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Homogeneous Riemannian spaces of four dimensions.

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E. Cartan [1] gave a general method to determine homogeneous Riemannian spaces, i. e. those which admit a transitive group of isometries or motions. He determined, in particular, applying his own method, the topological structure of three dimensional homogeneous Riemannian spaces.

On the other hand, K. Yano [10] has given many beautiful theorems on groups of isometries in a Riemannian space. But unfortunately four dimensional spaces are excluded in most of his theorems.

The purpose of the present paper is to determine, by the same method as Cartan's, the topological structure of four dimensional homogeneous Riemannian spaces which are connected and simply connected. Following the programme indicated in [1], we shall first determine in § 1 all the types of subgroups of the proper orthogonal group $R(4)$ in four variables. In § 2, we shall explain, for the sake of completeness, the general method of E. Cartan [1]. In § 3, we shall determine homogeneous Riemannian spaces admitting each subgroup obtained in § 1 as the group of stability. In most cases we determine the local structure of homogeneous Riemannian spaces by the method of moving frames, and from the local structure obtained we shall find the topological structure of the spaces.

Our main result is the following

THEOREM. *Any four dimensional homogeneous Riemannian space which is connected and simply connected is homeomorphic to one of the following manifolds:*

Euclidean space of four dimensions,
Sphere of four dimensions,
Complex projective space of two complex dimensions,
Product space of two spheres of two dimensions,

Product space of a straight line and a sphere of three dimensions,
Product space of a Euclidean plane and a sphere of two dimensions.

In § 4 we shall consider the cases which are exceptional in Yano's theorems. § 5 is devoted to the discussions of homogeneous Riemannian spaces of four dimensions whose fundamental group is compact. Finally in § 6, we shall consider homogeneous Kählerian spaces of two complex dimensions.

§ 1. Subgroups of the proper orthogonal group $R(4)$ in four variables. In this section we shall determine the types of connected subgroups of $R(4)$. It is well known that the universal covering group of the proper orthogonal group $R(4)$ in four variables is the product group of the universal covering group of the proper orthogonal group $R(3)$ in three variables by itself [3]. In a Euclidean space E^4 of four dimensions, there exist two families $\mathfrak{E}$ and $\mathfrak{E}'$ of oriented orthogonal frames (e_0, e_1, e_2, e_3) of unit vectors at the origin of E^4. With respect to one of these two families, say $\mathfrak{E}$, we define

$$\theta_{ij} = de_i \cdot e_j \qquad (i, j, k = 0, 1, 2, 3),$$

where

$$\theta_{ij} + \theta_{ji} = 0 .$$

The differential forms θ_{ij} satisfy the following equations of structure of our group $R(4)$:

$$d\theta_{ij} = \textstyle\sum_k \theta_{ik} \wedge \theta_{kj} . \tag{1.1}$$

Introducing the following differential forms:

$$\begin{aligned} &\varphi_1 = \theta_{01} - \theta_{23}, \quad \varphi_2 = \theta_{02} - \theta_{31}, \quad \varphi_3 = \theta_{03} - \theta_{12}; \\ &\psi_1 = -\theta_{01} - \theta_{23}, \quad \psi_2 = -\theta_{02} - \theta_{31}, \quad \psi_3 = -\theta_{03} - \theta_{12}, \end{aligned} \tag{1.2}$$

we have the relations

$$\begin{aligned} &d\varphi_1 = \varphi_2 \wedge \varphi_3, \quad d\varphi_2 = \varphi_3 \wedge \varphi_1, \quad d\varphi_3 = \varphi_1 \wedge \varphi_2; \\ &d\psi_1 = \psi_2 \wedge \psi_3, \quad d\psi_2 = \psi_3 \wedge \psi_1, \quad d\psi_3 = \psi_1 \wedge \psi_2 \end{aligned} \tag{1.3}$$

by virtue of (1.1). From (1.3), we have the decomposition of the universal covering group of $R(4)$. If we use the family $\mathfrak{E}'$ of frames instead of $\mathfrak{E}$, the systems of forms φ_p and ψ_p $(p = 1, 2, 3)$ are interchanged mutually. From these preliminary remarks we get

PROPOSITION. *Let $\mathfrak{H}$ be a non-trivial subalgebra of the Lie algebra $\mathfrak{G}$ of $R(4)$. Then $\mathfrak{H}$ is equivalent, under an adjoint transformation induced on $\mathfrak{G}$ by an element of the orthogonal group $O(4)$, to one of the subalgebras defined by the following equations:*

1°. $\varphi_2=\varphi_3=0$;

2°. $\varphi_1-\psi_1=0, \quad \varphi_2-\psi_2=0, \quad \varphi_3-\psi_3=0$;

3°. $\varphi_1=\varphi_2=\varphi_3=0$;

4°. $\varphi_2=\varphi_3=\psi_2=\psi_3=0$;

5°. $\varphi_1=\varphi_2=\varphi_3=\psi_2=\psi_3=0$;

6°. $\varphi_2=\varphi_3=\psi_2=\psi_3=0, \quad m\varphi_1=\psi_1 \quad (m>0)$.

Each of these equations can be expressed by the corresponding relations among θ's according to the definition (1.2). From these equations, we are able to construct the matrix-representation of a connected subgroup of $R(4)$ corresponding to each of the subalgebras given in the above list, and we have a familiar linear group in each case.

By virtue of this proposition, we have a lemma concerning subgroups of $R(4)$.

LEMMA. *Let g be a connected Lie subgroup of the proper orthogonal group $R(4)$ in four variables. Then g is one of the eight subgroups $g_1, g_2, \cdots, g_8$ of $R(4)$ up to conjugation with respect to $O(4)$, where the subgroups $g_1, g_2, \cdots, g_8$ are defined as follows:*

(I) $g_1=R(4)$ *and* $\dim g_1=6$;

(II) g_2 *has the Lie algebra defined by* 1° *and* $\dim g_2=4$;

(III) g_3 *has the Lie algebra defined by* 2° *and* $\dim g_3=3$;

(IV) g_4 *has the Lie algebra defined by* 3° *and* $\dim g_4=3$;

(V) g_5 *has the Lie algebra defined by* 4° *and* $\dim g_5=2$;

(VI) g_6 *has the Lie algebra defined by* 5° *and* $\dim g_6=1$;

(VII) g_7 *has the Lie algebra defined by* 6° *and* $\dim g_7=1$;

(VIII) g_8 *is composed only of the identity element.*

It is to be noted that $R(4)$ *has no subgroup of order* 5 *and has subgroups of order* 4. In general, D. Montgomery and H. Samelson [7] have proved that *in an n-dimensional Euclidean space, for* $n \neq 4$, *there exists no proper subgroup of the rotation group of order greater than* $(n-1)(n-2)/2$.

§ **2. General method of E. Cartan.** The homogeneous Riemannian space M is naturally identified with the coset space G/g of the fundamental group G of M, where G is a Lie group and g is the subgroup of stability at a point $O \in M$. *Let us suppose hereafter that the group G is effective on M.* The group of stability at the point O induces a group $\tilde{g}$ of orthogonal transformations on the tangent space of M at O. Since G is effective on M, g is isomorphic to $\tilde{g}$. It is not an essential restriction, for our problem, *to assume that g is connected.* Therefore the subgroup g is assumed to be one of the subgroups $g_1, g_2, \cdots, g_8$ contained in the lemma of § **1**.

We shall begin with a sketch of the Cartan's method [1]. An element of the Lie algebra of $R(4)$ can be expressed by

$$\sum_{i<j} \xi_{ij} X_{ij}, \qquad \xi_{ij}+\xi_{ji}=0;$$

where ξ's are real numbers and X's are the infinitesimal operators such that $X_{ij} f = x^i \partial f/\partial x^j - x^j \partial f/\partial x^i$. Here, Latin indices $i, j, k, \cdots$ run over the range $0, 1, 2, 3$ in this section. Let us suppose that the Lie algebra of the group g of stability, which is considered as a subgroup of $R(4)$, is expressed by some linear equations

$$\sum_{i<j} A_{aij} \xi_{ij} = 0 \qquad (\alpha = 1, 2, \cdots, m), \tag{2.1}$$

where m is an integer and A_{aij} are constants.

On the other hand, let $\mathfrak{F}$ be a family of adapted frames (P, e_i) of G on M, where the point P runs over M. In usual way, the variation of frames of $\mathfrak{F}$ can be given by the equations

$$dP = \sum_k \omega_k e_k,$$
$$de_i = \sum_k \omega_{ik} e_k, \qquad \omega_{ij}+\omega_{ji}=0.$$

These equations give the Euclidean connection of the Riemannian space M with respect to the family $\mathfrak{F}$ of adapted frames. The Riemannian metric on the space M, which is invariant under the fundamental group G, is given by

$$ds^2 = \sum_k \omega_k^2 .$$

Moreover, the relative components of the group G are given by some independent linear combinations of the forms ω_i and ω_{ij}.

On the subgroup g of G, which is defined by $\omega_i = 0$, the forms ω_{ij} satisfy the relations

$$\sum_{i<j} A_{aij}\, \omega_{ij} = 0 \qquad (\alpha = 1, 2, \cdots, m) .$$

Then, on the group G we have

$$\sum_{i<j} A_{aij}\, \omega_{ij} = \sum_i c_{ai}\, \omega_i \qquad (\alpha = 1, 2, \cdots, m) , \tag{2.2}$$

where c_{ai} are constants to be determined.

An element $X = \sum_{i<j} \xi_{ij} X_{ij}$ of the Lie algebra of g induces an infinitesimal transformation on the family $\mathfrak{F}$ of adapted frames, and then X induces variations of ω_i and ω_{ij}. The variations of the forms ω_i and ω_{ij} induced by X are given by

$$\delta\omega_i = \sum_k \xi_{ik}\, \omega_k ,$$
$$\delta\omega_{ij} = \sum_k \xi_{ik}\, \omega_{kj} + \sum_k \xi_{jk}\, \omega_{ik} .$$

Since the relations (2.2) are invariant, we have

$$\sum_{i<j} A_{aij}\, \delta\omega_{ij} = \sum_i c_{ai}\, \delta\omega_i \qquad (\alpha = 1, 2, \cdots, m) . \tag{2.2$'$}$$

Thus, substituting the above variations of ω_i and ω_{ij} in both sides of (2.2)$'$, we have the following relations:

$$\sum_{i<j} \sum_k A_{aij} (\xi_{ik}\, \omega_{kj} + \xi_{jk}\, \omega_{ik}) = \sum_{i,k} c_{ai}\, \xi_{ik}\, \omega_k \qquad (\alpha = 1, 2, \cdots, m) , \tag{2.3}$$

where

$$\sum_{i<j} A_{aij}\, \xi_{ij} = 0 .$$

From (2.3), we have some relations among the constants c_{ai}. These relations are very useful for us to determine the constants c_{ai}.

The equations of structure of the Riemannian space M are, as is well known, given by

$$\begin{aligned} d\omega_i &= \sum_k \omega_{ik} \wedge \omega_k , \\ d\omega_{ij} &= \sum_k \omega_{ik} \wedge \omega_{kj} + \Omega_{ij} , \end{aligned} \tag{2.4}$$

where $\Omega_{ij} = -\Omega_{ji}$ are the *curvature forms* of M. Let us here remember well-known identities

$$\sum_k \Omega_{ik} \wedge \omega_k = 0 , \tag{2.5}$$

and the identities of Bianchi

$$d\,\Omega_{ij} + \sum_k \Omega_{ik} \wedge \omega_{kj} - \sum_k \omega_{ik} \wedge \Omega_{kj} = 0 . \tag{2.6}$$

If the constants c_{ai} are thus determined, we can write down the equations (2.4) of structure of the Riemannian space M. Consequently, when the group g of stability is given, the structure of the Riemannian space M is obtained. Since M is complete, we can determine the topological structure of M, using the local structure of the Riemannian space M and the following lemma [1].

LEMMA. *Let M be an n-dimensional complete Riemannian space which is connected and simply connected. Then M is homeomorphic to a sphere of n dimensions, if its sectional curvature is a positive constant. When M has non-positive sectional curvature, it is homeomorphic to a Euclidean space of n dimensions.*

Furthermore, some additional remarks are needed. If the fundamental group G of the homogeneous space M is locally isomorphic to a direct product of two groups G_1 and G_2, and if, under the same local isomorphism, the subgroup g of stability is so, then the space M is locally isometric to a product space of two homogeneous Riemannian spaces M_1 and M_2 whose fundamental groups are G_1 and G_2 respectively. In addition, if M_1 and M_2 are simply connected, then M is the direct product of M_1 and M_2, since M is simply connected.

Let us now introduce some notations about linear groups and manifolds which will be used frequently in the paper.

$R(n)$ is the proper orthogonal group in n real variables.

$L(n)$ is the connected component of the identity in the group of all non-singular linear transformations in n real variables $(x_1, x_2, \cdots, x_n)$ which leave invariant the quadratic form

$$x_1^2 + x_2^2 + \cdots + x_{n-1}^2 - x_n^2 .$$

$U(n)$ is the unitary group in n complex variables, and

$SU(n)$ is the group of all unitary transformations whose determinants are equal to 1.

Λ_n is the subgroup of $SU(n)$ which is composed of n matrices

$I_n, \lambda_1 I_n, \cdots, \lambda_{n-1} I_n$, where I_n is the unit n-matrix and $1, \lambda_1, \lambda_2, \cdots, \lambda_{n-1}$ are distinct n roots of the equation $x^n - 1 = 0$.

$\mathfrak{L}(n)$ is the group of all non-singular linear transformations in n complex variables $(z_1, z_2, \cdots, z_n)$ which leave invariant the form

$$z_1 \bar{z}_1 + z_2 \bar{z}_2 + \cdots + z_{n-1} \bar{z}_{n-1} - z_n \bar{z}_n .$$

$S\mathfrak{L}(n)$ is the group of all linear transformations of the group $\mathfrak{L}(n)$ whose determinants are equal to 1.

$\mathfrak{M}(n)$ is the group of proper motions in an n-dimensional Euclidean space.

$\mathfrak{M}_H(n)$ is the group of all martices which have the form

$$\begin{pmatrix} & & & a_1 \\ & A & & \vdots \\ & & & a_n \\ 0 & \cdots & 0 & 1 \end{pmatrix},$$

where the matrix A runs over $U(n)$ and $a_1, a_2, \cdots, a_n$ take all complex numbers.

$S\mathfrak{M}_H(n)$ is the subgroup of $\mathfrak{M}_H(n)$ for which A runs over $SU(n)$.

E^n is a Euclidean space of n dimensions.

S^n is a sphere of n dimensions.

$P(C, n)$ is a complex projective space of n complex dimensions. (It is well known that $P(C, n)$ *is simply connected.*)

$C(+, n)$ is a Riemannian space of n dimensions with positive constant curvature.

$C(-, n)$ is a Riemannian space of n dimensions with negative constant curvature.

$C(0, n)$ is a locally flat Riemannian space of n dimensions.

§ **3. Determination of the space.** We shall determine, for each type of groups of stability, the local structure of the Riemannian space M following the Cartan's method explained in § **2.** In this section, the head-number of CASE denotes the type of the groups of stability appearing in the lemma of § **1.** During the discussions developed here, we talk frequently about such familiar linear groups that are given at the end of § **2.** Therefore, some appendices are added at the end of the paper about the structure of these groups.

In subsequent paragraphs indices run over the following ranges:

$$i, j, k, \cdots = 0, 1, 2, 3; \qquad p, q, r, \cdots = 1, 2, 3;$$
$$\alpha, \beta, \gamma, \cdots = 0, 1.$$

3.1. CASE (I). In this case $g = R(4)$. Then the Riemannian space M admits free mobility around any point of the space. Thus, M is $C(+, 4)$, $C(-, 4)$ or $C(0, 4)$ [1]. Consequently, the group G is locally isomorphic to $R(5)$, $L(5)$ or $\mathfrak{M}(4)$ respectively (Appendices **1, 2** and **5**) and hence *M is homeomorphic to S^4 or E^4.*

REMARK. M. Obata [8] has proved the following fact without the additional condition that M is simply connected. That is, *if M is $C(-, 4)$ or $C(0, 4)$, then it is homeomorphic to E^4.*

3.2. CASE (II). In this case, the equations (2.1) become

$$\xi_{02} - \xi_{31} = \xi_{03} - \xi_{12} = 0.$$

As a consequence of (2.3), we have

$$\omega_{02} = \omega_{31}, \qquad \omega_{03} = \omega_{12}.$$

If we now put

$$\pi_0 = \omega_0 + \sqrt{-1}\,\omega_1, \qquad \pi_1 = \omega_2 - \sqrt{-1}\,\omega_3,$$

then, by virtue of the first system of equations (2.4), we have

$$\begin{aligned} d\pi_0 &= \pi_{00} \wedge \pi_0 + \pi_{01} \wedge \pi_1, \\ d\pi_1 &= \pi_{10} \wedge \pi_0 + \pi_{11} \wedge \pi_1, \end{aligned} \tag{3.1}$$

where

$$\pi_{00} = -\sqrt{-1}\,\omega_{01}, \qquad \pi_{11} = \sqrt{-1}\,\omega_{23},$$
$$\pi_{01} = -\bar{\pi}_{10} = \omega_{02} + \sqrt{-1}\,\omega_{03}.$$

Here the conjugate complex of the form $\pi_{\alpha\beta}$ is denoted by $\bar{\pi}_{\alpha\beta}$, and analogous notations will be used for complex valued differential forms throughout the paper. Then we are able to see that

$$\pi_{\alpha\beta} + \bar{\pi}_{\beta\alpha} = 0.$$

If we now put

$$\Phi_{\alpha\beta}=d\pi_{\alpha\beta}-\sum_{\gamma}\pi_{\alpha\gamma}\wedge\pi_{\gamma\beta}, \tag{3.2}$$

then we have

$$\Phi_{\alpha\beta}+\overline{\Phi}_{\beta\alpha}=0.$$

Hence M is an Hermitian space without torsion, that is, a *Kählerian space.*

Moreover, in this case, the group g of stability induces a complex linear group $\tilde{g}$ on the complex tangent space T of M at the point $O\in M$, and the group g is isomorphic to $U(2)$. Thus, the group $\tilde{g}$ transforms the (complex) directions on T transitively; in other words, *M admits free mobility around any point in the sense of the Hermitian geometry.* Then M is a Kählerian space with constant holomorphic sectional curvature. Therefore, the curvature tensor of M is given by

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}}=K(g_{\alpha\bar{\beta}}\,g_{\gamma\bar{\delta}}+g_{\alpha\bar{\delta}}\,g_{\gamma\bar{\beta}})$$

according to K. Yano [9], where K is a real constant, and

$$ds^2=\sum_{\alpha,\beta}g_{\alpha\bar{\beta}}\,\pi_\alpha\,\overline{\pi}_\beta\ (=\sum_{\gamma}\pi_\gamma\,\overline{\pi}_\gamma).$$

On the other hand, since $\Phi_{\alpha\beta}$ are given by

$$\Phi_{\alpha\beta}=\sum_{\gamma,\delta}R_{\alpha}{}^{\beta}{}_{\gamma\bar{\delta}}\,\pi_\gamma\wedge\overline{\pi}_\delta,$$

then we obtain

$$\begin{aligned}\Phi_{00}&=K(2\,\pi_0\wedge\overline{\pi}_0+\pi_1\wedge\overline{\pi}_1),\\ \Phi_{11}&=K(\pi_0\wedge\overline{\pi}_0+2\,\pi_1\wedge\overline{\pi}_1),\\ \Phi_{01}&=-\overline{\Phi}_{10}=K\,\pi_1\wedge\overline{\pi}_0.\end{aligned} \tag{3.3}$$

Now we shall consider three cases. In the case $K=0$, we have that M is $C(0,4)$. Hence *M is homeomorphic to E^4* and G is locally isomorphic to $\mathfrak{M}_H(2)$ (Appendix **6**). In the case where M is $C(0,4)$, the group G contains a subgroup which is isomorphic to $S\mathfrak{M}_H(2)$ (§ **3.4.** CASE (IV)). Then *G is isomorphic to $\mathfrak{M}_H(2)$, when M is $C(0,4)$.*

When $K\neq 0$, replacing $\sqrt{|K|}\,\pi_0$ and $\sqrt{|K|}\,\pi_1$ by π_0 and π_1 respectively, we have

$$\Phi_{00}=\varepsilon(2\,\pi_0\wedge\overline{\pi}_0+\pi_1\wedge\overline{\pi}_1),$$

$$\Phi_{11}=\varepsilon(\pi_0\wedge\overline{\pi}_0+2\pi_1\wedge\overline{\pi}_1),$$

$$\Phi_{01}=-\overline{\Phi}_{10}=\varepsilon\pi_1\wedge\overline{\pi}_0,$$

where ε is the sign of K. Hence, if we replace $\overline{\pi}_\alpha$ and $\overline{\pi}_{\alpha\beta}$ by π_α and $\pi_{\alpha\beta}$ respectively, we have the following results.

When $K>0$, G is locally isomorphic to $SU(3)$ and *M is homeomorphic to $P(C, 2)$* (Appendix **3**). Moreover, *in this case G is isomorphic to the factor group $SU(3)/\Lambda_3$*.

When $K<0$, the general sectional curvature of M is always negative according to K. Yano [9]. Consequently, *M is homeomorphic to E^4* and G is locally isomorphic to $S\mathfrak{L}(3)$ (Appendix **4**).

As a consequence, we see that *M is homeomorphic to E^4 or $P(C, 2)$*.

REMARK. The following fact holds good, as is known in § **3.1,** without the additional condition that M is simply connected. That is, *if the constant K is negative or zero, M is homeomorphic to E^4*.

We can easily generalize the results obtained in this paragraph. That is, we obtain the following theorem.

THEOREM 1. *Let g be the group of stability of a $2n$-dimensional homogeneous Riemannian space $V^{2n}=G/g$. If g is isomorphic to the real representation of $U(n)$, then V^{2n} is a Kählerian space with constant holomorphic sectional curvature. If V^{2n} is connected and simply connected, and if g is as above, then V^{2n} is homeomorphic to $P(C, n)$ or E^{2n} and the fundamental group G is isomorphic to one of two groups $SU(n+1)/\Lambda_{n+1}$, $\mathfrak{M}_H(n)$, or locally isomorphic to $S\mathfrak{L}(n+1)$.*

3.3. CASE (III). In this case, the equations (2.1) become

$$\xi_{01}=\xi_{02}=\xi_{03}=0.$$

As a consequence of (2.3), we have

$$\omega_{0p}=c\,\omega_p,$$

where c is a constant.

According to K. Yano [10], if $c=0$, the Riemannian space M is one of the following spaces:

$$V^1\times C(+,3),\qquad V^1\times C(-,3),\qquad C(0,4),$$

where V^1 is a straight line with its natural Riemannian metric.

Consequently, G is locally isomorphic to one of the following groups:

$$A_1 \times R(4), \qquad A_1 \times L(4), \qquad A_1 \times \mathfrak{M}(3)$$

respectively, where A_1 is a 1-dimensional vector group (Appendices **1, 2** and **5**). Hence *M is homeomorphic to $E^1 \times S^3$ or E^4.*

If $c \neq 0$, according to K. Yano [10], M is $C(-,4)$. Hence *M is homeomorphic to E^4.* Here, we can assume $c=1$ without loss of generality. Then the group G is locally isomorphic to a subgroup G' of $L(5)$ by virtue of (2.4). Moreover, the group G' leaves invariant every point on the straight line defined by

$$x_2 = x_3 = x_4 = 0, \qquad x_1 + x_5 = 0$$

in a space of five variables $(x_1, x_2, x_3, x_4, x_5)$ on which the group $L(5)$ operates (Appendix **2**).

Consequently, we find that *M is homeomorphic to E^4 or $E^1 \times S^3$.*

REMARK. M. Obata [8] has proved the following two facts without the additional condition that M is simply connected. That is, *when M is $C(-,4)$, it is homeomorphic to E^4. When M is locally isometric to $V^1 \times C(-,3)$ or $C(0,4)$, M is homeomorphic to E^4 or $S^1 \times E^3$.*

3.4. CASE (IV). In this case, the equations (2.1) become

$$\xi_{01} - \xi_{23} = \xi_{02} - \xi_{31} = \xi_{03} - \xi_{12} = 0.$$

As a consequence of (2.3), we obtain

$$\omega_{01} = \omega_{23}, \qquad \omega_{02} = \omega_{31}, \qquad \omega_{03} = \omega_{12}.$$

Putting now

$$\pi_0 = \omega_0 + \sqrt{-1}\,\omega_1, \qquad \pi_1 = \omega_2 - \sqrt{-1}\,\omega_3,$$

we see that the first system of (2.4) is reduced to (3.1), where

$$\pi_{00} = -\sqrt{-1}\,\omega_{01}, \qquad \pi_{11} = \sqrt{-1}\,\omega_{01},$$

$$\pi_{01} = -\overline{\pi}_{10} = \omega_{02} - \sqrt{-1}\,\omega_{03}.$$

Evidently we have

$$\pi_{\alpha\beta} + \overline{\pi}_{\beta\alpha} = 0, \qquad \pi_{00} + \pi_{11} = 0.$$

If we introduce the curvature forms $\overset{*}{\Phi}_{\alpha\beta}$ by (3.2), then we can easily see that

$$\Phi_{\alpha\beta}+\overline{\Phi}_{\beta\alpha}=0\,, \qquad \Phi_{00}+\Phi_{11}=0\,.$$

Therefore, M is a Kählerian space. The group g of stability behaves in the same manner as in § **3.2** CASE (II). Thus, the holomorphic sectional curvature of M is constant. Then we have the same relations as (3.3). On the other hand, since $\Phi_{00}+\Phi_{11}=0$ as mentioned above, the constant K in (3.3) is equal to zero. Consequently, M is $C(0,4)$ and G is locally isomorphic to $S\mathfrak{M}_H(2)$ (Appendix **6**). Hence *M is homeomorphic to E^4.*

REMARK. The results in this paragraph hold good without the condition that M is simply connected.

In this case, we can determine globally the fundamental group G as follows. Since the given space $M=G/g$ is simply connected and g is isomorphic to $SU(2)$ which is simply connected, the fundamental group G is also simply connected. Thus *G is isomorphic to $S\mathfrak{M}_H(2)$*, which is simply connected.

We can easily generalize the results obtained in this paragraph. That is, we can prove the following

THEOREM 2. *Let g be the group of stability of a $2n$-dimensional homogeneous Riemannian space $V^{2n}=G/g$. If g is isomorphic to the real representation of $SU(n)$, then V^{2n} is flat for $n\neq 3$. If V^{2n} is connected, and if g is as above, then V^{2n} is homeomorphic to E^4 and the fundamental group G is isomorphic to $S\mathfrak{M}_H(n)$ for $n\neq 3$.*

3.5. CASE (V). In this case, the equations (2.1) become

$$\xi_{02}=\xi_{03}=\xi_{31}=\xi_{12}=0\,.$$

By virtue of (2.3), we have

$$\omega_{02}=\omega_{03}=\omega_{31}=\omega_{12}=0\,.$$

Hence, if we now put $\omega_{01}=\omega$ and $\omega_{23}=\tilde{\omega}$, we can easily see that the forms Ω_{ij} are zero except

$$\Omega_{01}=d\,\omega\,, \qquad \Omega_{23}=d\,\tilde{\omega}\,.$$

Since it follows from (2.5) that

$$d\,\omega\wedge\omega_0=d\,\omega\wedge\omega_1=0\,,$$
$$d\,\tilde{\omega}\wedge\omega_2=d\,\tilde{\omega}\wedge\omega_3=0\,,$$

we have naturally

$$d\,\omega=K\,\omega_0\wedge\omega_1\,,\qquad d\,\tilde{\omega}=K'\,\omega_2\wedge\omega_3\,.$$

In the above equations K and K' are constants. Consequently, the given space M is a product space of two Riemannian spaces M_1 and M_2 of two dimensions whose curvatures are constants.

If $K=0$, the space M_1 is flat and then M_1 is homeomorphic to E^2. Thus G_1 is locally isomorphic to $\mathfrak{M}(2)$, where G_1 is the fundamental group of M_1 (Appendix **5**). In the case where $K<0$, M_1 is $C(+,2)$. Then M_1 is homeomorphic to S^2 and G_1 is locally isomorphic to $R(3)$ (Appendix **1**). When $K>0$, M is $C(-,2)$. Then M_1 is homeomorphic to E^2 and G_1 is locally isomorphic to $L(3)$ (Appendix **2**). Moreover, in the same way, the space M_2 and its fundamental group G_2 are obtained. Consequently, *the given space M is homeomorphic to*

$$S^2\times S^2\,,\qquad E^2\times S^2\quad \textit{or}\quad E^4\,.$$

3.6. CASE (VI). In this case, the equations (2.1) become

$$\xi_{02}=\xi_{03}=\xi_{31}=\xi_{12}=0\,,\qquad \xi_{23}=m\,\xi_{01}\qquad (m>0)\,.$$

As a consequence of (2.3), we have

$$\omega_{02}=\omega_{03}=\omega_{31}=\omega_{12}=0\,,\qquad \omega_{23}=m\,\omega_{01}\qquad (m>0)\,.$$

Then every form Ω_{ij} in (2.4) is zero except

$$\Omega_{01}=d\,\omega_{01}\,,\qquad \Omega_{23}=m\,d\,\omega_{01}\,.$$

Furthermore, using (2.5), we get $d\,\omega_{01}\wedge\omega_i=0$, since $m\neq0$. Then we obtain automatically

$$d\,\omega_{01}=0\,.$$

Therefore we have $\Omega_{ij}=0$. Hence M is $C(0,4)$, and so *M is homeomorphic to E^4*. Observing (2.4), we can easily find that the group G is locally isomorphic to a subgroup of $\mathfrak{M}(4)$ whose rotation-part is the group defined by the equations 5^0 in Proposition in § **1** (Appendix **5**).

3.7. CASE (VII)[1]. In this case, the equations (2.1) become

$$\xi_{02}=\xi_{03}=\xi_{12}=\xi_{23}=\xi_{31}=0.$$

By virtue of (2.3), we have

$$\omega_{02}=a\,\omega_0+b\,\omega_1, \qquad \omega_{03}=\alpha\,\omega_0+\beta\,\omega_1,$$

$$\omega_{12}=-b\,\omega_0+a\,\omega_1, \qquad \omega_{31}=\beta\,\omega_0-\alpha\,\omega_1, \qquad \omega_{23}=r\,\omega_2+t\,\omega_3,$$

where a, b, α, β, r and t are constants which must be determined.

According to (2.5), we obtain

$$\begin{aligned} &2ab+r\beta=0, \qquad 2\alpha b-rb=0, \\ &2a\beta+t\beta=0, \qquad 2\alpha\beta-tb=0, \qquad ar+\alpha t=0, \\ &d\omega_{01}=2K\omega_0\wedge\omega_1+(br+\beta t)\,\omega_2\wedge\omega_3, \end{aligned} \tag{3.4}$$

where K is a constant. If we consider the identity (2.6) of Bianchi for $i=0$, $j=1$, it is easily seen that

$$\begin{aligned} &2aK+\beta(br+\beta t)=0, \\ &2\alpha K-b(br+\beta t)=0. \end{aligned} \tag{3.5}$$

These relations will be frequently used in the following discussions.

Since the treatment in this step is more complicated, we shall consider following four cases:

$$\text{(i)}\ \ b\neq 0,\ \ \beta\neq 0; \qquad \text{(ii)}\ \ b\neq 0,\ \ \beta=0;$$

$$\text{(iii)}\ \ b=0,\ \ \beta\neq 0; \qquad \text{(iv)}\ \ b=0,\ \ \beta=0.$$

3.7, (i). **The case where $b\neq 0$, $\beta\neq 0$.** Evidently, from (3.4), we have

$$r=2\alpha, \qquad t=-2a, \qquad ab+\alpha\beta=0. \tag{3.6}$$

1) Prof. T. Ōtsuki has given the author many valuable suggestions concerning the treatments of this section.

In this case, the equations (3.5) are reduced to

$$aK+\beta(b\alpha-a\beta)=0\,,$$

$$\alpha K-b(b\alpha-a\beta)=0\,.$$

As a consequence of these relations and (3.6), it is easily seen that $K=b^2+\beta^2$, if $a\neq 0$ ($\alpha\neq 0$ is a natural consequence of $a\neq 0$ and $\beta\neq 0$). Then we have

$$d\,\omega_{01}=2(b^2+\beta^2)\,\omega_0\wedge\omega_1+2(b\alpha-a\beta)\,\omega_2\wedge\omega_3\,.$$

On the other hand, according to (3.6) and equations (2.4) of structure of the space M, we obtain

$$d\,\omega_2=2b\,\omega_0\wedge\omega_1+2\alpha\,\omega_2\wedge\omega_3\,,$$

$$d\,\omega_3=2\beta\,\omega_0\wedge\omega_1-2a\,\omega_2\wedge\omega_3\,.$$

By virtue of these relations, if we put $\omega=\omega_{01}-b\,\omega_2-\beta\omega_3$, we get easily

$$d\,\omega=0\,.$$

Therefore, the equation $\omega=0$ defines a subgroup G' of the fundamental group G. Obviously G' is transitive on the given space M. Hence M is homeomorphic to G', since M is simply connected. Moreover, *G' is homeomorphic to E^4 or $E^1\times S^3$*, because it has four parameters and is simply connected. (The proof of this statement will be given in § **3.8.**). Consequently, *M is homeomorphic to E^4 or $E^1\times S^3$*. In particular, it is obtained by more detailed calculations on the structure of G' that *M is homeomorphic to E^4.*

When $a=0$ ($\alpha=0$ is an obvious consequence of $a=0$ and $\beta\neq 0$), it is easily proved by virtue of (2.4) that

$$d\,\omega_3=2\beta\omega_0\wedge\omega_1\,,\qquad\qquad d\,\omega_{01}=2K\,\omega_0\wedge\omega_1\,.$$

Then, if we put $\omega=\beta\omega_{01}-K\,\omega_3$, we have

$$d\,\omega=0\,.$$

Therefore, the equation $\omega=0$ defines a subgroup G' of the group G, and the group G' is transitive on the space M, since $\beta\neq 0$. Hence

M is homeomorphic to E^4 *or* $E^1 \times S^3$, by the same reason as above.

3.7, (ii). **The case where** $b \neq 0$, $\beta = 0$. It is easily seen from (3.4) that

$$a = t = 0, \qquad r = 2\alpha,$$

$$d\,\omega_{01} = 2K\,\omega_0 \wedge \omega_1 + 2\alpha b\,\omega_2 \wedge \omega_3 .$$

By virtue of (3.5), we have $\alpha K = \alpha b^2$. Then $K = b^2$, if $\alpha \neq 0$. On the other hand, according to (2.4) we obtain

$$d\,\omega_2 = 2b\,\omega_0 \wedge \omega_1 + 2\alpha\,\omega_2 \wedge \omega_3 .$$

Consequently, if we put $\omega = \omega_{01} - b\,\omega_2$, we have

$$d\,\omega = 0 .$$

Therefore, the equation $\omega = 0$ defines a subgroup G' of the group G, and the group G' is evidently transitive on the space M. Thus *M is homeomorphic to* E^4 *or* $E^1 \times S^3$ by the same reason as in § **3.7,** (i).

When $\alpha = r = 0$, it is easily seen that

$$d\,\omega_2 = 2b\,\omega_0 \wedge \omega_1 , \qquad d\,\omega_{01} = 2K\,\omega_0 \wedge \omega_1 .$$

Consequently, if we put $\omega = b\,\omega_{01} - K\,\omega_2$, we have

$$d\,\omega = 0 .$$

Therefore, a subgroup G' of the group G is defined by the equation $\omega = 0$ and the group G' is transitive on M by virtue of $b \neq 0$. Then *M is homeomorphic to* E^4 *or* $E^1 \times S^3$ by the same reason as in § **3.7,** (i). In particular, by more detailed consideration about the structure of G', it is obtained that *M is homeomorphic to* E^4.

3.7, (iii). **The case where** $b = 0$, $\beta \neq 0$. In this case, the same results as in § **3.7,** (ii) are obtained by an analogous process.

3.7, (iv). **The case where** $b = \beta = 0$. By virtue of (3.4), we have

$$d\,\omega_{01} = 2K\,\omega_0 \wedge \omega_1 .$$

Observing (3.5) again, we can easily see that $aK = \alpha K = 0$. Then $K = 0$, if $a \neq 0$ (or $\alpha \neq 0$). That is, we have

$$d\,\omega_{01}=0\,.$$

Therefore, a subgroup G' of the group G is defined by $\omega_{01}=0$, and the group G' is evidently transitive on M. Thus, *if* $a\neq 0$ (*or* $\alpha\neq 0$), M *is homeomorphic to* E^4 *or* $E^1\times S^3$ by the same reason as in § **3.7,** (i). In particular, from more detailed consideration on the structure of G', it follows that M *is homeomorphic to* E^4.

When $a=\alpha=0$, the first system in (2.4) and the curvature forms are reduced to

$$\begin{aligned} &d\,\omega_0=\omega_{01}\wedge\omega_1\,, \qquad d\,\omega_1=\omega_{10}\wedge\omega_0\,, \\ &\qquad\qquad \Omega_{01}=d\,\omega_{01}=2K\,\omega_0\wedge\omega_1\,; \\ &d\,\omega_2=r\,\omega_2\wedge\omega_3\,, \qquad d\,\omega_3=t\,\omega_2\wedge\omega_3\,, \\ &\qquad\qquad \Omega_{23}=(t^2+r^2)\,\omega_2\wedge\omega_3\,, \end{aligned}$$

where all forms Ω_{ij}, except Ω_{01} and Ω_{23}, are equal to zero.

Therefore, the space M is a product of two spaces M_1 and M_2 of two dimensions whose structures are given respectively by the first and the second systems in the above equations.

Now let us consider the spaces M_1 and M_2. When $K<0$, M_1 is $C(+,2)$ and the fundamental group G_1 of M_1 is locally isomorphic to $R(3)$ (Appendix **1**). In the case where $K>0$, M_1 is $C(-,2)$ and G_1 is locally isomorphic to $L(3)$ (Appendix **2**). If $K=0$, M_1 is $C(0,2)$ and G_1 is locally isomorphic to $\mathfrak{M}(2)$ (Appendix **5**). Considering the space M_2, it is $C(-,2)$ or $C(0,2)$, as a consequence of $t^2+r^2\geqq 0$, and its fundamental group G_2 is locally isomorphic to a 2-dimensional vector group A_2 or the group Γ composed of all non-singular matrices of the form $\begin{pmatrix} a & b \\ 0 & 1\end{pmatrix}$ in which a and b are real.

Summing up the results in this step, we see that M *is homeomorphic to* $E^2\times S^2$ *or* E^4, and the group G is locally isomorphic to one of the following groups:

$$R(3)\times A_2\,,\qquad L(3)\times A_2\,,\qquad \mathfrak{M}(2)\times A_2\,,\qquad R(3)\times\Gamma\,,$$
$$L(3)\times\Gamma\,,\qquad \mathfrak{M}(2)\times\Gamma\,.$$

We have here come to an end in CASE (VII), and, consequently, we can say that *M is homeomorphic to*

$$E^4,\ E^2\times S^2 \quad \textit{or} \quad E^1\times S^3$$

in CASE (VII).

3.8. CASE (VIII). In this case the group G is simply transitive on M locally. Since M is simply connected, M is homeomorphic to G. Then, if we know every topological type of simply connected groups with four parameters, our problem is completely solved.

First of all, it is easily seen that there is no semi-simple group with four parameters. If we now denote the Lie algebra of the group G by $\mathfrak{G}$ and the radical of $\mathfrak{G}$ by $\mathfrak{r}$, then it is well known that the factor algebra $\mathfrak{S}=\mathfrak{G}/\mathfrak{r}$ is semi-simple or $\{0\}$ and $\mathfrak{G}$ is the direct sum of $\mathfrak{r}$ and $\mathfrak{S}$ as a vector space. When $\mathfrak{S}\neq\{0\}$, we have $\dim\mathfrak{r}=1$ and $\dim\mathfrak{S}=3$, since $\dim\mathfrak{S}\neq 4$ and $\dim\mathfrak{S}\geqq 3$.

Hence, according to E. Cartan [2], the group G is topologically a product space of E^1 and a semi-simple group H with three parameters which is simply connected, as G is simply connected. On the other hand, if $\dim\mathfrak{S}=3$, the Lie algebra $\mathfrak{S}$ has one of the following structures:

$$d\theta_1=\theta_2\wedge\theta_3, \qquad d\theta_2=\theta_3\wedge\theta_1, \qquad d\theta_3=\theta_1\wedge\theta_2;$$

and

$$d\theta_1=\theta_2\wedge\theta_3, \qquad d\theta_2=\theta_3\wedge\theta_1, \qquad d\theta_3=-\theta_1\wedge\theta_2,$$

where θ_1, θ_2 and θ_3 are suitably chosen relative components of H whose Lie algebra is $\mathfrak{S}$. When $\mathfrak{S}$ has the first structure, it generates a compact group H which is homeomorphic to S^3. If $\mathfrak{S}$ is of the second structure, then it is the Lie algebra of the unimodular group in two variables. On the other hand, the group H is homeomorphic to E^3 in the second case, since the unimodular group in two variables is topologically the product space of E^2 and S^1 ([5], p. 14).

From these considerations, we see that the group G is homeomorphic to E^4 or $E^1\times S^3$ when G is not solvable. On the other hand, E. Cartan [2] has proved that the group G is homeomorphic to E^4, if it is solvable and simply connected.

Summing up the above results, the group G and also *the space M is homeomorphic to E^4 or $E^1\times S^3$.*

3.9. Conclusion. In the preceding paragraphs § **3.1**–**3.8**, we have obtained several results. From these results, the following Theorems 3 and 4 are established, and the former is the main result in this paper.

THEOREM 3. *The homogeneous Riemannian space of four dimensions, which is connected and simply connected, is homeomorphic to one of the following manifolds:*

$$E^4, \quad S^4, \quad P(C,2), \quad S^2\times S^2, \quad E^1\times S^3, \quad E^2\times S^2.$$

THEOREM 4. *Let $G/g=M$ be a homogeneous Riemannian space of four dimensions. Then the Riemannian space M and the local structure of the group G are completely determined, if $r=\dim G\geqq 6$, as follows:*

If $r=10$, M is of constant sectional curvature and

$$G\cong R(5), \quad L(5) \quad or \quad \mathfrak{M}(4).$$

If $r=8$, M is a 2-dimensional Kählerian space with constant holomorphic sectional curvature and

$$G\cong SU(3), \quad S\mathfrak{L}(3) \quad or \quad G=\mathfrak{M}_H(2).$$

If $r=7$, M is the product space of a straight line and a 3-dimensional Riemannian space with constant sectional curvature, or M is a 4-dimensional Riemannian space with constant negative sectional curvature, and in this case

$$G\cong A_1\times R(4), \quad A_1\times L(4), \quad A_1\times\mathfrak{M}(3)$$

or a subgroup of $L(5)$ (§ **3.3**),

or $G=S\mathfrak{M}_H(2)$. Here, A_1 is a 1-dimensional vector group.

If $r=6$, M is a product space of two Riemannian spaces of two dimensions each of which is of constant curvature and

$$G\cong R(3)\times R(3), \quad R(3)\times L(3), \quad R(3)\times\mathfrak{M}(2),$$
$$L(3)\times L(3), \quad L(3)\times\mathfrak{M}(2) \quad or \quad \mathfrak{M}(2)\times\mathfrak{M}(2).$$

Here, $\cong$ means local isomorphism.

§ 4. Some remarks concerning Yano's theorems.

Recently, K. Yano [10] has proved the following theorems.

THEOREM A. *In an n-dimensional Riemannian space, for $n \neq 4$, there exists no group of motions of order r such that*

$$n(n+1)/2 > r > n(n-1)/2+1 .$$

THEOREM B. *If an n-dimensional Riemannian space, for $n \neq 4$, admits a group of motions of order $n(n-1)/2+1$, then the group is transitive.*

THEOREM C. *A necessary and sufficient condition that an n-dimensional Riemannian space V^n for $n>4$, $n \neq 8$ admits a group of motions of order $r=n(n-1)/2+1$ is that the space be the product of a straight line and an $(n-1)$-dimensional Riemannian space of constant curvature or that the space be of negative constant curvature.*

For the exceptional case of Theorem A, the following theorem holds good.

THEOREM A′. *In any 4-dimensional Riemannian space there exists no group of motions of order 9. If a 4-dimensional Riemannian space admits a group of motions of order 8, then the group of motions is transitive and the space is a Kählerian space whose holomorphic sectional curvature is constant.*

The first part of this theorem is derived from the fact that $R(4)$ has no subgroup of order 5. The second part is a consequence of the discussions given in § **3.2**, CASE (II). It is evident that the converse of the second part holds good locally.

It is easily seen that Theorem B holds good also for $n=4$, and, according to CASE (III) and (II), we have Theorem C for $n=4$.

REMARK. The exceptional case in Theorem C for $n=8$ has been recently removed by M. Obata [8].

§ 5. The case where G is compact.

We consider the case where G is compact in this section. The homogeneous Riemannian space $M=G/g$ has one of the structures given in the following Theorem 5, when the group G is compact. For brevity's sake, we shall introduce some notations.

$K(+,n)$ and $K(-,n)$ are Kählerian spaces of n complex dimensions whose holomorphic sectional curvatures are positive and negative constants respectively.

V^1 is a straight line with its natural Riemannian metric.

A_r is a vector group of r dimensions.

T^r is a toroidal group of r dimensions.

In this section the local isomorphism between two groups H and K is denoted by $H \cong K$.

THEOREM 5. *Let $M = G/g$ be a homogeneous Riemannian space of four dimensions whose fundamental group G is compact. Then G and M have one of the following structures:*

If $\dim G = 10$, $G \cong R(5)$ *and M is* $C(+, 4)$.

If $\dim G = 8$, $G \cong SU(3)$ *and M is* $K(+, 2)$.

If $\dim G = 7$, $G \cong A_1 \times R(4)$ *and M is the product of V^1 and* $C(+, 3)$ *locally.*

If $\dim G = 6$, $G \cong R(3) \times R(3)$ *and M is locally the product of* $C(+, 2)$ *by itself.*

If $\dim G = 5$, $G \cong A_2 \times R(3)$. *Then M is the product of* $C(0, 2)$ *and* $C(+, 2)$ *locally.*

If $\dim G = 4$, *then* $G = T^4$ *and M is flat; or* $G \cong A_1 \times R(3)$ *and M is the product of V^1 and* $C(+, 3)$ *locally.*

There is no group G of dimension 9.

The following lemma is required to prove Theorem 5 [5].

LEMMA. *Let us suppose that G is a connected Lie group and there is a connected Lie subgroup H of G which is solvable and whose closure $\overline{H}$ is equal to G. Then the group G is solvable also.*

PROOF of Theorem 5. Theorem 5 for $\dim G \geqq 6$ is obvious as a consequence of § **3.1**–**3.5**. Thus we shall develop here the proof for $\dim G = 5$ and for $\dim G = 4$.

When $\dim G = 5$, CASE (VI) in § **3.6** and CASE (VII) in § **3.7** must be discussed. In CASE (VI), G is locally isomorphic to the subgroup of $\mathfrak{M}(4)$ given in § **3.6**, and hence CASE (VI) does not occur for compact G.

In CASE (VII), there is a subgroup G' of G which is transitive on M and whose dimension is equal to 4, if it does not happen that

$$a = b = \alpha = \beta = 0.$$

For a, b, α, $\beta \neq 0$, G' is solvable but not Abelian. Thus G' is not closed in G, since a solvable and compact group is Abelian. Since $\dim G' = \dim G + 1$, then $\overline{G'} = G$. Hence G itself is solvable by virtue of the above lemma. Consequently, G is Abelian, since G is solvable and compact. This contradicts the fact that G is effective on M. Thus this case does not occur for compact G.

For $b, \beta \neq 0$ and $a=\alpha=0$, G is locally the product of A_2 and a semi-simple group of three dimensions by virtue of its structure, if $K \neq b^2+\beta^2$. Then there is a covering group $\tilde{G}$ of G which is the product of T^2 and a compact semi-simple group of three dimensions, since G is compact. Hence $G \cong A_2 \times R(3)$, and consequently $K > b^2+\beta^2$. Moreover, M is locally the product of $C(0,2)$ and $C(+,2)$ by virtue of the fact that $K > b^2+\beta^2$. When $K=b^2+\beta^2$, G is solvable but not Abelian. Then G must be non-compact, if $K=b^2+\beta^2$.

For $b \neq 0$ and $a=\alpha=\beta=0$, we have the same results as above.

For $a, b, \alpha \neq 0$ and $\beta=0$, G' is solvable but not Abelian. Then G is not compact in this case.

Finally, for $a=b=\alpha=\beta=0$, $G \cong A_2 \times R(3)$ is easily obtained from the fact that G is compact. Then M is locally the product of $C(0,2)$ and $C(+,2)$ by virtue of the structure of the Riemannian space M.

Here, we have the required theorem for $\dim G=5$, summing up these results.

When G is compact and of dimension 4, G is Abelian or $G \cong A_1 \times R(3)$. Therefore, M is flat or locally the product of V^1 and $C(+,3)$. Hence we have the required results for $\dim G=4$. Thus Theorem 5 is proved completely.

REMARK. By definition, a homogeneous Riemannian space $M=G/g$ is called *locally symmetric*, if there is an involutive antomorphism σ of the Lie algebra $\mathfrak{G}$ of G and the subalgebra $\mathfrak{G}'$ of $\mathfrak{G}$ corresponding to g is the one composed of all elements invariant by σ. It is easily seen from Theorem 4 that *any homogeneous Riemannian space $M=G/g$ of four dimensions is locally symmetric, if* $\dim G$ *is greater than* 5.

§ 6. Homogeneous Kählerian spaces of two complex dimensions.

A homogeneous Riemannian space $M=G/g$ of even dimensions is called *a homogeneous Kählerian space* by definition, if M is Kählerian and the Kählerian structure of M is invariant under the fundamental group G. Then we have the following theorem from Theorem 5.

THEOREM 5'. *Let $M=G/g$ be a homogeneous Kählerian space of two complex dimensions and G be compact. Then the group G and the* *space M have one of the following structures.*

If $\dim G=8$, $G\cong SU(3)$ *and M is* $K(+,2)$.

If $\dim G=6$, $G\cong SU(2)\times SU(2)$ *and M is locally the product of* $K(+,1)$ *by itself.*

If $\dim G=5$, $G\cong A_2\times SU(2)$ *and M is the product of* $C(0,2)$ *and* $K(+,1)$ *locally.*

If $\dim G=4$, $G=T^4$ *and M is locally unitary.*

There is no homogeneous Kählerian space G/g of two complex dimensions whose fundamental group G is of dimension 7.

Here, $\cong$ means the local isomorphism.

It is easily seen from Theorem 5′ that *any homogeneous Kählerian space of two complex dimensions is locally symmetric, if G is compact and* $\dim G\geqq 5$. This has been already given by A. Lichnerowicz [6].

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Appendix

The structure of the familiar linear groups, which are used in the present paper, is as follows. Indices run over the following ranges:

$$i, j, k, \cdots =1, 2, \cdots, n\,;$$
$$\alpha, \beta, \gamma, \cdots =1, 2, \cdots, n-1\,.$$

1. $R(n)$. Let θ_{ij} be the relative components of $R(n)$ as usual. Then we have

$$d\,\theta_{ij}=\sum_k \theta_{ik}\wedge\theta_{kj}\,, \qquad \theta_{ij}+\theta_{ji}=0\,,$$

Putting $\theta_\alpha=\theta_{\alpha n}$, we have

$$d\,\theta_{\alpha\beta}=\sum_\gamma \theta_{\alpha\gamma}\wedge\theta_{\gamma\beta}-\theta_\alpha\wedge\theta_\beta\,,$$
$$d\,\theta_\alpha=\sum_\gamma \theta_{\alpha\gamma}\wedge\theta_\gamma\,, \qquad \theta_{\alpha\beta}+\theta_{\beta\alpha}=0\,.$$

These equations are those of structure of an $(n-1)$*-dimensional Riemannian space with constant positive sectional curvature.*

2. $L(n)$. Let θ_{ij} be the relative components of $L(n)$ as usual. Then we have

$$d\theta_{ij}=\sum_k \theta_{ik}\wedge\theta_{kj},$$

$$\theta_{\alpha\beta}+\theta_{\beta\alpha}=0, \qquad \theta_{\alpha n}-\theta_{n\alpha}=0, \qquad \theta_{nn}=0.$$

If we put $\theta_\alpha=\theta_{\alpha n}$, it follows that

$$d\theta_{\alpha\beta}=\sum_\gamma \theta_{\alpha\gamma}\wedge\theta_{\gamma\beta}+\theta_\alpha\wedge\theta_\beta,$$

$$d\theta_\alpha=\sum_\gamma \theta_{\alpha\gamma}\wedge\theta_\gamma, \qquad \theta_{\alpha\beta}+\theta_{\beta\alpha}=0.$$

These equations are those of structure of an $(n-1)$*-dimensional Riemannian space with constant negative sectional curvature.*

3. $U(n)$ **and** $SU(n)$. Let θ_{ij} be the complex relative components of $U(n)$ as usual. Then we obtain

$$d\theta_{ij}=\sum_k \theta_{ik}\wedge\theta_{kj}, \qquad \theta_{ij}+\bar\theta_{ji}=0,$$

where $\bar\theta_{ij}$ denotes the complex conjugate of θ_{ij}.

Putting $\pi_\alpha=\theta_{\alpha n}$, $\pi_{\alpha\beta}=\theta_{\alpha\beta}\ (\alpha\neq\beta)$ and $\pi_{\alpha\alpha}=\theta_{\alpha\alpha}-\theta_{nn}$, it is easily seen that

$$d\pi_{\alpha\beta}=\sum_\gamma \pi_{\alpha\gamma}\wedge\pi_{\gamma\beta}-\pi_\alpha\wedge\bar\pi_\beta \qquad (\alpha\neq\beta),$$

$$d\pi_{\alpha\alpha}=\sum_\gamma \pi_{\alpha\gamma}\wedge\pi_{\gamma\alpha}-(\pi_\alpha\wedge\bar\pi_\alpha+\sum_\gamma \pi_\gamma\wedge\bar\pi_\gamma),$$

$$d\pi_\alpha=\sum_\gamma \pi_{\alpha\gamma}\wedge\pi_\gamma,$$

$$\pi_{\alpha\beta}+\bar\pi_{\beta\alpha}=0.$$

For $SU(n)$, the relation $\sum_k \theta_{kk}=0$ must be added.

These equations for $SU(n)$ *are those of structure of an* $(n-1)$*-dimensional Kählerian space with constant positive holomorphic sectional curvature.*

4. $\mathfrak{L}(n)$ **and** $S\mathfrak{L}(n)$. Let θ_{ij} be the relative components of $\mathfrak{L}(n)$ as usual. Then

$$d\theta_{ij}=\sum_k \theta_{ik}\wedge\theta_{kj},$$

$$\theta_{\alpha\beta}+\bar\theta_{\beta\alpha}=0, \qquad \theta_{\alpha n}-\bar\theta_{n\alpha}=0, \qquad \theta_{nn}+\bar\theta_{nn}=0.$$

If we put $\pi_\alpha = \theta_{\alpha n}$, $\pi_{\alpha\beta} = \theta_{\alpha\beta}$ $(\alpha \neq \beta)$ and $\pi_{\alpha\alpha} = \theta_{\alpha\alpha} - \theta_{nn}$, then it follows that

$$d\,\pi_{\alpha\beta} = \sum_\gamma \pi_{\alpha\gamma} \wedge \pi_{\gamma\beta} + \pi_\alpha \wedge \bar{\pi}_\beta \qquad (\alpha \neq \beta)\,,$$

$$d\,\pi_{\alpha\alpha} = \sum_\gamma \pi_{\alpha\gamma} \wedge \pi_{\gamma\alpha} + (\pi_\alpha \wedge \bar{\pi}_\alpha + \sum_\gamma \pi_\gamma \wedge \bar{\pi}_\gamma)\,,$$

$$d\,\pi_\alpha = \sum_\gamma \pi_{\alpha\gamma} \wedge \pi_\gamma\,, \qquad \pi_{\alpha\beta} + \bar{\pi}_{\beta\alpha} = 0\,.$$

For $S\mathfrak{L}(n)$, the condition $\sum_k \theta_{kk} = 0$ must be added.

These equations for $S\mathfrak{L}(n)$ *are those of structure of an* $(n-1)$-*dimensional Kählerian space with constant negative holomorphic sectional curvature.*

5. $\mathfrak{M}(n)$. Let θ_i and θ_{ij} be the relative components of $\mathfrak{M}(n)$ as usual. Then we have

$$d\,\theta_{ij} = \sum_k \theta_{ik} \wedge \theta_{kj}\,, \qquad d\,\theta_i = \sum_k \theta_{ik} \wedge \theta_k\,.$$

These equations are those of structure of a flat Riemannian space of n dimensions.

6. $\mathfrak{M}_H(n)$ **and** $S\mathfrak{M}_H(n)$. Let π_i and π_{ij} be the relative components of $\mathfrak{M}_H(n)$ as usual. Then we have

$$d\,\pi_{ij} = \sum_k \pi_{ik} \wedge \pi_{kj}\,, \qquad \pi_{ij} + \bar{\pi}_{ji} = 0\,,$$

$$d\,\pi_i = \sum_k \pi_{ik} \wedge \pi_k\,.$$

These equations are those of structure of an n-dimensional Kählerian space which is locally unitary.

For $S\mathfrak{M}_H(n)$, the conditions $\sum_k \pi_{kk} = 0$ must be added.

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