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A renewal theorem.

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1. Introduction Let $X_i\ (i=1, 2,\cdots)$ be independent identically distributed random variables, having the mean values $E(X_i)=m$. Then it holds

$$\lim_{x\to\infty}\sum_{n=1}^{\infty}P_r(x<S_n\leqq x+h)=\frac{h}{m}, \tag{1.1}$$

where $S_n=\sum_{i=1}^{n}X_i$, under some restrictions. This is known as renewal theorem.

Feller [7] and Täcklind [10] proved (1.1) under some conditions in the case $X_i\geqq 0$, $(i=1, 2,\cdots)$.

Blackwell [1] has proved (1.1) with the only condition that $E(X_n)<\infty$, when $X_i\geqq 0$ and X_i has not the lattice distribution. Chung-H. Pollard [3] imposed the restriction that the distribution of X_i possesses an absolutely continuous part when X_i has not a lattice distribution and is not necessarily non-negative. T. H. Harriss by written communication and Blackwell [2] have shown that the restriction is unnecessary. Doob [6] discussed (1.1) from another point of view. Cox-Smith [4] have proved, under certain assumptions, in the case where X_i has a probability density, that

$$\lim_{n\to\infty}\sum_{n=1}^{\infty}h_n(x)=\frac{1}{m}, \tag{1.2}$$

where $h_n(x)$ is the probability density of S_n and we suppose $m>0$. Cox-Smith have discussed (1.1) where the distributions of X_i are not necessarily identical. Recently S. Karlin [9] has shown the renewal theorem (1.1) in either cases lattice or continuous, where X_i are not necessarily non-negative.

We shall treat the case the distributions of X_i are not necessarily identical. Then it would be natural to expect that

(1.3) $$\lim_{T\to\infty}\frac{1}{T}\int_0 \sum_{n=1}^{\infty} P_r(x<S_n\leqq x+h)dx=\frac{h}{m},$$

when

(1.4) $$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} E(X_i)=m$$

exists. We shall prove this in § 3 under some conditions on the distribution functions of X_i.

2. Lemmas. We state first some lemmas.

LEMMA 1. *Let* $f(t)\geqq 0$,

(2.1) $$\int_{-\infty}^{0} e^{-st}f(t)\,dt<\infty\,, \qquad \textit{for} \quad 0\leqq s\leqq s_0\,,$$

and

(2.2) $$\int_{-\infty}^{\infty} e^{-st}f(t)\,dt\sim\frac{A}{s^{\gamma}}, \qquad \textit{as} \quad s\to 0\,,$$

for some positive $\gamma>0$, *then*

$$\int_{-\infty}^{t} f(u)\,du\sim\frac{At^{\gamma}}{\Gamma(\gamma+1)}, \qquad \textit{as} \qquad t\to\infty\,.$$

PROOF. We have

$$\int_0^{\infty} e^{-st}f(t)\,dt=\int_{-\infty}^{\infty} e^{-st}f(t)\,dt-\int_{-\infty}^{0} e^{-st}f(t)\,dt$$
$$\sim\frac{A}{s^{\gamma}}-C$$
$$\sim\frac{A}{s^{\gamma}},$$

for $s\to 0$, C being $\int_{-\infty}^{0} f(t)\,dt$.

Hence by a well-known theorem, it results

$$\int_0^{t} f(u)\,du\sim\frac{At^{\gamma}}{\Gamma(\gamma+1)}, \qquad t\to\infty$$

and this is equivalent to

$$\int_{-\infty}^{t} f(u)\,du \sim \frac{At^{\gamma}}{\Gamma(\gamma+1)}.$$

LEMMA 2. *Let* X_i $(i=1,2,\cdots)$ *be independent random variables such that* $E(X_i)=m_i>0$. *Suppose that the distribution function* $F_n(x)$ *of* X_n *satisfies*

$$\int_{-\infty}^{0} e^{-sx}\,dF_n(x)<\infty\,, \qquad \textit{for} \quad 0\leqq s\leqq s_0 \tag{2.3}$$

for some s_0 *and further that*

$$\lim_{A\to\infty}\int_{A}^{\infty} x dF_n(x)=0\,, \tag{2.4}$$

$$\lim_{A\to\infty}\int_{-\infty}^{-A} e^{-sx}\,dF_n(x)=0 \tag{2.5}$$

hold uniformly with respect to n *and* $0<s\leqq s_0$.

If

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} m_i=m\,, \qquad (>0) \tag{2.6}$$

then

$$\lim_{s\to 0} s\sum_{n=1}^{\infty}\varphi_n(s)=\frac{1}{m}\,, \tag{2.7}$$

where

$$\varphi_n(s)=\int_{-\infty}^{\infty} e^{-sx}\,d\sigma_n(x)\,,$$

$\sigma_n(x)$ *being the distribution function of* S_n.

PROOF. We notice that there exists a constant C_1 independent of n such that

$$\int_{-\infty}^{\infty} |x|\,dF_n(x)<C_1\,. \tag{2.8}$$

This is immediate by (2.4) and (2.5). For, $|x|<e^{-s_0x}$ $(x<0)$ for large $|x|$ and, (2.4) and (2.5) show that

$$\int_{|x|>A} |x|\,dF_n(x)<a\,, \tag{2.9}$$

where A and a are some constants independent of n, and

$$\left|\int_{-\infty}^{\infty} x dF_n(x)\right| \leqq \int_{|x|>A} |x|\, dF_n(x) + \int_{|x|\leqq A} |x|\, dF_n(x)\,.$$

Put

$$(2.10) \qquad f_n(s) = \int_{-\infty}^{\infty} e^{-sx}\, dF_n(x)\,, \qquad 0 \leqq s \leqq s_0\,.$$

Let ε be any given positive number. Take A so large that

$$(2.11) \qquad \int_{|x|<A} |x|\, dF_n(x) < \varepsilon\,,$$

$$(2.12) \qquad \int_{-\infty}^{-A} e^{-s_0 x}\, dF_n(x) < \varepsilon\,,$$

which are possible by (2.4) and (2.5). Now we determine s_1 so that

$$(2.13) \qquad \int_{-\infty}^{-A} |x| e^{-sx}\, dF_n(x) < \int_{-\infty}^{-A} e^{-s_0 x}\, dF_n(x) < \varepsilon\,, \qquad \text{for} \quad 0 \leqq s \leqq s_1 < s_0\,.$$

Further we take s_2 so that

$$(2.14) \qquad |1 - e^{sA}| < \varepsilon\,, \qquad \text{for} \quad 0 \leqq s \leqq s_2 \leqq s_1\,.$$

Then we have

$$(2.15) \qquad \begin{aligned} f_n(s) &= f_n(0) + s f_n'(\theta s)\,, \qquad 0 < \theta < 1 \\ &= 1 + s f_n'(0) + s[f_n'(\theta s) - f_n'(0)]\,, \end{aligned}$$

and

$$\begin{aligned} |f_n'(\theta s) - f_n'(0)| &\leqq \left| \left(\int_{|x|>A} + \int_{|x|\leqq A} \right) (e^{-\theta s x} - 1)\, x dF_n(x) \right| \\ &\leqq \int_{x>A} |x|\, dF_n(x) + \int_{x<-A} |x| e^{-sx}\, dF_n(x) + \int_{|x|\leqq A} (e^{sA} - 1) |x|\, dF_n(x) \\ &< \varepsilon + \varepsilon + (e^{sA} - 1) \int_{-\infty}^{\infty} |x|\, dF_n(x) \\ &< 2\varepsilon + \varepsilon C_1 \end{aligned}$$

by (2.11), (2.13), (2.14) and (2.8). Hence (2.15) shows that we can write:

$$(2.16) \qquad f_n(s) = 1 - s m_n + s \eta_n\,,$$

where

(2.17) $$|\eta_n| < \varepsilon (C_1+2), \qquad \text{for} \quad 0 \leqq s \leqq s_2,$$

uniformly with respect to n. Thus

$$\log f_n(s) = \log(1 - sm_n + s\eta_n)$$

(2.18) $$= -sm_n + s\eta_n - \frac{s^2}{2}(m_n - \eta_n)^2 + \cdots$$

$$= -sm_n - s\xi_n,$$

say. Then there exists s_2 such that

(2.19) $$|\xi_n| < \varepsilon, \qquad \text{for} \quad 0 \leqq s \leqq s_2, \text{ uniformly for } n,$$

noticing that m_n is uniformly bounded by (2.8).

Now we have

$$\varphi_n(s) = \prod_{i=1}^{n} f_i(s) = e^{-s\Sigma_1^n (m_i+\xi_i)},$$

which we can represent as

(2.20) $$\varphi_n(s) = e^{-sn(m+\delta_n+\zeta_n)},$$

putting

$$\sum_{i=1}^{n} m_i = nm + n\delta_n, \qquad \sum_{i=1}^{n} \xi_i = n\zeta_n.$$

From (2.6), there exists an N for which

$$|\delta_n| < \varepsilon, \qquad \text{for} \quad n > N.$$

And

$$|\zeta_n| < \varepsilon, \qquad 0 \leqq s \leqq s_2$$

uniformly for n.

Hence we have

$$\sum_{n=1}^{\infty} \varphi_n(s) = \sum_{n=1}^{\infty} e^{-ns(m+\delta_n+\zeta_n)},$$

$$s\sum_{n=1}^{\infty} \varphi_n(s) = s\sum_{n=1}^{N} \varphi_n(s) + s\sum_{n=N+1}^{\infty} \varphi_n(s)$$

$$\leqq sN + s\sum_{n=N+1}^{\infty} e^{-ns(m-2\varepsilon)}$$

$$\leqq sN + \frac{se^{-s(m-2\varepsilon)}}{1-e^{-s(m-2\varepsilon)}}.$$

Thus

$$\limsup_{s\to 0} s\sum_{n=1}^{\infty}\varphi_n(s) \leqq \frac{1}{m-2\varepsilon}$$

and since ε is arbitrary we get

$$\limsup_{s\to 0} s\sum_{n=1}^{\infty}\varphi_n(s) \leqq \frac{1}{m}. \tag{2.21}$$

On the other hand

$$\begin{aligned} s\sum_{n=1}^{\infty}\varphi_n(s) &\geqq s\sum_{n=N+1}^{\infty}\varphi_n(s) \\ &\geqq s\sum_{n=N+1}^{\infty} e^{-ns(m+2\varepsilon)} \\ &= s\left(\sum_{n=0}^{\infty}-\sum_{n=0}^{N}\right) \\ &\geqq \frac{s}{1-e^{-s(m+2\varepsilon)}} - sN. \end{aligned}$$

Hence

$$\liminf_{s\to 0} s\sum_{n=1}^{\infty}\varphi_n(s) \geqq \frac{1}{m+2\varepsilon}$$

from which it results

$$\liminf_{s\to 0} s\sum_{m=1}^{\infty}\varphi_n(s) \geqq \frac{1}{m}. \tag{2.22}$$

(2.21) and (2.22) show (2.7).

3. Theorem. *Let* X_i $(i=1,2,\cdots)$ *be independent random variables. Suppose that* (2.3) *holds, and* (2.4) *and* (2.5) *hold uniformly with respect to* n *and* $0\leqq s\leqq s_0$. *If* (2.6) *is satisfied, then*

$$\lim_{x\to\infty}\frac{1}{x}\int_{-\infty}^{x}\sum_{n=1}^{\infty}P_r(x<S_n\leqq x+h)\,dx = \frac{h}{m}, \tag{3.1}$$

where $S_n=\sum_{i=1}^{n} X_i$.

PROOF. We put

$$G_N(x)=\sum_{n=1}^{N} P_r(x<S_n\leqq x+h)$$

and form

$$\int_{-\infty}^{\infty} e^{-sx}\, dG_N(x)=\sum_{n=1}^{N}\left(\int_{-\infty}^{\infty} e^{-sx}\, d\sigma_n(x+h)-\int_{-\infty}^{\infty} e^{-sx}\, d\sigma_n(x)\right),$$

where $\sigma_n(x)$ is, as before, the distribution function of S_n. Using the notations in **2**, the last expression is

$$\sum_{n=1}^{N}(e^{sh}-1)\int_{-\infty}^{\infty} e^{-sx}\, d\sigma_n(x)=(e^{sh}-1)\sum_{n=1}^{N}\varphi_n(s).$$

By Lemma 3, $\sum_{1}^{\infty}\varphi_n(s)$ is convergent and thus

$$\lim_{N\to\infty}\int_{-\infty}^{\infty} e^{-sx}\, dG_N(x)$$

exists and we have

$$\lim_{N\to\infty}\int_{-\infty}^{\infty} e^{-sx}\, dG_N(x)=(e^{sh}-1)\sum_{n=1}^{\infty}\varphi_n(s). \tag{3.2}$$

Similarly, putting

$$H_N(x)=\sum_{n=1}^{N} P_r(S_n\leqq x)$$

we have

$$\int_{-\infty}^{\infty} e^{-sx}\, dH_N(x)=\sum_{n=1}^{N}\int_{-\infty}^{\infty} e^{-sx}\, d\sigma_n(x)$$

$$=\sum_{n=1}^{N}\varphi_n(s) \tag{3.3}$$

and

$$\lim_{N\to\infty}\int_{-\infty}^{\infty} e^{-sx}\, dH_N(x) \tag{3.4}$$

exists.

Since $\varphi_n(s)$ is uniformly bounded for $0 \leqq s \leqq s_2$ (see (2.20)), we have: $\int_{-\infty}^{\infty} e^{-sx}\,dH_N(x) \leqq NC_2$ putting $\varphi_n(s) \leqq C_2$. Therefore

$$\int_{-\infty}^{-A} e^{-sx}\,dH_N(x) \leqq NC_2\,,$$

for any positive A. Taking t less than s_2 and $s = s_2$,

$$\begin{aligned} NC_2 &\geqq \int_{-\infty}^{-A} e^{-(s_2-t)x}\,e^{-tx}\,dH_N(x) \\ &\geqq e^{(s_2-t)A} \int_{-\infty}^{-A} e^{-tx}\,dH_N(x) \\ &\geqq e^{(s_2-t)A} \int_{-B}^{-A} e^{-tx}\,dH_N(x) \end{aligned}$$

which is

$$\int_{-B}^{-A} e^{-tx}\,dH_N(x) \leqq e^{-(s_2-t)A}\,NC_2\,.$$

Letting $t \to 0$, and $B \to \infty$, we get

$$H_N(-A) \leqq e^{-s_2 A}\,NC_2\,.$$

Therefore if $0 \leqq s \leqq s_3 < s_2$

$$\lim_{x \to -\infty} e^{sx} H_N(x) = 0\,. \tag{3.5}$$

Thus integration by parts shows that for $0 \leqq s \leqq s_3$

$$\int_{-\infty}^{\infty} e^{-sx}\,dH_N(x) = s \int_{-\infty}^{\infty} e^{-sx} H_N(x)\,dx\,. \tag{3.6}$$

Since $H_N(x)$ increases as $N \to \infty$ and tends to a non-decreasing function the existence of the limit (3.4) and (3.6) show that

$$\lim_{N \to \infty} \int_{-\infty}^{\infty} e^{-sx} H_N(x)dx = \int_{-\infty}^{\infty} e^{-sx} H(x)\,dx \tag{3.7}$$

exists for $0 \leqq s \leqq s_3$. $H(x)$ equals to $\sum_{n=1}^{\infty} P_r(S_n \leqq x)$. The existence of the right side integral shows that for $0 \leqq s \leqq s_4 \leqq s_3$,

$$H(x) = o(e^{sx})\,, \qquad \text{for} \quad |x| \to \infty\,. \tag{3.8}$$

Hence

$$s \int_{-\infty}^{\infty} e^{-sx} H(x)\,dx = \int_{-\infty}^{\infty} e^{-sx}\,dH(x) \tag{3.9}$$

exists. Since $H(x+h)-H(x)=G(x)=\sum_{n=1}^{\infty}P_r(x<S_n\leqq x+h)$, combining (3.2), (3.6), (3.7) and (3.9), we get

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}e^{-sx}dG_n(x)=s\int_{-\infty}^{\infty}e^{-sx}G(x)\,dx$$

$$=(e^{-sh}-1)\sum_{n=1}^{\infty}\varphi_n(s)\,.$$

By Lemma 3, we have

(3.10) $$\int_{-\infty}^{\infty}e^{-sx}G(x)dx\sim\frac{h}{ms}\,,\qquad s\to 0\,.$$

By (3.8), $G(x)=o(e^{sx})$ for $|x|\to\infty$ and for fixed h.

$$\int_{-\infty}^{0}e^{-sx}G(x)\,dx<\infty\,,\qquad \text{for}\qquad 0\leqq s\leqq s_5<s_4\,.$$

Then by Lemma 2, we have finally

$$\int_{-\infty}^{x}G(x)\,dx\sim\frac{hx}{m}\qquad \text{as}\qquad x\to\infty$$

which proves the theorem.

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