*

PROOF. By lemma 2, there exist m function $\overline{\varphi}_i$, such that $A_i\cdot(\overline{\varphi}_i) = A_i\cdot(\varphi_i)$, and $A_i\cdot(\overline{\varphi}_i)_0$, $A_i\cdot(\overline{\varphi}_i)_\infty$ have no common component. $(A_i\cdot(\overline{\varphi}_i)_0)\cdot Y$ and $(A_i\cdot(\overline{\varphi}_i)_\infty)\cdot Y$ are defined at the same time as $(A_i\cdot(\overline{\varphi}_i))\,Y$, and our corollary follows from proposition 3.

PROPOSITION 4. *Let U be a non singular variety and V a complete non-singular variety. Let X be a cycle on $U\times V$ of dimension r, which is ~ 0 on $U\times V$, so that it can be written in the form $\sum_{i=1}^{m} a_iA_i\cdot(\varphi_i)$, where the projections A_i' of A_i on U are also non-singular. If the dimensions of A_i' are not equal to r, then $pr_U X\approx 0$ on U.*

PROOF. From lemma 3, there exist m functions ψ_i such that $pr_U A_i\cdot(\varphi_i) = A_i'\cdot(\psi_i)$. Then, by the linearity of algebraic projection, we have $pr_U X = \sum_{i=1}^{m} a_iA_i'\cdot(\psi_i)$.

COROLLARY. *Let Γ be a non-singular curve, and V a complete non-singular variety. Let X be a cycle of dimension zero, which is ~ 0 on $\Gamma\times V$. Then $pr_U X\sim 0$ on Γ.*

PROOF. By linearity of algebraic projection, we may assume that X has a form $A\cdot(\varphi)$, where A is a subvariety of $\Gamma\times V$ of dimension 1 and φ is a function on $\Gamma\times V$. If the projection of A on Γ has the dimension 1, our corollary follows immediately from proposition 4. If the projection of A on Γ has the dimension zero, the subvariety A is of the form $P\times B$, where P is a point of Γ, and B is a subvariety of V. As $(\varphi)\cdot(P\times B)$ is defined, by Weil [1] VIII th. 4, φ induces the function ψ on $P\times B$. Let k be a common field of definition of B, ψ, Γ, and V, and $P\times Q$ a generic point of $P\times B$ over k. Let θ be a function on B defined over $k(P)$ by $\theta(Q)=\psi(P\times Q)$, and let Λ_ψ, Λ_θ be graphs of ψ, θ respectively. Then we have $\Lambda_\psi = P\times\Lambda_\theta$, and hence, from Weil [1] VIII th. 4, we have $(\varphi)\cdot(P\times B) = pr_{\Gamma\times V}\Lambda_\psi\cdot(\Gamma\times V\times\Theta) = P\times pr_V\Lambda_\theta\cdot(V\times\Theta)$. On the other hand, deg $\Lambda_\theta\cdot(V\times\Theta)=0$ follows from Weil [1] VII th. 13. Hence we have $pr_\Gamma(\varphi)\cdot(P\times B)=0$.

PROPOSITION 5. *Let U be a variety of dimension n, V a complete* *non-singular variety, and X a cycle* ~ 0 *on U, so that it can be written in the form* $X=\sum_{i=1}^{m}(\varphi_i)\cdot A_i$. *Let W be a subvariety of dimension n of* $U\times V$ *with the projection U on U. If* $W\cdot(X\times V)$, *and every* $W\cdot(A_i\times V)$ *are defined, and all the components of* $W\cdot(A_i\times V)$ *are all non-singular and have the projection* A_i *on U for every i, then we have* $pr_V W\cdot(X\times V)\sim 0$ *on V.*

PROOF. As $W\cdot(A_i\times V)$ is defined, $(W\cdot(A_i\times V))\cdot((\varphi_i)\times V)$ is also defined. In fact, if it were not defined, then one of the components of $W\cdot(A_i\times V)$ should be contained in one of the components of $(\varphi_i)\times V$. In considering the projection on U, we see that A_i is contained in some component of (φ_i). But this is impossible since $A_i\cdot(\varphi_i)$ is defined. Therefore we have $W\cdot(X\times V)=\sum_{i=1}^{m} W\cdot((A_i\times V)\cdot((\varphi_i)\times V))$ $=\sum_{i=1}^{m}(W\cdot(A_i\times V))\cdot((\varphi_i)\times V)$, and $pr_V W\cdot(X\times V)\sim 0$ follows from proposition 4.

§ 2. Product varieties of complete non-singular curves

1. Let Γ_i $1\leqq i\leqq n$ be n complete non-singular curves. In this §, we have a generalization of Riemann-Roch theorem on a product variety $\Gamma_1\times\cdots\times\Gamma_n$ of these curves Γ_i in view. From now on we take the universal domain Ω as a field of constant functions.

DEFINITION. Let φ be a function on $\Gamma_1\times\cdots\times\Gamma_n$ defined over a field k over which $\Gamma_1\times\cdots\times\Gamma_n$ is defined, and let $M_1\times\cdots\times M_n$ be a generic point of $\Gamma_1\times\cdots\times\Gamma_n$ over k. Then there exists a uniquely determined function φ_{Γ_i} defined over $k(M_1,\cdots,M_{i-1},M_{i+1},\cdots,M_n)$ such that $\varphi_{\Gamma_i}(M_i)=\varphi(M_1\times\cdots\times M_n)$ (cf. Weil [1] VIII prop. 6). We call φ_{Γ_i} the restriction of φ on Γ_i.

Let $\mathfrak{a}$ be a cycle of dimension zero on $\Gamma_1\times\cdots\times\Gamma_n$, and $L(\mathfrak{a})$ the set of all functions φ on $\Gamma_1\times\cdots\times\Gamma_n$, such that $(\varphi_{\Gamma_i})\succ -\mathfrak{a}_i$ for every i, where φ_{Γ_i} is the restriction of φ on Γ_i, and $\mathfrak{a}_i$ is the projection of $\mathfrak{a}$ on Γ_i.

PROPOSITION 1. *Let* $\Gamma_1,\cdots,\Gamma_n$ *be n complete non-singular curves, and* $\mathfrak{a}$ *a cycle of dimension* 0 *on* $\Gamma_1\times\cdots\times\Gamma_n$. *Then* $L(\mathfrak{a})$ *is a* Ω*-module.*

PROOF. This is an immediate consequence of the fact that $(\varphi$

$+\psi)_{\Gamma_i}=\varphi_{\Gamma_i}+\psi_{\Gamma_i}$ and $(c\varphi)_{\Gamma_i}=c\varphi_{\Gamma_i}$.

LEMMA 1. *Let $V_1,\cdots,V_n$ be n varieties and θ_i a function on V_i for every i. Then there exists a function φ on $V_1\times\cdots\times V_n$, whose restriction on V_i has the same divisor as that of θ_i for every i.*

PROOF. Let k be a common field of definition of $\theta_1,\cdots,\theta_n$, and $V_1,\cdots,V_n$ and let $M_1\times\cdots\times M_n$ be a generic point of $V_1\times\cdots\times V_n$ over the field k, where M_i is a generic point of V_i. If we put $z_i=\theta_i(M_i)$, z_i is an element of $k(M_i)$, and its product $w=z_1\cdots z_n$ is an element of $k(M_1,\cdots,M_n)$.

Then there exists a function φ on $V_1\times\cdots\times V_n$, such that $\varphi(M_1\times\cdots\times M_n)=w$. By definition, φ_{V_i} is defined over the field $k(M_1,\cdots,M_{i-1},M_{i+1},\cdots,M_n)$. But as $M_1,\cdots,M_n$ are independent over k, M_i is a generic point of V_i over the field $k(M_1,\cdots,M_{i-1},M_{i+1},M_n)$. Therefore the functions ζ_j on V_i defined by $\zeta_j(M_i)=z_j$ over the field $k(M_1,\cdots,M_{i-1},M_{i+1},\cdots,M_n)$ are constant functions on V_i for all $j\neq i$, and φ_i can be written in the form $\varphi_i=\zeta_1\cdots\zeta_{i-1}\zeta_{i+1}\cdots\zeta_n\cdot\theta_i$. Therefore the divisor of φ_i is the same as that of θ_i.

From now on, we denote the rank of the module $L(\mathfrak{a})$ over Ω by $l(\mathfrak{a})$, and the rank of the complete linear system $L_i(\mathfrak{a}_i)$ on the curve Γ_i by $l_i(\mathfrak{a}_i)$ where $\mathfrak{a}_i$ is the projection of $\mathfrak{a}$ on Γ_i.

PROPOSITION 2. *Let $\Gamma_1,\cdots,\Gamma_n$ be n complete non-singular curves and $\mathfrak{a}$ a cycle of dimension zero on $\Gamma_1\times\cdots\times\Gamma_n$, then we have $l(\mathfrak{a})=l_1(\mathfrak{a}_1)\cdots l_n(\mathfrak{a}_n)$, where $\mathfrak{a}_i$ is the projection of $\mathfrak{a}$ on Γ_i.*

PROOF. Let k be a field of definition of $\Gamma_1\times\cdots\times\Gamma_n$, over which $\mathfrak{a}$ is rational, then every $\mathfrak{a}_i$ is also rational over k. Therefore there exist $l_i(\mathfrak{a}_i)$ functions θ^i_j defined over k on Γ_i, such that they form a basis of $L_i(\mathfrak{a}_i)$. By lemma 1, there exists a function on $\Gamma_1\times\cdots\times\Gamma_n$, whose restriction on Γ_i has the same divisor as θ^i_j, and is a constant on $\Gamma_j (j\neq i)$. We denote this function by the same notation. If we put $\varphi_{j_1\cdots j_n}=\theta^1_{j_1}\cdots\theta^n_{j_n}$, then every $\varphi_{j_1\cdots j_n}$ is an element of $L(\mathfrak{a})$; because $(\varphi_{j_1,\cdots,j_n\Gamma_i})=((\theta'_{j_1}\cdots\theta^n_{j_n})_{\Gamma_i})=(\theta^i_{j_i})$. And these $l_1(\mathfrak{a}_1)\cdots l_n(\mathfrak{a}_n)$ functions are linearly independent over Ω. In fact, if these functions were not linearly independent over Ω, then there should exist $l_1(\mathfrak{a}_1)\cdots l_n(\mathfrak{a}_n)$ elements $c_{j_1\cdots j_n}$ of Ω, being not all zero, such that $\sum c_{j_1\cdots j_n}\varphi_{j_1,\cdots,j_n}=0$.

For instance, if we assume $c_{1,\cdots,1}\neq 0$, then $\varphi_{1\cdots 1}=-\sum_{\substack{j_1\neq 1\\ \vdots \\ j_n\neq 1}}\frac{c_{j_1\cdots j_n}}{c_{1\cdots 1}}\varphi_{j_1\cdots j_n}$.

$$(\varphi_{1\cdots 1})_{\Gamma_1}=\theta^1_{1\Gamma_1}\cdots\theta^1_{n\Gamma_1}=-\sum_{\substack{j_1\neq 1\\ \vdots \\ j_n\neq 1}}\frac{c_{j_1\cdots j_n}}{c_{1\cdots 1}}(\varphi_{j_1\cdots j_n})_{\Gamma_1}=-\sum_{\substack{j_1\neq 1\\ \vdots \\ j_n\neq 1}}\frac{c_{j_1\cdots j_n}}{c_{1\cdots 1}}\theta^1_{j_1\Gamma_1}\theta^2_{j_2\Gamma_1}\cdots\theta^n_{j_n\Gamma_1},$$

where $(\varphi_{1\cdots 1})_{\Gamma_1}$, $\theta^i_{j_i\Gamma_1}$ are restriction of $\varphi_{1\cdots 1}$, $\theta^i_{j_i}$ on Γ_i respectively. As every $\theta^2_{j_2},\cdots,\theta^n_{j_n}$ are all constants other than zero, then we have

$$\theta^1_{1\Gamma_1}=-\sum_{\substack{j_1\neq 1\\ \vdots \\ j_n\neq 1}}\frac{c_{j_1\cdots j_n}\theta^2_{j_2\Gamma_1}\cdots\theta^n_{j_n\Gamma_1}}{c_{1\cdots 1}\theta^2_{1\Gamma_1}\cdots\theta^n_{1\Gamma_1}}\theta^1_{j_1\Gamma_1}.$$

But this is impossible because θ^1_j are linearly independent over Ω.

Finally we prove that $L(\mathfrak{a})$ is spanned by these $l_1(\mathfrak{a}_1)\cdots l_n(\mathfrak{a}_n)$ functions. For this purpose, we use the induction with respect to the dimension n of $\Gamma_1\times\cdots\times\Gamma_n$. If $n=1$, it is obvious. We assume this is also true for the case of dimension $n-1$. Let φ be a function on $\Gamma_1\times\cdots\times\Gamma_n$ belonging to $L(\mathfrak{a})$, and K a field of definition of φ containing k. Let $M_1\times\cdots\times M_n$ be a generic point of $\Gamma_1\times\cdots\times\Gamma_n$ over K. Then $w=\varphi(M_1\cdots M_n)$ can be written in the form $\sum t_jz_j$, where t_j are linearly independent elements of $K(M_1)$ over K, and z_j are elements of $K(M_2,\cdots,M_n)$. In fact, let φ_1 be a restriction of φ on $\Gamma_2\times\cdots\times\Gamma_n$, and $\sum_\alpha a_\alpha A_\alpha+\sum b_\beta B_\beta$ a reduced expression of $(\varphi_1)_\infty$, where every component of $\sum_\alpha a_\alpha A_\alpha$ has the projection $\Gamma_2\times\cdots\times\Gamma_{i-1}\times\Gamma_{i+1}\times\cdots\times\Gamma_n$ on $\Gamma_2\times\cdots\times\Gamma_{i-1}\times\Gamma_{i+1}\times\cdots\times\Gamma_n$, and every component of $\sum b_\beta B_\beta$ has not a projection $\Gamma_2\times\cdots\times\Gamma_{i-1}\times\Gamma_{i+1}\times\cdots\times\Gamma_n$ on $\Gamma_2\times\cdots\times\Gamma_{i-1}\times\Gamma_{i+1}\times\cdots\times\Gamma_n$ for fixed i. Then we have $(\varphi_1)_\infty\cdot(M_2\times\cdots\times M_{i-1}\times\Gamma_i\times M_{i+1}\times\cdots\times M_n)=(\sum_\alpha a_\alpha A_\alpha)\cdot(M_2\times\cdots\times M_{i-1}\times\Gamma_i\times M_{i+1}\times\cdots\times M_n)$. As left side is equal to the cycle $M_2\times\cdots\times M_{i-1}\times(\varphi_{1\Gamma_i})_\infty\times M_{i+1}\times\cdots\times M_n$, and as $\varphi_{1\Gamma_i}=\varphi_{\Gamma_i}$, we have, from the assumption, $(\sum_\alpha a_\alpha A_\alpha)\cdot(M_2\times\cdots\times M_{i-1}\times\Gamma_i\times M_{i+1}\times\cdots\times M_n)\prec M_2\times\cdots\times M_{i-1}\times\mathfrak{a}_i^+\times M_{i+1}\times\cdots\times M_n$, where $\mathfrak{a}_i^+$ is the positive part of $\mathfrak{a}_i$. This shows that every component of $(\sum_\alpha a_\alpha A_\alpha)\cdot(M_2\times\cdots\times M_{i-1}\times\Gamma_i\times M_{i+1}\times\cdots\times M_n)$ is algebraic over $K(M_2,\cdots,M_{i-1},M_{i+1},\cdots,M_n)$. Let $M_2\times\cdots\times M_{i-1}\times Q_i\times M_{i+1}\times\cdots\times M_n$ be a component of $A_\alpha \cdot (M_2 \times \cdots \times M_{i-1} \times \Gamma_i \times M_{i+1} \times \cdots \times M_n)$, then it is a generic point of A_α over the algebraic closure of $K(M_1)$. Let D be a locus of $M_2 \times \cdots \times M_{i-1} \times Q_i \times M_{i+1} \times \cdots \times M_n$ over the algebraic closure $\bar{K}$ of K, then it contains A. But the dimension of D is the same that of A, therefore, $A_\alpha = D$, and A_α is algebraic over K. As every component of $(\varphi_1)_\infty$ has the projection $\Gamma_2 \times \cdots \times \Gamma_{i-1} \times \Gamma_{i+1} \times \cdots \times \Gamma_n$ on $\Gamma_1 \times \cdots \times \Gamma_{i-1} \times \Gamma_{i+1} \times \cdots \times \Gamma_n$ for some i, every component of $(\varphi_1)_\infty$ is algebraic over K. As $(\varphi_1)_\infty$ is rational over $K(M_1)$, it is also rational over K. In fact, let σ be an automorphism of $\bar{K}$ over K. As $K(M_1)$ is a regular extension of K, $K(M_1) \cap \bar{K} = K$. Hence σ can be extended to an automorphism of $\overline{K(M_1)}$ over $K(M_1)$. Therefore $(\varphi_1)_\infty$ is invariant by every automorphism of $\bar{K}$ over K. Let $(\varphi_1)_\infty = \sum a_\alpha A_\alpha$ be the reduced expression of $(\varphi_1)_\infty$, and P_α a generic point of A_α over the algebraic closure $\overline{K(M_1)}$ of $K(M_1)$. Then a_α is a multiple of $[K(M_1, P_\alpha) : K(M_1)]_\iota$. As $K(P_\alpha)$ and $K(M_1)$ are independent over K and as $K(M_1)$ is a regular extension of K, $K(P_\alpha)$ and $K(M_1)$ are linearly disjoint over K. Therefore we have $[K(M_1, P_\alpha) : K(M_1)]_\iota = [K(P_\alpha) : K]_\iota$. And this shows that $(\varphi_1)_\infty$ is rational over K. Hence, by the same method of proof of Weil VIII th. 10, there exist t_λ, z_λ such that t_λ are linearly independent elements of $K(M_1)$ over K, and z_λ are elements of $K(M_2, \cdots, M_n)$ and w can be written in the form $w = \sum t_\lambda z_\lambda$. Every t_λ defines a function ξ_λ on $\Gamma_1 \times \cdots \times \Gamma_n$ such that $\xi_\lambda(M_1 \times \cdots \times M_n) = t_\lambda$, and every z_λ defines a function η_λ on $\Gamma_1 \times \cdots \times \Gamma_n$ such that $\eta_\lambda(M_1 \times \cdots \times M_n) = z_\lambda$. Then the function φ can be written in the form $\varphi = \sum \xi_\lambda \eta_\lambda$ and we have $\varphi_{\Gamma_i} = \sum \xi_{\lambda \Gamma_i} \eta_{\lambda \Gamma_i}$ $(i \neq 1)$. Every $\eta_{\lambda \Gamma_i}$ is defined over $K(M_2, \cdots, M_{i-1}, M_{i+1}, \cdots, M_n)$ and $\xi_{\lambda \Gamma_i} = t_\lambda$ are linearly independent over $K(M_2, \cdots, M_{i-1}, M_{i+1}, \cdots, M_n)$. Hence we have, from Weil [1] VIII th. 10, $(\eta_{\lambda \Gamma_i}) \succ -\mathfrak{a}_i$ for every $i \neq 1$, because $\mathfrak{a}_i$ is rational over $K(M_2, \cdots, M_{i-1}, M_{i+1}, \cdots, M_n)$.

If the elements z_j are linearly independent over K, then we have $(\xi_{\lambda \Gamma_1}) \succ -\mathfrak{a}_1$ by the same theorem. If z_j are not linearly independent over K, for instance, only r elements $z_1, \cdots, z_r$ are linearly independent over K, then the other elements $z_{r+\mu}$ can be written in the form $z_{r+\mu} = \sum_{\lambda=1}^{r} c_{\mu\lambda} z_\lambda$, where $c_{\mu\lambda}$ are elements of K.

Hence we have $w = \sum_{\lambda=1}^{r} (t_\lambda + \sum_\mu c_{\mu\lambda} t_{r+\mu}) z_\lambda$. If we denote $t_\lambda + \sum_\mu c_{\mu\lambda} t_{r+\mu}$

anew by the same t_λ again, and the function defined by this new t_λ over K again by ξ_λ, then we have $(\xi_{\lambda\Gamma_1}) > -\mathfrak{a}_1$, from the fact that $\mathfrak{a}_1$ is rational over K, and Weil [1] VIII th. 10.

Therefore the restriction of ξ_λ on Γ_1 have the form $\xi_{\lambda\Gamma_1} = \sum c_{\lambda j_1}\theta^1_{j_1}$ and we have $\xi_\lambda = \sum c_{\lambda j_1}\theta^1_{j_1}$.

As $\eta_{\lambda\Gamma_2\times\cdots\times\Gamma_n}$ is an element of $L(\mathfrak{a}_{\Gamma_2\times\cdots\times\Gamma_n})$ where $\mathfrak{a}_{\Gamma_2\times\cdots\times\Gamma_n}$ is the projection of $\mathfrak{a}$ on $\Gamma_2\times\cdots\times\Gamma_n$, we have from the assumption induction $\eta_{\lambda\Gamma_2\times\cdots\times\Gamma_n} = \sum c_{\lambda j_2\cdots j_n}\theta^2_{j_2}\cdots\theta^n_{j_n}$, and then we have $\eta_\lambda = \sum c_{\lambda j_2\cdots j_n}\theta^2_{j_2}\cdots\theta^n_{j_n}$.

Therefore we have $\varphi = \sum \xi_\lambda\eta_\lambda = \sum c_{\lambda j_1}c_{\lambda j_2\cdots j_n}\theta^1_{j_1}\cdots\theta^n_{j_n} = \sum c_{\lambda j_1}c_{\lambda j_n}\varphi_{j_1\cdots j_n}$.

Hence the module $L(\mathfrak{a})$ is spanned by $l_1(\mathfrak{a}_1)\cdots l_n(\mathfrak{a}_n)$ functions

PROPOSITION 3. *Let $\Gamma_1,\cdots,\Gamma_n$ be n complete non-singular curves and $\mathfrak{a},\mathfrak{b}$ be two cycles over the product variety $\Gamma_1\times\cdots\times\Gamma_n$ such that $\mathfrak{a}\sim\mathfrak{b}$, then we have $l(\mathfrak{a}) = l(\mathfrak{b})$.*

PROOF. From the cor. of proposition 4 of § 1, and the fact that the product of complete varieties is also complete, we have $pr_{\Gamma_i}\mathfrak{a}-\mathfrak{b} \sim 0$ on Γ_i. Therefore, $l_i(\mathfrak{a}_i) = l_i(\mathfrak{b}_i)$ for every i, and our proposition follows from proposition 2.

The value $l(0)$ of zero cycle 0 is 1. For, let φ be an element of $L(0)$, and K be a field of definition of φ, and $M_1\times\cdots\times M_n$ be a generic point of $\Gamma_1\times\cdots\times\Gamma_n$ over K. Then $w = \varphi(M_1\times\cdots\times M_n)$ is an element of $K(M_1, M_2,\cdots, M_{i-1}, M_{i+1},\cdots, M_n)$ for every i, namely w is an element of the intersection of n fields $K(M_1,\cdots, M_{i-1}, M_{i+1},\cdots, M_n)$. As $M_1,\cdots, M_n$ are independent over K, and as $K(M_1,\cdots, M_n)$ is a regular extension of K, the intersection $\bigcap_i K(M_1,\cdots, M_{i-1}, M_{i+1},\cdots, M_n)$ is K. Therefore the function φ is a constant, and we have $l(0) = 1$.

THEOREM 1. *Let $\Gamma_1\times\cdots\times\Gamma_n$ be a product variety of n complete non-singular curves Γ_i. If a degree of a cycle $\mathfrak{a}$ of dimension zero on $\Gamma_1\times\cdots\times\Gamma_n$ is sufficiently large, then $l(\mathfrak{a})$ is a polynomial of a degree of cycle $\mathfrak{a}$ with integral coefficients, independent of $\mathfrak{a}$.*

PROOF. By Riemann-Roch theorem, there exists an integer N_i for every Γ_i, such that if the degree of a cycle $\mathfrak{b}$ is larger than N_i, then $l(\mathfrak{b}) = \deg\mathfrak{b} - g_i + 1$, where g_i is the genus of Γ_i. Let $\mathfrak{a}$ be a cycle of $\Gamma_1\times\cdots\times\Gamma_n$ of dimension zero, whose degree is larger than the maximum of N_i's, then we have $l(\mathfrak{a}_i) = \deg\mathfrak{a}_i - g_i + 1$ for every i, where $\mathfrak{a}_i$ is the projection of $\mathfrak{a}$ on Γ_i. From prop. 2, we have

$$l(\mathfrak{a})=\prod_{i=1}^{n}(\deg \mathfrak{a}_i-g_i+1)=(\deg \mathfrak{a})^n+(-1)(\sum_{i=1}^{n}(g_i-1))(\deg \mathfrak{a})^{n-1}$$
$$+\cdots+(-1)^j(\sum(g_{i_1}-1)\cdots(g_{i_j}-1))(\deg \mathfrak{a})^{n-j}+\cdots+(-1)^n\prod_{i=1}^{n}(g_i-1)\,.$$

If we put $c_1=(-1)\sum(g_i-1),\cdots,c_n=(-1)^n\prod_{i=1}^{n}(g_i-1)$, we have

$$l(\mathfrak{a})=(\deg \mathfrak{a})^n+c_1(\deg \mathfrak{a})^{n-1}+\cdots+c_n \qquad \cdots(*)\,.$$

COROLLARY. *The genera of curves Γ_i are determined by the coefficients $c_1,\cdots,c_n$ of polynomial* (*).

PROOF. Let g_i be genus of curves Γ_i, then (g_i-1) is a root of the polynomial $x^n+c_1x^{n-1}+\cdots+c_n$.

2. Hereafter we shall confine ourselves to the case where the genera of Γ_i are the same. Let $\Gamma_1,\cdots,\Gamma_n$ be n complete non-singular curves which have the same genera, and let ω_i be a function on $\Gamma_i\times\Gamma_i$ such that $v_{\Delta_i}(\omega_i)=1$ where Δ_i is the diagonal of $\Gamma_i\times\Gamma_i$. By lemma 1, for every set of n functions $\omega_1,\cdots,\omega_n$ defined above, there exists a function ω of $\Gamma_1\times\Gamma_1\times\Gamma_2\times\Gamma_2\times\cdots\times\Gamma_n\times\Gamma_n$, such that $(\omega_{\Gamma_i\times\Gamma_i})=(\omega_i)$ for every i. Let k be a common field of definition of ω and $\Gamma_1\times\Gamma_1\times\cdots\times\Gamma_n\times\Gamma_n$, X_i a variety of the form $\Gamma_1\times\Gamma_1\times\cdots\times\Gamma_{i-1}\times\Gamma_{i-1}\times\Delta_i\times\Gamma_{i+1}\times\Gamma_{i+1}\times\cdots\times\Gamma_n\times\Gamma_n$ and let $M_1\times M_1'\times\cdots\times M_n\times M_n'$ be a generic point of $\Gamma_1\times\Gamma_1\times\cdots\times\Gamma_n\times\Gamma_n$ over the field k.

As the projection of X_i on $\Gamma_1\times\Gamma_1\times\cdots\times\Gamma_{j-1}\times\Gamma_{j-1}\times\Gamma_{j+1}\times\Gamma_{j+1}\times\cdots\times\Gamma_n\times\Gamma_n$ is not the variety $\Gamma_1\times\Gamma_1\times\cdots\times\Gamma_{j-1}\times\Gamma_{j-1}\times\Gamma_{j+1}\times\Gamma_{j+1}\times\cdots\times\Gamma_n\times\Gamma_n$, if $j\neq i$. Therefore we have $X_i\cdot(M_1\times M_1'\times\cdots\times M_{j-1}\times M_{j-1}'\times\Gamma_j\times\Gamma_j\times M_{j+1}\times M_{j+1}'\times\cdots\times M_n\times M_n')=0$. If we put $X=\sum_{i=1}^{n}X_i$, then $X\cdot((\omega)-X)$ is defined. In fact, if $X\cdot((\omega)-X)$ were not defined, then there would exist a component of X, such that it is also a component of $((\omega)-X)$. If we assume that X_i is such a component of X, then the integer $d=v_{X_i}(\omega)$ is other than 1. But the intersection product $(\omega)\cdot(M_1\times M_1'\times\cdots\times M_{i-1}\times M_{i-1}'\times\Gamma_i\times\Gamma_i\times M_{i+1}\times M_{i+1}'\times\cdots\times M_n\times M_n')$ is defined, and has the form $dM_1\times M_1\times\cdots\times M_{i-1}\times M_{i-1}'\times\Delta_i\times M_{i+1}\times M_{i+1}'\times\cdots\times M_n\times M_n'+Z$, where Z is a cycle which does not contain $M_1\times M_1'\times\cdots\times M_{i-1}\times M_{i-1}'\times\Delta_i\times M_{i+1}\times M_{i+1}'\times\cdots\times M_n\times M_n'$ as its component. Thus we have $v_{\Delta_i}(\omega_{\Gamma_i\times\Gamma_i})=d$. But this contradicts to our assumption.

Let Z be a cycle of the form $Z=\sum_{i=1}^{n} M_1\times M_1'\times\cdots\times M_{i-1}\times M_{i-1}'\times\Gamma_i\times\Gamma_i\times M_{i+1}\times M_{i+1}'\times\cdots\times M_n\times M_n'$ and Y a cycle of the form $Y=(n-1)g\,M_1\times M_1'\times\cdots\times M_n\times M_n'$, where g is the common genus of Γ_i. Then the intersection-product $X\cdot((\omega)-X)\cdot Z$ is defined. From now on, the cycles with the same form as these cycles Z, Y, we call (Z)-cycles, (Y)-cycles. The projection of $X\cdot((\omega)-X)\cdot Z-Y$ on the product variety $\Gamma_1\times\Gamma_2\times\cdots\times\Gamma_n$ of the 1st, 3rd,$\cdots$, $(2n-1)$-th components of $\Gamma_1\times\Gamma_1\times\cdots\times\Gamma_n\times\Gamma_n$ will be called a canonical cycle of dimension zero on $\Gamma_1\times\Gamma_2\cdots\times\Gamma_n$.

The projection of a canonical cycle on the curve Γ_i is the canonical divisor on Γ_i. The definition of canonical cycle depends not only on the function ω, but also the cycles Z and Y; we have however:

PROPOSITION 4. *Let $\mathfrak{k}_1$ and $\mathfrak{k}_2$ be any two canonical cycles on $\Gamma_1\times\cdots\times\Gamma_n$. Then there exist two canonical cycles $\mathfrak{k}_1'$ and $\mathfrak{k}_2'$, such that $\mathfrak{k}_1, \mathfrak{k}_2$ are isomorphic to $\mathfrak{k}_1', \mathfrak{k}_2'$ respectively, and $\mathfrak{k}_1'\sim\mathfrak{k}_2'$ on $\Gamma_1\times\cdots\times\Gamma_n$.*

PROOF. From the definition, there exist fields k, k', and functions ω_1, ω_2 defined over k, k' respectively, and cycles $Z_1, Y_1; Z_2, Y_2$ defined by using the fields k, k' respectively, such that

$$\mathfrak{k}_1=pr_{\Gamma_1\times\cdots\times\Gamma_n}(X\cdot((\omega_1)-X)\cdot Z_1-Y_1)\,,$$
$$\mathfrak{k}_2=pr_{\Gamma_1\times\cdots\times\Gamma_n}(X\cdot((\omega_2)-X)\cdot Z_2-Y_2)\,.$$

Let K be a composite of k and k' and Z_3, Y_3 a (Z)-cycle and a (Y)-cycle over K.

Then there exist two isomorphism σ_1 and σ_2, such that $Z_3^{\sigma_1}=Z_1$, $Z_3^{\sigma_2}=Z_2$, $Y_3^{\sigma_1}=Y_1$ and $Y_3^{\sigma_2}=Y_2$. If we put $\mathfrak{k}_1'=pr_{\Gamma_1\times\cdots\times\Gamma_n}(X\cdot((\omega_1)-X)\cdot Z_3$ $Y_3)$, and $\mathfrak{k}_2'=pr_{\Gamma_1\times\cdots\times\Gamma_n}(X\cdot((\omega_2)-X)\cdot Z_3-Y_3)$, then these $\mathfrak{k}_1'$ and $\mathfrak{k}_2'$ are canonical cycles, and we have $\mathfrak{k}_1'^{\sigma_1}=(pr_{\Gamma_1\times\cdots\times\Gamma_n}(X\cdot((\omega_1)-X)\cdot Z_3-Y_3))^{\sigma_1}$ $=pr_{\Gamma_1\times\cdots\times\Gamma_n}(X\cdot((\omega_1)-X)\cdot Z_1-Y_1)$, and $\mathfrak{k}_2'^{\sigma_2}=(pr_{\Gamma_1\times\cdots\times\Gamma_n}(X\cdot((\omega_2)-X)\cdot Z_3-Y_3))^{\sigma_2}=pr_{\Gamma_1\times\cdots\times\Gamma_n}(X\cdot((\omega_2)-X)\cdot Z_2-Y_2)$.

On the other hand, we have $X\cdot((\omega_1)-X)\cdot Z_3-Y_3-X\cdot((\omega_2)-X)\cdot Z_3-Y_3=X\cdot((\omega_1)-X)\,Z_3-X((\omega_2)-X)\cdot Z_3=X\cdot((\omega_1)-(\omega_2))\cdot Z_3=X\cdot(\omega_1/\omega_2)\cdot Z_3=(X\cdot Z_3)\cdot(\omega_1/\omega_2)$. Operating $pr_{\Gamma_1\times\cdots\times\Gamma_n}$ on both side of this equation, we have $\mathfrak{k}_1'-\mathfrak{k}_2'\sim 0$, from prop. 4 of § 1.

Now we get the following theorem:

THEOREM 2. *Let $\Gamma_1\times\cdots\times\Gamma_n$ be the product variety of n complete non-singular curves Γ_i, which have the same genus g. Then we have*

$$l(\mathfrak{a})=(\deg\mathfrak{a})^n+\beta_1(\mathfrak{a})(\deg\mathfrak{a})^{n-1}+\cdots+\beta_i(\mathfrak{a})(\deg\mathfrak{a})^{n-1}+\cdots+\beta_n(\mathfrak{a}).$$

$$\beta_i(\mathfrak{a})=\sum_{j=0}^{i}(-(g-1))^j\binom{n-i+j}{j}H(i-j,\mathfrak{a}),\ \textit{and}$$

$$H(i,\mathfrak{a})=\sum l(\Pi_{j_1\cdots j_i}(\mathfrak{k}-\mathfrak{a})),$$

where $\mathfrak{k}$ is a canonical cycle of dimension zero on $\Gamma_1\times\cdots\times\Gamma_n$, $\Pi_{j_1\cdots j_i}(\mathfrak{k}-\mathfrak{a})$ is the projection of $\mathfrak{k}-\mathfrak{a}$ on $\Gamma_{j_1}\times\cdots\times\Gamma_{j_i}$, $l(\Pi_{j_1\cdots j_i}(\mathfrak{k}-\mathfrak{a}))$ is the rank of the module defined on $\Gamma_{j_1}\times\cdots\times\Gamma_{j_i}$ by $\Pi_{j_1\cdots j_i}(\mathfrak{k}-\mathfrak{a})$, and the last sum is taken over all the combinations of i elements from the set $(1, 2,\cdots, n)$.

PROOF. From proposition 2, and theorem of Riemann-Roch, we have

$$l(\mathfrak{a})=\prod_{i=1}^{n}(\deg\mathfrak{a}_i-g+1+r_i(\mathfrak{a}_i)),$$

where $\mathfrak{a}_i$ is the projection of $\mathfrak{a}$ on Γ_i, and every $r_i(\mathfrak{a}_i)$ is the index of speciality of $\mathfrak{a}_i$ on Γ_i. If we put $\beta_i(\mathfrak{a})=\sum_{\binom{n}{i}}\prod_{j_1\cdots j_i}(r_{j_k}(\mathfrak{a}_{j_k})-g+1)$, then we have $l(\mathfrak{a})=\sum_{i=0}^{n}\beta_i(\mathfrak{a})(\deg\mathfrak{a})^{n-i}$. Every $\beta_i(\mathfrak{a})$ can be written in the form; $\beta_i(\mathfrak{a})=\sum_{j=0}^{i}\binom{n-i+j}{j}(-(g-1))^jH(i-j,\mathfrak{a})$, if we put $H(i,\mathfrak{a})=\sum_{\binom{n}{i}}l(\Pi_{j_1\cdots j_i}(\mathfrak{k}-\mathfrak{a}))$.

COROLLARY. 1. *Let $\mathfrak{k}$ be a canonical cycle, then we have*

$$l(\mathfrak{k})=g^n,\quad \beta_i(\mathfrak{k})=\binom{n}{i}(-g+1)^i,\ \textit{and}\ \deg\mathfrak{k}=2g-2.$$

This is an immediate consequence of prop. 2 and th. 2.

COROLLARY. 2. *If $\mathfrak{a}$ and $\mathfrak{b}$ are two cycles of dimension zero, such that $\mathfrak{a}\sim\mathfrak{b}$ on $\Gamma_1\times\cdots\times\Gamma_n$, then we have*

$$\beta_i(\mathfrak{a})=\beta_i(\mathfrak{b})\ \textit{for every } i,\ \textit{and}\ \deg\mathfrak{a}=\deg\mathfrak{b}.$$

PROOF. Let $\mathfrak{a}_j, \mathfrak{b}_j$ be the projections of $\mathfrak{a}, \mathfrak{b}$ on Γ_j respectively; then we have $\mathfrak{a}_j-\mathfrak{b}_j\sim 0$ for every j. (cf. § 1. prop. 4 cor.). Let $r_j(\mathfrak{a}_j)$, $r_j(\mathfrak{b}_j)$ be indices of speciality of $\mathfrak{a}_j, \mathfrak{b}_j$ respectively, then we have

$r_j(\mathfrak{a}_j)=r_j(\mathfrak{b}_j)$, and consequently $H(i,\mathfrak{a})=H(i,\mathfrak{b})$. Hence we have $\beta_i(\mathfrak{a})=\beta_i(\mathfrak{b})$. Therefore we have $\deg\mathfrak{a}=\deg\mathfrak{b}$, from th. 2.

Two canonical cycles of dimension zero of the form $pr_{\Gamma_1\times\cdots\times\Gamma_n} X\cdot[(\omega)-X]\cdot Z_1-Y_1$, $pr_{\Gamma_1\times\cdots\times\Gamma_n} X\cdot[(\omega)-X]\cdot Z_2-Y_2$ with the same $X[(\omega)-X]$ will be called equivalent to each other. Every class of canonical cycles by this equivalence relation is called a subcanonical class of dimension zero.

Now we introduce pseudo-differentials on $\Gamma_1\times\cdots\times\Gamma_n$ as follows.

Let $\mathfrak{D}_1,\cdots,\mathfrak{D}_n$ be the additive groups of differentials on $\Gamma_1,\cdots,\Gamma_n$ respectively. $\mathfrak{D}_i$ forms a module over the field of functions on Γ_i (cf. Weil [2]). Let $\mathfrak{D}$ be the direct sum of $\mathfrak{D}_1,\cdots,\mathfrak{D}_n$. Then $\mathfrak{D}$ can be considered as a module over the field of functions on $\Gamma_1\times\cdots\times\Gamma_n$, for the operation by a function φ on $\Gamma_1\times\cdots\times\Gamma_n$ on $\mathfrak{D}$ being defined by $\varphi(\alpha_1+\cdots+\alpha_n)=\varphi_{\Gamma_1}\cdot\alpha_1+\cdots+\varphi_{\Gamma_n}\cdot\alpha_n$ where $\alpha_i\in\mathfrak{D}_i$.

We call an element $\alpha=\sum_{i=1}^{n}\alpha_i$ of this module $\mathfrak{D}=\sum_{i=1}^{n}\mathfrak{D}_i$ a pseudo-differential on $\Gamma_1\times\cdots\times\Gamma_n$.

PROPOSITION 1. *Let $\alpha=\sum_{i=1}^{n}\alpha_i$, $\alpha'=\sum_{i=1}^{n}\alpha'_i$ be two pseudo-differentials on $\Gamma_1\times\cdots\times\Gamma_n$. Let ω_i, ω'_i be two functions on $\Gamma_i\times\Gamma_i$ such that $\alpha_i=\{\omega_i\}$, $\alpha'_i=\{\omega'_i\}$ (cf. Weil [2] p. 21), and put $\varphi_i=\dfrac{\{\omega_i\}}{\{\omega'_i\}}$. By lemma 1, there exist two functions ω, ω' such that $(\omega_{\Gamma_i\times\Gamma_i})=(\omega_i)$, $(\omega'_{\Gamma_i\times\Gamma_i})=(\omega'_i)$ for every i.*

Let furthermore $\mathfrak{k}$, $\mathfrak{k}'$ be two canonical cycles defined by

$$\mathfrak{k}=pr_{\Gamma_1\times\cdots\times\Gamma_n}(X\cdot[(\omega)-X]\cdot Z-Y),\ \mathfrak{k}'=pr_{\Gamma_1\times\cdots\times\Gamma_n}(X\cdot[(\omega')-X]\cdot Z-Y).$$

Then we have $\mathfrak{k}-\mathfrak{k}'=(pr_{\Gamma_1\times\cdots\times\Gamma_n}X\cdot Z)\cdot(\varphi)$, where φ is a function on $\Gamma_1\times\cdots\times\Gamma_n$ such that $(\varphi_{\Gamma_i})=(\varphi_i)$.

PROOF. From the definition, we get by a simple calculation $\mathfrak{k}-\mathfrak{k}'=\sum M_1\times\cdots\times M_{i-1}\times(\varphi_i)\times M_{i+1}\times\cdots\times M_n$. By lemma 1, there exists a function φ on $\Gamma_1\times\cdots\times\Gamma_n$ such that $(\varphi_{\Gamma_i})=(\varphi_i)$ for every i. Then we have $\mathfrak{k}-\mathfrak{k}'=pr_{\Gamma_1\times\cdots\times\Gamma_n}(X\cdot Z)\cdot(\varphi)$ with this φ.

Therefore, for every pseudo-differential α, there corresponds a canonical class $\bar{\mathfrak{k}}$ uniquely.

Let φ be a function on $\Gamma_1\times\cdots\times\Gamma_n$, and $M_1\times\cdots\times M_n$ and $M'_1\times\cdots$

$\times M_n'$ be independent generic points of $\Gamma_1\times\cdots\times\Gamma_n$ over a common field of definition of φ and $\Gamma_1\times\cdots\times\Gamma_n$.

Let $\varphi_{\partial_i}(M_i, M_i')$ be a function on $\Gamma_i\times\Gamma_i$ defined over $K(M_1, M_1', M_{i-1}, M_{i-1}', M_{i+1}, M_{i+1}',\cdots, M_n, M_n')$ by $\varphi_{\partial_i}(M_i, M_i')=\varphi(M_1\times\cdots\times M_{i-1}\times M_i\times M_{i+1}\times\cdots\times M_n)-\varphi(M_1\times\cdots\times M_{i-1}\times M_i'\times M_{i+1}\times\cdots\times M_n)$ and let α_i be the pseudo differential on Γ_i determined by $\varphi_{\partial_i}(M_i, M_i')$. Then we define the pseudo differential of φ as $\sum\alpha_i$ and denote it by $d\varphi$.

PROPOSITION 2. *Let φ, ψ be two functions on $\Gamma_1\times\cdots\times\Gamma_n$. Then we have $d(\varphi+\psi)=d\varphi+d\psi$, $d(\varphi\psi)=\psi d\varphi+\varphi d\psi$.*

This follows immediately from the definition and Weil [2].

§ 3. On abelian varieties

We shall now deal with cycles of dimension zero on an abelian variety.

THEOREM 1. *Let A be an abelian variety of dimension n, then there exist n complete non-singular curves $\Gamma_1,\cdots,\Gamma_n$, which are subvarieties of A, such that the set of all points of the form $\sum x_i$ coincides with A, where every x_i is a point of curve Γ_i (cf. Weil* [3] *prop.* 30).

PROOF. As A is an abelian variety, it is imbedded in the projective space P^N (Chevalley [1]). Let $L_1=L_1^{N-n+1}$ be a generic linear variety of dimension $N-n+1$ in P^N over the field k, over which A is defined. Then $A\cdot L_1$ is defined and by virtue of theorem of Bertini (Matusaka [1], Zariski [1]) $A\cdot L_1$ is a curve Γ_1, which is, like A, complete and non-singular (Weil [4], Nakai [1]).

Now we assume that there are r complete non-singular curves $\Gamma_1,\cdots,\Gamma_r$ on A such that the variety Y_r spanned by $\Gamma_1,\cdots,\Gamma_r$ has the dimension r.

Let k_r be a field of definition of A and Y_r, containing k. Let L_{r+1} be a generic linear variety of dimension $N-n+1$ in P^N over k_r, and let Γ_{r+1} be the intersection product of A and L_{r+1}. Then Γ_{r+1} is a complete non-singular curve. We shall show that Γ_{r+1} is not contained in Y_r, if $r\neq n$. Let $\sum_{i=0}^{N}u_{ij}X_j=0$ $1\leqq i\leqq n-1$ be the defining equation of L_{r+1}, then there exists a generic point M of Γ_{r+1} over $k(u_{ij})$ such that M is also a generic point of A over k_r. If Y_r contains Γ_{r+1}, then M is contained in Y_r, but as k_r is a field of definition of Y_r, Y_r contains the locus of M over k_r, namely A. But this is impossible, unless $r \neq n$. Therefore Γ_{r+1} is not contained in Y_r, and the variety, spanned by $\Gamma_1, \cdots, \Gamma_{r+1}$ has the dimension $r+1$. Thus we have proved this theorem by induction.

Let A be an abelian variety of dimension n defined over a field K and P^N a projective space in which A is imbedded. Let $L_1, \cdots, L_n$ be n generic linear varieties of dimension $N-n+1$ over the field k, such that all the coefficients of defining equations of $L_1, \cdots, L_n$ are algebraically independent over the field k. Let Γ_i be the intersection of A and L_i; then every Γ_i is a complete non-singular curve. We call the set of these n curves Γ_i a generic system of curves of A over k, the set of n generic linear varieties L_i a set of generic linear varieties for $\{\Gamma_i\}$, and a set of coefficients of the defining equations of $L_1, \cdots, L_n$, a set of coefficients for $\{\Gamma_i\}$.

REMARK. If $\{\Gamma_i\}, \{\Gamma'_i\}$ are two generic systems of curves of an abelian variety, then $\Gamma_1 \times \cdots \times \Gamma_n$ is isomorphic to $\Gamma'_1 \times \cdots \times \Gamma'_n$. This follows immediately from the following fact, which is itself easy to prove.

Now let V be a variety defined over a field k, and X be a cycle on V, defined over k, with expression $X = \sum_i a_i A_i$. (Here and hereafter we call a cycle X on V 'defined over k', if every component of X is defined over k.) Let σ be an isomorphism of k on a field k'. If we put $X^\sigma = \sum_i a_i A_i^\sigma$, then X^σ is a cycle on V^σ. We call this cycle X^σ a transform of X by σ. If X is rational over k, then X is defined over the algebraic closure $\bar{k}$ of k. Let $\bar{\sigma}_1$ and $\bar{\sigma}_2$ be extensions of σ to isomorphisms of $\bar{k}$ on $\bar{k}'$, then we have $X^{\bar{\sigma}_1} = X^{\bar{\sigma}_2}$. Therefore, as the transform of X by $\bar{\sigma}$ does not depend on the choice of extension $\bar{\sigma}$ of σ, we shall denote it by X^σ and call it a transform of X by σ. We shall say also X is isomorphic to X^σ.

PROPOSITION 1. *Let Γ_1 be a complete non-singular curve defined over a field k_1. Let k_2 be a field which is isomorphic to k_1 by an isomorphism σ. Let $\mathfrak{a}_1$ be a rational cycle on Γ_1 of dimension zero over k_1, then $\mathfrak{a}_2 = \mathfrak{a}_1^\sigma$ is a cycle on $\Gamma_2 = \Gamma_1^\sigma$ and we have $l(\mathfrak{a}) = l(\mathfrak{a}^\sigma)$.*

PROOF. Let φ be a function on Γ_1 defined over the field k_1, and M a generic point of Γ_1 over k_1. Let M^σ be the corresponding generic point of Γ_2 over k_2 If we put $\varphi^\sigma(M^\sigma)=\varphi(M)^\sigma$, then φ^σ is a function on Γ_2 defined over k_2, and the divisor of φ^σ is isomorphic to that of φ. In fact, let Λ_φ and Λ_φ^σ be graphs of φ and φ^σ, respectively, then we have $\Lambda_\varphi^\sigma=\Lambda_{\varphi^\sigma}$. If we put $\Theta=(0)-(\infty)$, then we have $(\Lambda_\varphi\cdot(\Gamma_1\times\Theta))^\sigma=\Lambda_\varphi^\sigma\cdot(\Gamma_1\times\Theta)^\sigma=\Lambda_\varphi^\sigma\cdot(\Gamma_2\times\Theta)=\Lambda_{\varphi^\sigma}\cdot(\Gamma_2\times\Theta)$. Hence we get $(\varphi)^\sigma=pr_{\Gamma_2}(\Lambda_\varphi\cdot(\Gamma_1\times\Theta))^\sigma=pr_{\Gamma_2}(\Lambda_\varphi^\sigma\cdot(\Gamma_2\times\Theta))=(\varphi^\sigma)$.

The mapping ψ which maps φ to φ^σ induces an isomorphism of $L(\mathfrak{a}_1,k_1)$ on $L(\mathfrak{a}_2,k_2)$, where every $L(\mathfrak{a}_i,k_i)$ is the set of all elements of $L(\mathfrak{a}_i)$ defined over k_i. But by virtue of Weil [1] VIII th. 10, $l(\mathfrak{a}_1)=$ rank of $L(\mathfrak{a}_1,k_1)$, and $l(\mathfrak{a}_2)=$ the rank of $L(\mathfrak{a}_2,k_2)$. Therefore we have $l(\mathfrak{a}_1)=l(\mathfrak{a}_2)$.

PROPOSITION 2. *Let $\Gamma_1,\cdots,\Gamma_n$ and $\Gamma'_1,\cdots,\Gamma'_n$ be two generic systems of curves of an abelian variety A over the field k, over which A and its law of composition are defined. Let (u) and (u') be sets of coefficients of $\{\Gamma_i\}$ and $\{\Gamma'_i\}$ respectively, and let $\mathfrak{a}$ be a cycle of $\Gamma_1\times\cdots\times\Gamma_n$ of dimension zero, which is rational over the field $k(u)$.*

Let σ be an isomorphism of $k(u)$ onto $k(u')$ over k, such that $\sigma(\Gamma_i)=\Gamma'_i$ for every i, then we have $l(\mathfrak{a})=l(\mathfrak{a}^\sigma)$.

PROOF. Let φ be a function on $\Gamma_1\times\cdots\times\Gamma_n$ defined over $k(u)$, $M_1\times\cdots\times M_n$ a generic point of $\Gamma_1\times\cdots\times\Gamma_n$ over $k(u)$, where every M_i is a generic point of Γ_i over $k(u)$, and φ^σ a function on $\Gamma'_1\times\cdots\times\Gamma'_n$ defined over $k(u')$ by $\varphi^\sigma(M_1^\sigma\times\cdots\times M_n^\sigma)=\varphi(M_1\times\cdots\times M_n)^\sigma$. Then we have clearly $(\varphi)^\sigma=(\varphi^\sigma)$. Let φ_{Γ_i} be the restriction of φ on Γ_i, and $\varphi^\sigma_{\Gamma'_i}$ be the restriction of φ^σ on Γ'_i. Then we have $(\varphi_{\Gamma_i})^\sigma=(\varphi^\sigma_{\Gamma'_i})$. In fact, as $(\varphi)\cdot(M_1\times\cdots\times M_{i-1}\times\Gamma_i\times M_{i+1}\times\cdots\times M_n)=M_1\times\cdots\times M_{i-1}\times(\varphi_{\Gamma_i})\times M_{i+1}\times\cdots\times M_n$, we have $((\varphi)\cdot(M_1\times\cdots\times M_{i-1}\times\Gamma_i\times M_{i+1}\cdots\times M_n))^\sigma=(\varphi)^\sigma\cdot(M_1^\sigma\times\cdots\times M_{i-1}^\sigma\times\Gamma'_i\times M_{i+1}^\sigma\times\cdots\times M_n^\sigma)=(\varphi^\sigma)\cdot(M_1^\sigma\times\cdots\times M_{i-1}^\sigma\times\Gamma'_i\times M_{i+1}^\sigma\times\cdots\times M_n^\sigma)=M_1^\sigma\times\cdots\times M_{i-1}^\sigma\times(\varphi^\sigma_{\Gamma'_i})\times M_{i+1}^\sigma\times\cdots\times M_n^\sigma$. Operating the algebraic projection on both side of this equality, we have $(\varphi_{\Gamma_i})^\sigma=(\varphi^\sigma_{\Gamma'_i})$. Thus we have $l(\mathfrak{a}_i)=l(\mathfrak{a}_i^\sigma)$ where $\mathfrak{a}_i$ is the projection of $\mathfrak{a}$ on Γ_i. Therefore from proposition 2 of § 2, follows $l(\mathfrak{a})=l(\mathfrak{a}^\sigma)$.

Let $\mathfrak{a}$ be a cycle of dimension zero on an abelian variety A, b a point of A, and T_b a translation defined by b. Then we denote $T_b(\mathfrak{a})$ by $\mathfrak{a}_b$.

PROPOSITION 3. *Let $\Gamma_1,\cdots,\Gamma_n$ and $\Gamma'_1,\cdots,\Gamma'_n$ be two generic systems of curves of an abelian variety A over the field k, over which A and* *the law of composition are defined. Let* $\mathfrak{a}$ *be a rational cycle of dimension* 0 *on* A *over* k. *Let* F *and* F' *be loci of points* $M_1\times\cdots M_n\times\sum_{i=1}^{n}M_i$ *and* $M'_1\times\cdots\times M'_n\times\sum_{i=1}^{n}M'_i$ *over the fields* $k(u)$ *and* $k(u')$ *respectively, where* (u), (u') *are the sets of coefficients of* $\{\Gamma_i\}$, $\{\Gamma'_i\}$ *respectively, and every* M_i *is a generic point of* Γ_i *over* $k(u)$, *while every* M'_i *is a generic point of* Γ'_i *over* $k(u')$. *Let* t *be a generic point of* A *over* $k(u)$. *If we put* $\bar{\mathfrak{a}}_t=pr_{\Gamma_1\times\cdots\times\Gamma_n}F\cdot(\Gamma_1\times\cdots\times\Gamma_n\times\mathfrak{a}_t)$, *and* $\bar{\mathfrak{a}}_{t'}=pr_{\Gamma_1'\times\cdots\times\Gamma_n'}F\cdot(\Gamma'_1\times\cdots\times\Gamma'_n\times\mathfrak{a}_{t'})$, *then* $\bar{\mathfrak{a}}_t$ *is isomorphic to* $\bar{\mathfrak{a}}'_{t'}$, *where* t' *is a certain generic point of* A *over* $k(u')$.

PROOF. Let σ be an isomorphism of $k(u)$ on $k(u')$. Then σ transform $\Gamma_1\times\cdots\times\Gamma_n$ on $\Gamma'_1\times\cdots\times\Gamma'_n$, and this σ can be extended to an isomorphism of $k(M_1,\cdots,M_n,u)$ on $k(M'_1,\cdots,M'_n,u')$. Let t' be the isomorphic image of t by an extension of σ to $k(t,M_1,\cdots,M_n\,u)$. Then t' is a generic point of A over $k(u')$. We have $F^\sigma=F'$ for this σ. In fact, let Λ be a law of composition of A for n elements, then $\Lambda\cdot(M_1\times\cdots\times M_n\times A)=M_1\times\cdots\times M_n\times\sum_{i=1}^{n}M_i$, and $\Lambda\cdot(M'_1\times\cdots\times M'_n\times A)=M'_1\times\cdots\times M'_n\times\sum_{i=1}^{n}M'_i$. Hence we have $M'_1\times\cdots\times M'_n\times(\sum_{i=1}^{n}M_i)^\sigma=(\Lambda\cdot(M_1\times\cdots\times M_n\times A))^\sigma=\Lambda\cdot(M'_1\times\cdots\times M'_n\times A)=M'_1\times\cdots\times M'_n\times\sum_{i=1}^{n}M'_i$, and this shows that $F^\sigma=F'$. Thus, we have $(F\cdot(\Gamma_1\times\cdots\times\Gamma_n\times\mathfrak{a}_t))^\sigma=F'\cdot(\Gamma'_1\times\cdots\times\Gamma'_n\times\mathfrak{a}_{t'})$, and operating $pr_{\Gamma_1'\times\cdots\times\Gamma_n'}$ on both sides, we get our proposition.

$l(\bar{\mathfrak{a}}_t)$ is determined by a generic system of curves $\{\Gamma_i\}$, a cycle $\mathfrak{a}$ of dimension zero, a generic point t of A, and a field k. To show this dependence on these variables, we denote $l(\bar{\mathfrak{a}}_t)$ by $l(\{\Gamma_i\},\mathfrak{a},t,k)$.

PROPOSITION 4. *Let* A *be an abelian variety defined over a field* k, $\mathfrak{a}$ *a rational cycle of dimension zero on* A *over* k, *a generic system of curves of* A *over* k, *and* (u) *the set of coefficients of* $\{\Gamma_i\}$. *If* t *and* t' *are generic points of* A *over* $k(u)$, *then we have* $l(\{\Gamma_i\},\mathfrak{a},t,k)=l(\{\Gamma_i\},\mathfrak{a},t',k)$.

PROOF. Let σ be an isomorphism of $k(u,t)$ on $k(u,t')$ over $k(u)$, Λ the law of composition of A.

Then we have $(\Lambda\cdot(\mathfrak{a}\times t\times A))^\sigma=(\mathfrak{a}\times t\times\mathfrak{a}_t)^\sigma=\mathfrak{a}^\sigma\times t'\times\times(\mathfrak{a}_t)^\sigma=\Lambda\cdot(\mathfrak{a}^\sigma\times t'\times A)=\mathfrak{a}\times t'\times\mathfrak{a}^\sigma_{t'}$. The algebraic projection of this equality on A

shows $(\mathfrak{a}_t)^\sigma = \mathfrak{a}_{t'}$.

Therefore our proposition follows from proposition 2.

Propositions 2, 3 show that $l(\{\Gamma_i\}, \mathfrak{a}, t, k)$ is independent of choice of $\{\Gamma_i\}, t$. And as for k, the independency of $l(\{\Gamma_i\}, \mathfrak{a}, t, k)$ of k is easily shown from the fact that $l(\{\Gamma_i\}, \mathfrak{a}, t, k) = l(\{\Gamma_i\}, \mathfrak{a}, t, K)$, where K is a field containing k, $\{\Gamma_i\}$ a generic system of curves over K and t a generic point of A over $K(u)$. Therefore $l(\{\Gamma_i\}, \mathfrak{a}, t, k)$ depends only on $\mathfrak{a}$, and we denote it by $l(\mathfrak{a})$.

PROPOSITION 5. *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *two cycles of dimension zero on an abelian variety. If* $\mathfrak{a} \approx \mathfrak{b}$ *over* A, *then we have* $l(\mathfrak{a}) = l(\mathfrak{b})$.

PROOF. Let k be a common field of definition of A and of the law of composition of A, over which $\mathfrak{a}$ and $\mathfrak{b}$ are rational.

Let $\{\Gamma_i\}$ be a generic system of curves of A over k and (u) the set of coefficients of $\{\Gamma_i\}$, and t a generic point of A over $k(u)$.

Let F be a variety defined in proposition 3, then we have $l(\mathfrak{a}) = l(\bar{\mathfrak{a}}_t)$ and $l(\mathfrak{b}) = l(\bar{\mathfrak{b}}_t)$, where $\bar{\mathfrak{a}}_t = pr_{\Gamma_1 \times \cdots \times \Gamma_n} F \cdot (\Gamma_1 \times \cdots \times \Gamma_n \times \mathfrak{a}_t)$ and $\bar{\mathfrak{b}}_t = pr_{\Gamma_1 \times \cdots \times \Gamma_n} F \cdot (\Gamma_1 \times \cdots \times \Gamma_n \times \mathfrak{b}_t)$.

Now let T_t be a translation on A defined by t; then we have $\mathfrak{a}_t \approx \mathfrak{b}_t$. In fact, by proposition 1 of § 1, $\mathfrak{a} - \mathfrak{b} \approx 0$ implies $(\mathfrak{a} - \mathfrak{b}) \times A \approx 0$ on $A \times A$. As the translation T_t is everywhere biregular, we have $T_t \cdot ((\mathfrak{a} - \mathfrak{b}) \times A) \approx 0$, from proposition 3 of § 1. Hence $\mathfrak{a}_t - \mathfrak{b}_t \approx 0$, from proposition 4 of § 1.

By the definition of F and proposition 2 of § 1, we have $F \cdot (\Gamma_1 \times \cdots \times \Gamma_n \times (\mathfrak{a}_t - \mathfrak{b}_t)) \sim 0$ on $\Gamma_1 \times \cdots \times \Gamma_n \times A$.

The cor. of proposition 4 of § 1 shows that $pr_{\Gamma_i} F \cdot (\Gamma_1 \times \cdots \times \Gamma_n \times (\mathfrak{a}_t - \mathfrak{b}_t)) \sim 0$ on Γ_i, for every i, then our proposition follows from proposition 2 of § 2.

Now the genus of a curve belonging to a generic system of curves of A—we shall call it the pseudogenus of A—is a uniquely determined number depending only on A, i. e. if $\{\Gamma_i\}$ and $\{\Gamma'_i\}$ be any two generic system of curves of A, then genus of Γ_i is the same as that of Γ'_j for every i, j. In fact, let σ be an isomorphism of Γ_i on Γ'_j, and $\mathfrak{a}$ a cycle on Γ_i. Then we have $l(\mathfrak{a}) = l(\mathfrak{a}^\sigma)$ by proposition 2, and then Riemann-Roch theorem shows our assertion.

We therefore can speak of the canonical cycles over the product variety $\Gamma_1 \times \cdots \times \Gamma_n$, and we have the following proposition.

PROPOSITION 6. *Let* $\mathfrak{a}$ *be a cycle of dimension zero on an abelian variety* A *of dimension* n, *rational over the field* k, *over which* A *and the law of composition of* A *are defined. Let* $\{\Gamma_i\}$ *and* $\{\Gamma'_i\}$ *be two generic systems of curves of* A *over* k *and* (u) *and* (u') *be the sets of coefficients of generic systems* $\{\Gamma_i\}$ *and* $\{\Gamma'_i\}$ *respectively. Let furthermore* t, *and* t' *be generic points of* A *over the field* $k(u)$ *and* $k(u')$ *respectively, and* $\mathfrak{k}, \mathfrak{k}'$ *canonical cycles on* $\Gamma_1\times\cdots\times\Gamma_n$ *and* $\Gamma'_1\times\cdots\times\Gamma'_n$ *respectively. If we define* $\bar{\mathfrak{a}}_t$ *and* $\bar{\mathfrak{a}}'_{t'}$ *in the same way as in proposition 3, then we have*

$$l(\mathfrak{k}-\bar{\mathfrak{a}}_t)=l(\mathfrak{k}'-\bar{\mathfrak{a}}'_{t'})\,.$$

PROOF. Let t and s be two generic points of A over the field $k(u)$. Then there exists an isomorphism σ of $k(u,t)$ onto $k(u,s)$ over $k(u)$ which maps t onto s. From the definition, and the th. 9 of Weil [1] VIII, there exists a canonical cycle $\mathfrak{k}_1$ which can be written in the form,

$$\mathfrak{k}_1=\sum_{i=1}^{n} M_1\times\cdots\times M_{i-1}\times pr_{\Gamma_i}[\Delta_i\cdot[(\omega_i)-\Delta_i]]\times M_{i+1}\times$$
$$\cdots\times M_n-pr_{\Gamma_1\times\cdots\times\Gamma_n} Y$$

where $M_1\times\cdots\times M_n$ is a generic point of $\Gamma_1\times\cdots\times\Gamma_n$ over the field $k(u,t)$ and ω_i is a function defined over the field $k(u,t)$.

If we extend σ to an isomorphism of $k(u,t,M_i)$ onto $k(u,s,M'_i)$ where every M'_i is a generic point of Γ_i, then the image $\mathfrak{k}_1^\sigma$ of $\mathfrak{k}_1$ by σ is also canonical.

In fact, $\mathfrak{k}_1^\sigma=\sum_{i=1}^{n} M'_1\times\cdots\times M'_{i-1}\times pr_{\Gamma_i}\Delta_i\cdot[(\omega_i)^\sigma-\Delta_i]\times M'_{i+1}\times\cdots\times M'_n-pr_{\Gamma_1\times\cdots\times\Gamma_n} Y^\sigma=\sum_{i=1}^{n} M'_1\times\cdots\times M'_{i-1}\times pr_{\Gamma_i}\Delta_i\cdot[(\omega_i^\sigma)-\Delta_i]\times M'_{i+1}\times\cdots\times M'_n-pr_{\Gamma_1\times\cdots\times\Gamma_n} Y^\sigma$.

Therefore $l(\mathfrak{k}_1-\bar{\mathfrak{a}}_t)=l(\mathfrak{k}_1^\sigma-\bar{\mathfrak{a}}_t^\sigma)=l(\mathfrak{k}_1^\sigma-\bar{\mathfrak{a}}_{t^\sigma})=l(\mathfrak{k}_1^\sigma-\bar{\mathfrak{a}}_s)$. From proposition 2 of § 2, we have $l(\mathfrak{k}-\bar{\mathfrak{a}}_t)=l(\mathfrak{k}_1-\bar{\mathfrak{a}}_t)$, and $l(\mathfrak{k}-\bar{\mathfrak{a}}_s)=l(\mathfrak{k}_1^\sigma-\bar{\mathfrak{a}}_s)$. Hence we get $l(\mathfrak{k}-\bar{\mathfrak{a}}_t)=l(\mathfrak{k}-\bar{\mathfrak{a}}_{s'})$.

Let τ be the isomorphism of $k(u)$ onto $k(u')$ such that $\tau((u))=(u')$.

Let $\mathfrak{k}_1^\tau$ be the image of $\mathfrak{k}_1$ by τ, then $\mathfrak{k}_1^\tau$ is a canonical cycle of $\Gamma'_1\times\cdots\times\Gamma'_n$. In fact we have

$$\mathfrak{k}_1^\tau=\sum_{i-1}^{n} M_1^\tau\times\cdots\times M_{i-1}^\tau\times pr_{\Gamma_i'}\Delta_i^\tau\cdot[(\omega_i)^\tau-\Delta_i^\tau]\times M_{i+1}^\tau\times\cdots\times M_n^\tau$$
$$-pr_{\Gamma_1'\times\cdots\times\Gamma_n'}Y^\tau .$$

But, as every Δ_i^τ is the diagonal of $\Gamma_i'\times\Gamma_i'$ and $(\omega_i)^\tau$ is a divisor of ω_i^τ, $\mathfrak{k}_1^\tau$ is a canonical cycle of $\Gamma_1'\times\cdots\times\Gamma_n'$. And we have $(\mathfrak{k}_1-\bar{\mathfrak{a}}_t)^\tau=(\mathfrak{k}_1^\tau-\bar{\mathfrak{a}}'_{t^\tau})$. Then by prop. 2, we get $l(\mathfrak{k}_1-\bar{\mathfrak{a}}_t)=l(\mathfrak{k}_1^\tau-\bar{\mathfrak{a}}'_{t^\tau})$. But as $\mathfrak{k}_1^\tau$ is a canonical cycles of $\Gamma_1'\times\cdots\times\Gamma_n'$, we have $l(\mathfrak{k}_1-\bar{\mathfrak{a}}'_{t^\tau})=l(\mathfrak{k}_1^\tau-\bar{\mathfrak{a}}'_{t^\tau})$, and therefore $l(\mathfrak{k}'-\bar{\mathfrak{a}}'_{t^\tau})=l(\mathfrak{k}'-\bar{\mathfrak{a}}'_{t'})$. Hence we have $l(\mathfrak{k}-\bar{\mathfrak{a}}_t)=l(\mathfrak{k}'-\bar{\mathfrak{a}}'_{t'})$.

This proposition means that the value of $l(\mathfrak{k}-\bar{\mathfrak{a}}_t)$ depends only on the cycle $\mathfrak{a}$. If we put therefore $\mathfrak{r}(\mathfrak{a})=l(\mathfrak{k}-\bar{\mathfrak{a}}_t)$, then $\mathfrak{r}(\mathfrak{a})$ depends only on $\mathfrak{a}$.

PROPOSITION 7. *Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be two cycles of dimension zero on an abelian variety. If* $\mathfrak{a}\approx\mathfrak{b}$, *then we have* $\mathfrak{r}(\mathfrak{a})=\mathfrak{r}(\mathfrak{b})$.

This proposition is an obvious consequence of proposition 5.

Let F be a rational mapping of $\Gamma_1\times\cdots\Gamma_n$ on A, defined in proposition 3, then the number $d=[F:A]$ depends only on A, and does not depend on the choice of generic systems of curves of A. We call this number d the index of A.

THEOREM 2. *Let A be an abelian variety of pseudogenus g and of index d, and* $\mathfrak{a}$ *a cycle of dimension zero on A. Then we have*

$$l(\mathfrak{a})=d^n(\deg\mathfrak{a})^n+d^{n-1}\beta_1(\mathfrak{a})(\deg\mathfrak{a})^{n-1}+\cdots+d^{n-i}\beta_i(\mathfrak{a})(\deg\mathfrak{a})^{n-i}+$$
$$\cdots+\beta_n(\mathfrak{a}),$$
$$\beta_i(\mathfrak{a})=\sum_{j=0}^{i}(-(g-1))^j\binom{n-i+j}{j}H(i-j,\mathfrak{a})$$
$$H(i,\mathfrak{a})=\sum\mathfrak{r}(\pi_{j_1\cdots j_i}(\mathfrak{a})),$$

where $\pi_{j_1\cdots j_i}\mathfrak{a}$ *is the projection of* $\mathfrak{a}$ *on* $\Gamma_{j_1}\times\cdots\times\Gamma_{j_i}$, $\mathfrak{r}(\pi_{j_1\cdots j_i}(\mathfrak{a}))$ *is considered on* $\Gamma_{j_1}\times\cdots\times\Gamma_{j_i}$, *and the last sum is taken over all the combinations of i elements from the set* $(1,2,\cdots,n)$.

PROOF. Let k be a field of definition of A, over which $\mathfrak{a}$ is rational, $\{\Gamma_i\}$ a generic system of curves of A over k, (u) the set of coefficients of $\{\Gamma_i\}$, and t a generic point of A over $k(u)$. Let $\bar{\mathfrak{a}}_t$ be a cycle defined by $pr_{\Gamma_1\times\cdots\times\Gamma_n}F\cdot(\Gamma_1\times\cdots\times\Gamma_n\times\mathfrak{a}_t)$, then we have, by theorem 2 of § 2,

$$l(\bar{\mathfrak{a}}_t)=(\deg \bar{\mathfrak{a}}_t)^n+\beta_1(\bar{\mathfrak{a}}_t)(\deg \bar{\mathfrak{a}}_t)^{n-1}+\cdots+\beta_i(\bar{\mathfrak{a}}_t)\cdot(\deg \bar{\mathfrak{a}}_t)^{n-1}+$$
$$\cdots+\beta_n(\bar{\mathfrak{a}}_t)\,,$$

$$\beta_i(\bar{\mathfrak{a}}_t)=\sum_{j=0}^{n}(-(g-1))^j\binom{n-i+j}{j}H(i-j,\,\bar{\mathfrak{a}}_t)\,,\quad \text{and}$$

$$H(i,\bar{\mathfrak{a}}_t)=\sum l(\pi_{j_1,\dots,j_i}(\bar{\mathfrak{k}}-\bar{\mathfrak{a}}_t))\,,$$

where $\mathfrak{k}$ is a canonical cycle of dimension zero on $\Gamma_1\times\cdots\times\Gamma_n$. From the principle of conservation of number, we have $\deg\bar{\mathfrak{a}}_t=d\deg\bar{\mathfrak{a}}$. From proposition 7 follows $\beta_i(\mathfrak{a})=\beta_i(\mathfrak{a}_t)$, and $H(i,\mathfrak{a})=H(i,\bar{\mathfrak{a}}_t)$, and have our theorem.

COROLLARY 1. *Let A be an abelian variety, and* $\mathfrak{a},\mathfrak{b}$ *two cycles on A, such that* $\mathfrak{a}\approx\mathfrak{b}$, *then we have*

$$\deg\mathfrak{a}=\deg\mathfrak{b},\ \textit{and}\ \beta_i(\mathfrak{a})=\beta_i(\mathfrak{b})\,.$$

PROOF. From prop. 7, we get $\beta_i(\mathfrak{a})=\beta_i(\mathfrak{b})$. From this and prop. 5 and the above theorem, we have $\deg\mathfrak{a}=\deg\mathfrak{b}$.

COROLLARY 2. *Let A be an abelian variety,* $\mathfrak{a}$ *a cycle of dimension zero on A, and* $\beta_i(\mathfrak{a})$ *the function defined above.*

If the degree of $\mathfrak{a}$ *is sufficiently large, then we have*

$$\beta_i(\mathfrak{a})=(-1)^i\binom{n}{i}(g-1)^i\,.$$

It is an obious consequence of theorem 2 and prop. 6.

§ 4. Birational invariance

Here we shall prove the birational invariance of $l(\mathfrak{a}),\beta_i(\mathfrak{a})$ the pseudogenus and the index d of an abelian variety A. From now on, we assume that every abelian varieties to be considered is contained in a projective space P^N.

PROPOSITION 1. *Let A and A' be two birationally equivalent abelian varieties of dimension n. Then there exist generic systems of curves $\{\Gamma_i\}$ and $\{\Gamma'_i\}$ of A and A' respectively, such that every Γ_i corresponds to Γ'_i birationally.*

PROOF. Let T be a birational correspondence between A and A'. We may assume that A, A' and T have their representative A_0, A'_0

and T_0 in the same representative P_0 of P^N. Let k be a common field of definition of A, A' and T, and $x_1, \cdots, x_n$ are n independent generic points of A over k. If we put $x'_k = T(x_k)$, then $x'_1, \cdots, x'_n$ are n independent generic points of A' over k, and $k(x_k) = k(x'_k)$ for every i. Let $u_{ij}^{(1)}, \cdots, u_{ij}^{(n)}, (1 \leqq j \leqq N, 1 \leqq i \leqq N-n+1)$ be $nN(N-n+1)$ independent variables over $k(x_1, \cdots, x_n)$. If we put $u_{i,0}^{(k)} = \sum_{j=1}^{N} x_{k,j} u_{i,j}^{(k)}$, then $u_{i,j}^{(k)}$ $(0 \leqq j \leqq N, 1 \leqq i \leqq N-n+1)$ are $(N+1)(N-n+1)$ independent variables over k for every i, k. Let L_k be a linear variety defined by equations $\sum_{j=0}^{N} u_{ij}^{(k)} X_j = 0$ $(1 \leqq i \leqq N-n+1, 0 \leqq j \leqq N)$; then L_k is a generic linear variety, and $A \cdot L_k = \Gamma_k$ has a generic point x_k over $k(u_{i,j}^{(k)})$. By the same method, using $(u_{ij}^{(k)})$ $1 \leqq j \leqq N$, $1 \leqq i \leqq N-n+1$ and x'_k, we define generic linear varieties L'_k, and curves $\Gamma'_k = A' \cdot L'_k$ which have x'_k as a generic point over $k(u'^{(k)}_{i,j})$, where $u'^{(k)}_{i,j} = u_{i,j}^{(k)}$ for $1 \leqq j \leqq N$ and $u'^{(k)}_{i,0} = \sum_{j=1}^{N} x'_{k,j} u_{i,j}^{(k)}$. Then we have $k(x_k, u^k_{i,j}) = k(x'_k, u'^k_{i,j})$. Therefore, Γ_i corresponds to Γ'_i birationally for every i. As $x_1, \cdots, x_n$ are independent generic points of A and $x'_1, \cdots, x'_n$ are independent generic points of A' over k, $\{\Gamma_i\}$ and $\{\Gamma'_i\}$ are generic systems of curves of A and A' respectively over k.

This proposition shows the birational invariance of pseudogenus of abelian varieties.

REMARK. This curve Γ_i corresponds to Γ'_i everywhere biregularly by T for every i, because T is everywhere biregular.

Let A, A' and T be the same as defined above in prop. 1, let $\mathfrak{a}$ be a cycle of dimension zero on A, and $\mathfrak{a}'$ a cycle of dimension zero on A' defined by $\mathfrak{a}' = pr_{A'} T \cdot (\mathfrak{a} \times A')$.

Let K be a field containing k, such that $\mathfrak{a}$ and $\mathfrak{a}'$ are rational over K. If t is a generic point of A over K, then $t' = pr_{A'} T(t \times A')$ is a generic point of A' over K.

By prop. 1, there exist generic systems of curves $\{\Gamma_i\}, \{\Gamma'_i\}$ of A, and A' over K, such that Γ_i corresponds to Γ'_i birationally, and so $\Gamma_1 \times \cdots \times \Gamma_n$ corresponds to $\Gamma'_1 \times \cdots \times \Gamma'_n$ birationally. Let $x_1 \times \cdots \times x_n$ and $x'_1 \times \cdots \times x'$ be biregularly corresponding generic points of $\Gamma_1 \times \cdots \times \Gamma_n$ and $\Gamma'_1 \times \cdots \times \Gamma'_n$ over K, such that x_k corresponds to x'_k biregularly for every i. Let B be a locus of $x_1 \times \cdots \times x_n \times (\sum_{k=1}^{n} x_k) \times x'_1 \times$

$\cdots\times x'_n\times(\sum_{k=1}^{n}x')$ over k, then $B\cdot(\Gamma_1\times\cdots\times\Gamma_n\times\mathfrak{a}_t\times\Gamma'_1\times\cdots\times\Gamma'_n\times A')$ is defined. By Weil [3] th. 9, T can be written in the form $T=T^*+a$, where T^* is a homomorphism of A into A', and a is a rational point over K. As $T(x_i)=T^*(x_i)+a$, and $T(\sum_{i=1}^{n}x_i)=\sum_{i=1}^{n}T^*(x)=a$, we have $T(\sum_{i=1}^{n}x_i)=\sum_{i=1}^{n}T(x_i)-(n-1)a$. This proves that $K(\sum_{i=1}^{n}x_i)=K(\sum_{i=1}^{n}x'_i)$. Therefore we have $H\cdot(\bar{\mathfrak{a}}_t\times\Gamma'_1\times\cdots\times\Gamma'_n)=\bar{\mathfrak{a}}'_{t''}$, where $\bar{\mathfrak{a}}_t=pr_{\Gamma_1\times\cdots\times\Gamma_n}F\cdot(\Gamma_1\times\cdots\times\Gamma_n\times\mathfrak{a}_t)$, $\bar{\mathfrak{a}}'_{t'}=pr_{\Gamma_1'\times\cdots\times\Gamma_n'}F'\cdot(\Gamma'_1\times\cdots\times\Gamma'_n\times\bar{\mathfrak{a}}'_{t''})$, $t''=t'+(n-1)a$ and H is a locus of $x_1\times\cdots\times x_n\times x'_1\times\cdots\times x'_n$ over K.

Let $\bar{\mathfrak{a}}_{t,i}$, $\bar{\mathfrak{a}}'_{t'',i}$ be the projection of $\bar{\mathfrak{a}}_t$, $\bar{\mathfrak{a}}'_{t''}$ on Γ_i, Γ'_i respectively, and H_i be a locus of $x_i\times x'_i$ over K, then we have

$$pr_{\Gamma_i}H_i\cdot(\bar{\mathfrak{a}}_{t,i}\times\Gamma_1)=\bar{\mathfrak{a}}'_{t'',i}\,.$$

Let φ be a function on Γ_i, which is an element of a complete linear system $L(\bar{\mathfrak{a}}_{t,i})$ on Γ_i, and K' be a common field of definition of φ, Γ_i, Γ'_i, and H_i; let further $y_i\times y'_i$ be a generic point of H_i over K', where y_i and y'_i are generic points of Γ_i and Γ'_i respectively over K'. If we put $\varphi'(y_i)=\varphi(y_i)$, then φ' is a function on Γ'_i, and by Weil [1] VIII th. 7, we have $pr_{\Gamma_i'}\cdot H_i\cdot((\varphi)\times\Gamma'_i)=(\varphi')$. As φ' is obviously an element of a complete linear system $L(\bar{\mathfrak{a}}'_{t'',i})$, the mapping $\varphi\to\varphi'$ induces an isomorphism of $L(\bar{\mathfrak{a}}_{t,i})$ onto $L(\bar{\mathfrak{a}}'_{t'',i})$, and we have $l(\bar{\mathfrak{a}}'_{t,i})=l(\bar{\mathfrak{a}}'_{t'',i})$. Since, by proposition 2 of § 2, $l(\mathfrak{a})=\prod_{i=1}^{n}l(\bar{\mathfrak{a}}_{t,i})$, $l(\bar{\mathfrak{a}}')=\prod_{i=1}^{n}l(\bar{\mathfrak{a}}'_{t'',t})$, we have the following:

PROPOSITION 2. *Let A and A' be two birationally equivalent abelian varieties. Let $\mathfrak{a}$ be a cycle of dimension zero on A, and $\mathfrak{a}'$ the corresponding cycle of dimension zero on A'. Then we have*

$$l(\mathfrak{a})=l(\mathfrak{a}')\,,\ \beta_i(\mathfrak{a})=\beta_i(\mathfrak{a}')\,,\quad \textit{and}\quad d=d'\,,$$

where d and d' are indices of A and A' respectively.

PROOF. A canonical divisor $\mathfrak{k}_i$ on Γ_i can be written in the form $pr_{\Gamma_i}\Delta_i\cdot((\omega_i)-\Delta_i)$ where Δ_i is the diagonal of $\Gamma_i\times\Gamma_i$ and ω_i is a function on $\Gamma_i\times\Gamma_i$ such that $v_{\Delta_i}(\omega_i)=1$. Let R_i be a birational correspondence of $\Gamma_i\times\Gamma_i$ to $\Gamma'_i\times\Gamma'_i$, and K'' be a common field of definition of ω_i, $\Gamma_i \times \Gamma_i$, $\Gamma'_i \times \Gamma'_i$ and R, and let $y_{i,1} \times y_{i,2} \times y'_{i,1} \times y'_{i,2}$ be a generic point of R_i over K''. Let ω'_i be a function on $\Gamma'_i \times \Gamma'_i$, defined by $\omega'_i(y'_{i,1} \times y'_{i,2}) = \omega_i(y_{i,1} \times y_{i,2})$, then $\nu_{\Delta'_i}(\omega'_i) = 1$, and we have $pr_{\Gamma'_i \times \Gamma'_i} R_i \cdot ((\omega_i) - \Delta_i) \times \Gamma'_2 \times \Gamma'_2) = \Delta'_i((\omega'_i) - \Delta'_i)$. Therefore we have $\beta_i(\mathfrak{a}) = \beta_i(\mathfrak{a}')$.

As for d and d', $K(\sum_{i=1}^{n} x_i) = K(\sum_{i=1}^{n} x'_i)$, and $K(x_1, \cdots, x_n) = K(x'_1, \cdots, x'_n)$ prove $d = d'$.

References

[1] C. Chevalley, Imbedding theorem of group varieties in a projective space, Proc. of Intern. symp. on alg. number theory, forthcoming.

[2] T. Matusaka, The theorem of Bertini on linear system in modular fields, Mem. Coll. Sci. Univ. Kyoto, Ser. A. Math., 26 (1950), pp. 51-62.

[3] Y. Nakai, Note on the intersection of an algebraic variety with the generic hyperplane, Mem. Coll. Sci. Univ. Kyoto, Ser. A. Math., 26 (1951) pp. 185-186.

[4] A. Weil, Foundations of algebraic geometry, New York, 1946.

[5] ——— Sur les courbes algébriques et les variétés qui s'en déduisent, Paris, 1948.

[6] ——— Variétés abéliennes et courbes algébriques, Paris, 1948.

[7] ——— Sur les critères d'équivalence en géométrie algébrique, Math. Ann. B. 128 (1954) pp. 95-127.

[8] O. Zariski, Pencils on algebraic variety and a new proof of a theorem of Bertini, Trans. Amer. Math. Soc., 50 (1941) pp. 48-70.