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Proof of a special case of the fundamental conjecture of Takeuti's GLC.

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G. Takeuti [2] has generalized G. Gentzen's logic calculus LK (cf. [1]) to his *generalized logic calculus GLC*, and enounced the *fundamental conjecture of GLC:* Every provable sequence in GLC will be provable without cut. In [2] it is also shown that from this conjecture would follow the consistency of the analysis.

Some special cases of this conjecture have been proved by Takeuti [3], [4], [5], [6]. Another special case will be proved in this paper. After preparations in § 1, we shall formulate our theorem in § 2, and prove it by *reductions* indicated in § 3.

§ 1. Proof-figures in GLC.

We begin with listing the inference-schemata of GLC in a form slightly modified from those given in [2]. The equivalence of this system with that of [2] can be easily verified. For the meaning of terms such as "homologous", "t-variety", "f-variable" etc., we refer to [2], [3].

1.1. Inference-schemata

1.1.1. Inferences on structure

"Version"

left: $\dfrac{A, \Gamma \to \Delta}{\tilde{A}, \Gamma \to \Delta}$ right: $\dfrac{\Gamma \to \Delta, A}{\Gamma \to \Delta, \tilde{A}}$

where $\tilde{A}$ is a formula homologous to A.

"Weakening"

left: $\dfrac{\Gamma \to \Delta}{A, \Gamma \to \Delta}$ right: $\dfrac{\Gamma \to \Delta}{\Gamma \to \Delta, A}$

" Contraction "

left: $\dfrac{A, A, \Gamma \rightarrow \Delta}{A, \Gamma \rightarrow \Delta}$ right: $\dfrac{\Gamma \rightarrow \Delta, A, A}{\Gamma \rightarrow \Delta, A}$

" Exchange "

left: $\dfrac{\Gamma, A, B, \Pi \rightarrow \Delta}{\Gamma, B, A, \Pi \rightarrow \Delta}$ right: $\dfrac{\Gamma \rightarrow \Delta, A, B, \Lambda}{\Gamma \rightarrow \Delta, B, A, \Lambda}$

1.1.2. Cut

$$\frac{\Gamma \rightarrow \Delta, A \qquad A, \Pi \rightarrow \Lambda}{\Gamma, \Pi \rightarrow \Delta, \Lambda}$$

1.1.3. Inference on logical symbol

" $\neg$ "

left: $\dfrac{\Gamma \rightarrow \Delta, A}{\neg A, \Gamma \rightarrow \Delta}$ right: $\dfrac{A, \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, \neg A}$

" $\wedge$ "

left: $\dfrac{A, B, \Gamma \rightarrow \Delta}{A \wedge B, \Gamma \rightarrow \Delta}$ right: $\dfrac{\Gamma \rightarrow \Delta, A \qquad \Pi \rightarrow \Lambda, B}{\Gamma, \Pi \rightarrow \Delta, \Lambda, A \wedge B}$

" $\forall$ on t-variable "

left: $\dfrac{F(T), \Gamma \rightarrow \Delta}{\forall x F(x), \Gamma \rightarrow \Delta}$

where T is an arbitrary t-variety of the same type as x.

right: $\dfrac{\Gamma \rightarrow \Delta, F(a)}{\Gamma \rightarrow \Delta, \forall x F(x)}$

where a is a free t-variable of the same type as x, not contained in the lower sequence. (a is called the *eigenvariable* of this inference.)

" $\forall$ on f-variable "

left: $\dfrac{F(H), \Gamma \rightarrow \Delta}{\forall \varphi F(\varphi), \Gamma \rightarrow \Delta}$

where H is an arbitrary f-variety of the same type as φ.

right: $\dfrac{\Gamma \rightarrow \Delta, F(\alpha)}{\Gamma \rightarrow \Delta, \forall \varphi F(\varphi)}$

where α is a free f-variable of the same type as φ, not contained in the lower sequence. (α is called the eigenvarialle of this inference.)

1.2. In the above schemata, the formulas denoted by $A, B, F(T), F(a)$, $F(H)$ or $F(\alpha)$ in the upper sequence are called the *subformulas* of the inference, and the formulas denoted by $A, \tilde{A}, B, \neg A, A \wedge B, \forall x F(x)$, or $\forall \varphi F(\varphi)$ in the lower sequence are called the *chief formulas* of the inference. A subformula of a cut is called a *cut-formula*, and a chief formula of a weakening is called a *weakening formula.*

1.3. When a formula C is contained in the upper sequence of an inference which is represented by one of the above inference-schemata, the *successor* of C is defined as follows: if C is a cut-formula then there is no successor of C; if C is a subformula of an inference other than cut and exchange, then the successor of C is the chief formula of the inference; if C is a subformula denoted by A (resp. B) in the schemata of exchange, then the successor of C is a chief formula denoted by A (resp. B); if C is the k-th formula of Γ, Δ, Π or Λ in the upper sequence, then the successor of C is the k-th formula of Γ, Δ, Π or Λ respectively in the lower sequence. We define a *descendant* of a formula recursively as follows: the successor of a formula is a descendant of the formula; the successor of a descendant of a formula is a descendant of the formula.

1.4. A formula in a proof-figure is called *implicit* or *explicit* according as the formula has or has not a descendant, which is a cut-formula of a cut. An inference is called implicit or explicit according as the chief formula of the inference is implicit or explicit.

1.5. A sequence in a proof-figure is called *contained in the end-place* of the proof-figure, if and only if there is no implicit logical inference under the sequence. An inference in a proof-figure is called contained in the end-place of the proof-figure, if and only if the lower sequence of the inference is contained in the end-place. An inference in a proof-figure is called to *belong to the boundary* of the end-place, if and only if the lower sequence is contained in the end-place and the upper sequence is not contained in the end-place.

1.6. A cut in the end-place is called *suitable*, if and only if each cut-formula of the cut is a descendant of the chief formula of an inference, which belongs to the boundary of the end-place.

§ 2. The formulation of the theorem and the plan of its proof.

2.1. THEOREM If a proof-figure $\mathfrak{P}$ has no implicit contraction, then the end-sequence of $\mathfrak{P}$ is provable without cut.

2.2. In the following, we shall prove this theorem by the mathematical induction on the *grade*, which is the sum of the numbers of cuts and logical inferences contained in the proof-figure.

2.3. Let $\mathfrak{P}$ and $\mathfrak{Q}$ (resp. $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$) be proof-figures. We say that $\mathfrak{P}$ is *reduced* to $\mathfrak{Q}$ (resp. $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$), if the following conditions (1), (2), (3) are satisfied: (1) $\mathfrak{P}$ and $\mathfrak{Q}$ (resp. $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$) have no implicit contractions; (2) if the end-sequence of $\mathfrak{Q}$ (resp. $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$) is provable without cut then the end-sequence of $\mathfrak{P}$ is provable without cut; (3) the grade of $\mathfrak{Q}$ (resp. of $\mathfrak{Q}_1$ and of $\mathfrak{Q}_2$) is smaller than the grade of $\mathfrak{P}$.

2.4. Let $\mathfrak{P}$ be a proof-figure without implicit contraction with the grade not equal to zero. Our theorem will be proved, if we find a definite way of reduction for any such $\mathfrak{P}$. We may assume thereby, by a wellknown method of changing the free variables, that for every inference on $\forall$ right its eigenvariable is contained in $\mathfrak{P}$ only in sequences above the inference.

§ 3. Reductions

3.1. The case, where the end-place of $\mathfrak{P}$ has an explicit logical inference. Let $\mathfrak{J}$ be the undermost logical inference contained in the end-place of $\mathfrak{P}$.

3.1.1. If $\mathfrak{J}$ is an inference on $\neg$ left, we can assume that $\mathfrak{P}$ is of the form:

$$\begin{array}{c} \vdots \\ \dfrac{\Gamma \rightarrow \Delta, A}{\neg A, \Gamma \rightarrow \Delta}\ \mathfrak{J} \\ \vdots \\ \Upsilon \rightarrow \Omega \end{array}$$

Since there is no logical inference under $\mathfrak{J}$, Υ contains $\neg A$. Hence $\mathfrak{P}$ is reducible to the following proof-figure:

$$\begin{array}{c} \vdots \\ \Gamma \to \Delta, A \\ \hline \text{weakenings and exchanges} \\ \hline \neg A, \Gamma \to A, \Delta \\ \vdots \\ \Upsilon \to A, \Omega \end{array}$$

If $\mathfrak{J}$ is an inference on $\neg$ right, on $\wedge$ left or on $\forall$, the reduction is similar to the above.

3.1.2. If $\mathfrak{J}$ is an inference on $\wedge$ right, and $\mathfrak{P}$ is of the form

$$\begin{array}{c} \begin{array}{cc} \vdots & \vdots \\ \Gamma \to \Delta, A & \Pi \to \Lambda, B \end{array} \\ \hline \Gamma, \Pi \to \Delta, \Lambda, A \wedge B \\ \vdots \\ \Upsilon \to \Omega \end{array} \; \mathfrak{J}$$

We reduce $\mathfrak{P}$ to the following proof-figures:

$$\begin{array}{c} \vdots \\ \Gamma \to \Delta, A \\ \hline \text{weakenings and exchanges} \\ \hline \Gamma, \Pi \to A, \Delta, \Lambda, A \wedge B \\ \vdots \\ \Upsilon \to A, \Omega \end{array} \qquad \begin{array}{c} \vdots \\ \Pi \to \Lambda, B \\ \hline \text{weakenings and exchanges} \\ \hline \Gamma, \Pi \to B, \Delta, \Lambda, A \wedge B \\ \vdots \\ \Upsilon \to B, \Omega \end{array}$$

3.2. The case where the end-place contains a beginning sequence $\mathfrak{S}$ and no explicit logical inference. We can assume here that one of the two beginning formulas of $\mathfrak{S}$ is implicit. In fact, if both of them are explicit, the end-sequence of $\mathfrak{P}$ is obtained simply by some versions, weakenings and exchanges. Let $\mathfrak{P}$ be of the form:

$$\dfrac{\begin{array}{c} A \rightarrow A \quad \mathfrak{S} \\ \vdots \\ \Gamma, A, \Gamma' \rightarrow \Delta, A \end{array} \qquad \begin{array}{c} \vdots \\ A, \Pi \rightarrow \Lambda \end{array}}{\Gamma, A, \Gamma', \Pi \rightarrow \Delta, \Lambda} \text{ cut}$$

$$\vdots$$

$\mathfrak{P}$ is then reduced to the following proof-figure:

$$\begin{array}{c} \vdots \\ A, \Pi \rightarrow \Lambda \\ \hline \text{weakenings and exchanges} \\ \hline \Gamma, A, \Gamma', \Pi \rightarrow \Delta, \Lambda \\ \vdots \end{array}$$

3.3. The case, where the end-place contains an implicit weakening. Let $\mathfrak{P}$ be of the form:

$$\dfrac{\begin{array}{c} \vdots \\ \Xi \rightarrow \Sigma, A \end{array} \qquad \begin{array}{c} \vdots \\ \dfrac{\Gamma \rightarrow \Delta}{A, \Gamma \rightarrow \Delta} \text{ weakening} \\ \vdots \\ A, \Pi \rightarrow \Lambda \end{array}}{\Xi, \Pi \rightarrow \Sigma, \Lambda} \text{ cut}$$

$$\vdots$$

$\mathfrak{P}$ is then reduced to the following:

$$\begin{array}{c} \vdots \\ \Gamma \rightarrow \Delta \\ \vdots \\ \Pi \rightarrow \Lambda \\ \hline \text{weakenings and exchanges} \\ \hline \Xi, \Pi \rightarrow \Sigma \Lambda \\ \vdots \end{array}$$

3.4. The case, where the end-place contains no beginning sequence, no explicit logical inference and no implicit weakening. We can prove by the induction on the number of inferences contained in the end-place that there is a suitable cut. Let $\mathfrak{J}$ be the lowest cut in the end-place of $\mathfrak{P}$, and $\mathfrak{P}$ be of the form:

$$\begin{array}{c}\dfrac{\begin{array}{c}\Downarrow \mathfrak{P}_1\\ \Gamma\to\Delta, A\end{array} \qquad\qquad \begin{array}{c}\Downarrow \mathfrak{P}_2\\ A, \Pi\to\Lambda\end{array}}{\Gamma, \Pi\to\Delta, \Lambda}\ \mathfrak{J}\\ \Downarrow\end{array}$$

We see easily that the end-place of $\mathfrak{P}_i$ has a sequence not contained in the end-place of $\mathfrak{P}$, if and only if there is an inference, in the boundary of the end-place of $\mathfrak{P}$, and a cut-formula of $\mathfrak{J}$ is a descendant of the chief-formula of this inference. Therefore, if $\mathfrak{J}$ is not suitable, then the end-place $\mathfrak{E}$ of $\mathfrak{P}_1$ or $\mathfrak{P}_2$ is a subset of the end-place of $\mathfrak{P}$. Then, by the hypothesis of the induction, there exists a suitable cut $\mathfrak{K}$ in $\mathfrak{E}$. Clearly $\mathfrak{K}$ is a suitable cut of $\mathfrak{P}$.

Let $\mathfrak{J}$ be a suitable cut in the end-place of $\mathfrak{P}$.

3.4.1. The case, where the outermost logical symbol of the cut-formulas of $\mathfrak{J}$ is $\neg$. Let $\mathfrak{P}$ be of the form:

$$\begin{array}{c}\dfrac{\begin{array}{c}\Downarrow\\ \dfrac{A, \Gamma\to\Delta}{\Gamma\to\Delta, \neg A}\\ \Downarrow\\ \Xi\to\Sigma, \neg A\end{array} \qquad\qquad \begin{array}{c}\Downarrow\\ \dfrac{\Pi\to\Lambda, A}{\neg A, \Pi\to\Lambda}\\ \Downarrow\\ \neg A, \Phi\to\Psi\end{array}}{\Xi, \Phi\to\Sigma, \Psi}\ \mathfrak{J}\\ \Downarrow\end{array}$$

We reduce $\mathfrak{P}$ to the following proof-figure:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\vdots\\ \Pi \rightarrow \Lambda, A\end{array} \qquad \begin{array}{c}\vdots\\ A, \Gamma \rightarrow \Delta\end{array}}{\Pi, \Gamma \rightarrow \Lambda, \Delta}}{\text{exchanges}}}{\Gamma, \Pi \rightarrow \Lambda, \Delta}$$

$$\vdots$$

$$\dfrac{\dfrac{\Xi, \Pi \rightarrow \Lambda, \Sigma}{\text{exchanges}}}{\Pi, \Xi \rightarrow \Sigma, \Lambda}$$

$$\vdots$$

$$\dfrac{\dfrac{\Phi, \Xi \rightarrow \Sigma, \Psi}{\text{exchanges}}}{\Xi, \Phi \rightarrow \Sigma, \Psi}$$

$$\vdots$$

3.4.2. The case, where the outermost logical symbol of the cut-formulas of $\mathfrak{J}$ is $\wedge$. Let $\mathfrak{P}$ be of the form:

$$\dfrac{\begin{array}{c}\dfrac{\begin{array}{c}\vdots\\ \Gamma_1 \rightarrow \Delta_1, A\end{array} \qquad \begin{array}{c}\vdots\\ \Gamma_2 \rightarrow \Delta_2, B\end{array}}{\Gamma_1, \Gamma_2 \rightarrow \Delta_1, \Delta_2, A \wedge B}\\ \vdots \\ \Xi \rightarrow \Sigma, A \wedge B\end{array} \qquad \begin{array}{c}\dfrac{\begin{array}{c}\vdots\\ A, B, \Pi \rightarrow \Lambda\end{array}}{A \wedge B, \Pi \rightarrow \Lambda}\\ \vdots \\ A \wedge B, \Phi \rightarrow \Psi\end{array}}{\Xi, \Phi \rightarrow \Sigma, \Psi}\ \mathfrak{J}$$

$$\vdots$$

We reduce $\mathfrak{P}$ to the following proof-figure:

$$\dfrac{\begin{matrix}\vdots\\ \Gamma_2 \to \Delta_2, B\end{matrix} \qquad \dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\\ \Gamma_1 \to \Delta_1, A\end{matrix} \qquad \begin{matrix}\vdots\\ A, B, \Pi \to \Lambda\end{matrix}}{\Gamma_1, B, \Pi \to \Delta_1, \Lambda}}{\text{exchanges}}}{B, \Gamma_1, \Pi \to \Delta_1, \Lambda}}{\Gamma_2, \Gamma_1, \Pi \to \Delta_2, \Delta_1, \Lambda}$$

$$\dfrac{\text{exchanges}}{\Gamma_1, \Gamma_2, \Pi \to \Lambda, \Delta_1, \Delta_2}$$

$$\vdots$$

$$\dfrac{\dfrac{\Xi, \Pi \to \Lambda, \Sigma}{\text{exchanges}}}{\Pi, \Xi \to \Sigma, \Lambda}$$

$$\vdots$$

$$\dfrac{\dfrac{\Phi, \Xi \to \Sigma, \Psi}{\text{exchanges}}}{\Xi, \Phi \to \Sigma, \Psi}$$

$$\vdots$$

3.4.3. The case, where the outermost logical symbol of the cut-formulas of $\mathfrak{J}$ is $\forall$. Let $\mathfrak{P}$ be of the form:

$$\dfrac{\begin{matrix}\vdots\\ \dfrac{\Gamma \to \Delta, F(\alpha)}{\Gamma \to \Delta, \forall\varphi F(\varphi)}\\ \vdots\\ \Xi \to \Sigma, \forall\varphi F(\varphi)\end{matrix} \qquad \begin{matrix}\vdots\\ \dfrac{F(H), \Pi \to \Lambda}{\forall\varphi F(\varphi), \Pi \to \Lambda}\\ \vdots\\ \forall\varphi F(\varphi), \Phi \to \Psi\end{matrix}}{\Xi, \Phi \to \Sigma, \Psi}\ \mathfrak{J}$$

$$\vdots$$

We reduce $\mathfrak{P}$ to the following proof-figure:

$$\frac{\begin{array}{c}\vdots\\ \Gamma \to \Delta, F(H)\end{array} \qquad \begin{array}{c}\vdots\\ F(H), \Pi \to \Lambda\end{array}}{\Gamma, \Pi \to \Delta, \Lambda}$$

$$\frac{\Gamma, \Pi \to \Delta, \Lambda}{\Gamma, \Pi \to \Lambda, \Delta}\ \text{exchanges}$$

$$\vdots$$

$$\frac{\Xi, \Pi \to \Lambda, \Sigma}{\Pi, \Xi \to \Sigma, \Lambda}\ \text{exchanges}$$

$$\vdots$$

$$\frac{\Phi, \Xi \to \Sigma, \Psi}{\Xi, \Phi \to \Sigma, \Psi}\ \text{exchanges}$$

$$\vdots$$

where the part of the form:

$$\begin{array}{c}\vdots\\ \Gamma \to \Delta, F(H)\end{array}$$

is obtained from the corresponding part of $\mathfrak{P}$ by substituting H for α.

References

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