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The predicate calculus with ε-symbol.

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The purpose of this paper is to prove the following theorem:

"If a formal axiom system represented by formulae in the ordinary predicate calculus is consistent in the ordinary predicate calculus, it is consistent also in the predicate calculus with ε-symbol."[1]

By the ordinary predicate calculus we mean here Gentzen's 'Kalkül LK[2]', and what we call here 'ε-symbol' means the logical symbol 'ε' used in representing the quantifier 'εx' which was originally proposed by Hilbert and named 'transfinite logische Auswahlfunktion'. When $F(x)$ represents a proposition containing the variable x for an individual, as long as there exists at least such an x as makes $F(x)$ true, $\varepsilon x F(x)$ indicates such an x as makes $F(x)$ true. And if there exists no x such as makes $F(x)$ true, $\varepsilon x F(x)$ means an arbitrary individual.[3]

For obtaining the predicate calculus with ε-symbol (of first order), it is sufficient, as is well-known, to adjoin the logical axiom schema

$$F(a) \rightarrow F(\varepsilon x F(x))$$

and appropriate rules of inference to the propositional calculus. But, for the sake of convenience, we now use as the predicate calculus with ε-symbol the logical system obtained from the ordinary predicate calculus[4] by adjoining the above logical axiom schema to it.

In an Appendix, we shall consider the ε-symbol on propositions.

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§ 1. Terminologies and symbols.

1.1. 'Term' and 'formula'.

1.11. DEFINITION:

1.111. A free variable is a *term*.

1.112. If $\mathfrak{t}_1,\ldots,\mathfrak{t}_n$ are *terms*, and $\mathfrak{f}(*,\ldots,*)$ is a function of n argument places[5], then $\mathfrak{f}(\mathfrak{t}_1,\dots,\mathfrak{t}_n)$ is a *term* ($n=0, 1, 2,\dots$).

1.113. If $\mathfrak{t}_1,\dots,\mathfrak{t}_n$ are *terms*, and $\mathfrak{P}(*,\dots,*)$ is a predicate of n argument places[6], then $\mathfrak{P}(\mathfrak{t}_1,\dots,\mathfrak{t}_n)$ is a *formula* ($n=0, 1, 2,\dots$).

1.114. If $\mathfrak{A}$ is a *formula*, then $\neg\mathfrak{A}$ is a *formula*. If $\mathfrak{A}$ and $\mathfrak{B}$ are *formulae*, then $\mathfrak{A}\,\&\,\mathfrak{B}$, $\mathfrak{A}\vee\mathfrak{B}$ and $\mathfrak{A}\supset\mathfrak{B}$ are *formulae*.

1.115. If $\mathfrak{F}(\mathfrak{a})$ is a *formula*, $\mathfrak{a}$ is a free variable, $\mathfrak{x}$ is a bound variable, and $\mathfrak{F}(\mathfrak{x})$ is the result of substituting $\mathfrak{x}$ for $\mathfrak{a}$ throughout $\mathfrak{F}(\mathfrak{a})$, then $V\mathfrak{x}\mathfrak{F}(\mathfrak{x})$ and $\exists\mathfrak{x}\mathfrak{F}(\mathfrak{x})$ are *formulae*, and $\varepsilon\mathfrak{x}\mathfrak{F}(\mathfrak{x})$ is a *term*.

1.116. The only *terms* and *formulae* are those given by 1.111–1.115.

The ***grade*** of a term or formula is the number ($\geqq 0$) of occurrences of logical symbols ($\neg$, &, $\vee$, $\supset$, V, $\exists$, ε) in the term or formula.

1.12. *Abbreviations.*

1.121. When $V\mathfrak{x}\mathfrak{F}(\mathfrak{x})$ or $\exists\mathfrak{x}\mathfrak{F}(\mathfrak{x})$ is a formula, or $\varepsilon\mathfrak{x}\mathfrak{F}(\mathfrak{x})$ is a term of such form, $\mathfrak{F}(\mathfrak{t})$ means the result of substituting an arbitrary term $\mathfrak{t}$ for all occurrences[7] of the bound variable $\mathfrak{x}$ in $\mathfrak{F}(\mathfrak{x})$ which are contained in none of the scopes of the quantifiers $V\mathfrak{x}$, $\exists\mathfrak{x}$ and $\varepsilon\mathfrak{x}$.

1.122. We shall use Greek small letters to stand for finite sequences of zero or more terms or bound variables, when we wish to indicate sets of terms or bound variables without naming them individually, as in the following

EXAMPLE 1. If

$$\tau \text{ stands for a sequence of terms: } \mathfrak{t}_1,\dots,\mathfrak{t}_n,$$

and

$$\xi \text{ stands for a sequence of bound variables: } \mathfrak{x}_1,\dots,\mathfrak{x}_n,$$

then

$$\mathfrak{F}(\tau) \text{ means } \mathfrak{F}(\mathfrak{t}_1,\dots,\mathfrak{t}_n),$$

and

$$V\xi\mathfrak{F}(\xi) \text{ means } V\mathfrak{x}_1\dots V\mathfrak{x}_n\mathfrak{F}(\mathfrak{x}_1,\dots,\mathfrak{x}_n)$$

($n=0, 1, 2,\dots$).

EXAMPLE 2. If σ and τ stand for sequences of terms $\mathfrak{s}_1,\dots,\mathfrak{s}_n$ or $\mathfrak{t}_1,\dots,\mathfrak{t}_n$, respectively, then $\sigma=\tau$ means the formula

$$((\mathfrak{s}_1=\mathfrak{t}_1\ \&\ \mathfrak{s}_2=\mathfrak{t}_2)\ \&\dots)\ \&\ \mathfrak{s}_n=\mathfrak{t}_n$$

($n=1, 2,\dots$).

1.13. We define 'subterm' of a given term or formula thus:

1.131. If $\mathfrak{t}$ is a term, $\mathfrak{t}$ is a *subterm* of $\mathfrak{t}$.

1.132. If $\mathfrak{t}_1,\ldots,\mathfrak{t}_n$ are terms, and $\mathfrak{f}(*,\ldots,*)$ is a function of n argument places, the *subterms* of $\mathfrak{t}_i\,(i=1,2,\ldots,n)$ are *subterms* of $\mathfrak{f}(\mathfrak{t}_1,\ldots,\mathfrak{t}_n)$.

1.133. If $\mathfrak{t}_1,\ldots,\mathfrak{t}_n$ are terms, and $\mathfrak{P}(*,\ldots,*)$ is a predicate of n argument places, the *subterms* of $\mathfrak{t}_i\ (i=1,2,\ldots,n)$ are *subterms* of $\mathfrak{P}(\mathfrak{t}_1,\ldots,\mathfrak{t}_n)$.

1.134. If $\mathfrak{A}$ is a formula, the *subterms* of $\mathfrak{A}$ are *subterms* of $\neg\mathfrak{A}$. If $\mathfrak{A}$ and $\mathfrak{B}$ are formulae, the *subterms* of $\mathfrak{A}$ and the *subterms* of $\mathfrak{B}$ are *subterms* of $\mathfrak{A}\,\&\,\mathfrak{B}$, $\mathfrak{A}\vee\mathfrak{B}$ and $\mathfrak{A}\supset\mathfrak{B}$.

1.135. If $\mathfrak{F}(\mathfrak{a})$ is a formula, $\mathfrak{a}$ is a free variable, $\mathfrak{x}$ is a bound variable, and $\mathfrak{F}(\mathfrak{x})$ is the result of substituting $\mathfrak{x}$ for $\mathfrak{a}$ throughout $\mathfrak{F}(\mathfrak{a})$, the *subterms* of $\mathfrak{F}(\mathfrak{a})$ which do not contain $\mathfrak{a}$ are *subterms* of $\forall\mathfrak{x}\mathfrak{F}(\mathfrak{x})$, $\exists\mathfrak{x}\mathfrak{F}(\mathfrak{x})$ and $\varepsilon\mathfrak{x}\mathfrak{F}(\mathfrak{x})$.[8]

1.136. A term or formula has only the *subterms* required by 1.131–1.135.

A subterm of a term $\mathfrak{t}$ which is not identical with $\mathfrak{t}$ itself is called a *proper* subterm of $\mathfrak{t}$.

1.14. A term of the form $\varepsilon\mathfrak{x}\mathfrak{F}(\mathfrak{x})$ is called an *ε-term.* When and only when all of the proper subterms of an ε-term are free variables each of which occurs only once in the ε-term, the ε-term is called an *ε-type,* and the free variables contained in it are called the *arguments* of the ε-type. When and only when $\varepsilon\mathfrak{x}\mathfrak{F}(\mathfrak{x},\alpha)$ is an ε-type, and α indicates the arguments, the ε-type is called a *type of* any ε-term of the form $\varepsilon\mathfrak{x}\mathfrak{F}(\mathfrak{x},\tau)$, where τ stands for a sequence of arbitrary terms. Two ε-types are said to be *congruent* with each other, when and only when the each of them is a type of the other.

1.141. For an arbitrary ε-term, there exists a type of it.

1.142. Types of an ε-term are all congruent with each other.

1.143. A type of a type of an ε-term is also a type of the ε-term.

1.2. ‘Sequent’.

A *sequent* is a formal expression of the form

$$\mathfrak{A}_1,\ldots,\mathfrak{A}_m\rightarrow\mathfrak{B}_1,\ldots,\mathfrak{B}_n$$

where $m,n\geqq 0$ and $\mathfrak{A}_1,\ldots,\mathfrak{A}_m$, $\mathfrak{B}_1,\ldots,\mathfrak{B}_n$ are arbitrary formulae. The part $\mathfrak{A}_1,\ldots,\mathfrak{A}_m$ is called the *antecedent,* and $\mathfrak{B}_1,\ldots,\mathfrak{B}_n$ the *succedent* of the sequent.

We shall use Greek capitals, such as Γ, Δ, Θ, Λ, and so on, to stand for finite sequences of zero or more formulae with separating formal commas included, when we wish to indicate sequences of formulae in antecedent or succedent without naming the formulae individually.

1.3. 'Logical axiom'.

As *logical axioms* we use only the sequents of the form

$$\mathfrak{F}(\mathfrak{t}) \rightarrow \mathfrak{F}(\varepsilon\mathfrak{x}\mathfrak{F}(\mathfrak{x}))$$

where $\varepsilon\mathfrak{x}\mathfrak{F}(\mathfrak{x})$ is an arbitrary ε-term, $\mathfrak{t}$ is an arbitrary term, and $\mathfrak{F}(\mathfrak{t})$ and $\mathfrak{F}(\varepsilon\mathfrak{x}\mathfrak{F}(\mathfrak{x}))$ are the results of substituting $\mathfrak{t}$ or $\varepsilon\mathfrak{x}\mathfrak{F}(\mathfrak{x})$ for all free occurrences of the bound variable $\mathfrak{x}$ in $\mathfrak{F}(\mathfrak{x})$, respectively. Especially, when $\mathfrak{F}(\mathfrak{x})$ has no free occurrence of $\mathfrak{x}$, that logical axiom has the form

$$\mathfrak{D} \rightarrow \mathfrak{D}$$

like that of the 'logische Grundsequenz' in Gentzen's Kalkül*L K*.[9]

1.4. 'Rules of inference'.

As *rules of inference* we use ones for *LK* which are represented as the 'Schlussfiguren-Schemata' in Gentzen [1], thus:

1.41. Structural rules of inference:

	in antecedent.	in succedent.
Thinning:	$\dfrac{\Gamma \rightarrow \Theta}{\mathfrak{D}, \Gamma \rightarrow \Theta}$	$\dfrac{\Gamma \rightarrow \Theta}{\Gamma \rightarrow \Theta, \mathfrak{D}}$
Contraction:	$\dfrac{\mathfrak{D}, \mathfrak{D}, \Gamma \rightarrow \Theta}{\mathfrak{D}, \Gamma \rightarrow \Theta}$	$\dfrac{\Gamma \rightarrow \Theta, \mathfrak{D}, \mathfrak{D}}{\Gamma \rightarrow \Theta, \mathfrak{D}}$
Interchange:	$\dfrac{\Delta, \mathfrak{D}, \mathfrak{E}, \Gamma \rightarrow \Theta}{\Delta, \mathfrak{E}, \mathfrak{D}, \Gamma \rightarrow \Theta}$	$\dfrac{\Gamma \rightarrow \Theta, \mathfrak{E}, \mathfrak{D}, \Lambda}{\Gamma \rightarrow \Theta, \mathfrak{D}, \mathfrak{E}, \Lambda}$

Cut:

$$\frac{\Gamma \rightarrow \Theta, \mathfrak{D} \quad \mathfrak{D}, \Delta \rightarrow \Lambda}{\Gamma, \Delta \rightarrow \Theta, \Lambda}$$

1.42. Logical rules of inference:

Introduction of	in antecedent.	in succedent.
$\neg$:	$\dfrac{\Gamma \to \Theta, \mathfrak{A}}{\neg \mathfrak{A}, \Gamma \to \Theta}$	$\dfrac{\mathfrak{A}, \Gamma \to \Theta}{\Gamma \to \Theta, \neg \mathfrak{A}}$
& :	$\dfrac{\mathfrak{A}, \Gamma \to \Theta}{\mathfrak{A} \,\&\, \mathfrak{B}, \Gamma \to \Theta}$ $\quad$ $\dfrac{\mathfrak{B}, \Gamma \to \Theta}{\mathfrak{A} \,\&\, \mathfrak{B}, \Gamma \to \Theta}$	$\dfrac{\Gamma \to \Theta, \mathfrak{A} \quad \Gamma \to \Theta, \mathfrak{B}}{\Gamma \to \Theta, \mathfrak{A} \,\&\, \mathfrak{B}}$
$\vee$:	$\dfrac{\mathfrak{A}, \Gamma \to \Theta \quad \mathfrak{B}, \Gamma \to \Theta}{\mathfrak{A} \vee \mathfrak{B}, \Gamma \to \Theta}$	$\dfrac{\Gamma \to \Theta, \mathfrak{A}}{\Gamma \to \Theta, \mathfrak{A} \vee \mathfrak{B}}$ $\quad$ $\dfrac{\Gamma \to \Theta, \mathfrak{B}}{\Gamma \to \Theta, \mathfrak{A} \vee \mathfrak{B}}$
$\supset$:	$\dfrac{\Gamma \to \Theta, \mathfrak{A} \quad \mathfrak{B}, \Gamma \to \Theta}{\mathfrak{A} \supset \mathfrak{B}, \Gamma \to \Theta}$	$\dfrac{\mathfrak{A}, \Gamma \to \Theta, \mathfrak{B}}{\Gamma \to \Theta, \mathfrak{A} \supset \mathfrak{B}}$
$\forall$:	$\dfrac{\mathfrak{F}(\mathfrak{t}), \Gamma \to \Theta}{\forall \mathfrak{x} \mathfrak{F}(\mathfrak{x}), \Gamma \to \Theta}$	$\dfrac{\Gamma \to \Theta, \mathfrak{F}(\mathfrak{a})}{\Gamma \to \Theta, \forall \mathfrak{x} \mathfrak{F}(\mathfrak{x})}$
$\exists$:	$\dfrac{\mathfrak{F}(\mathfrak{a}), \Gamma \to \Theta}{\exists \mathfrak{x} \mathfrak{F}(\mathfrak{x}), \Gamma \to \Theta}$	$\dfrac{\Gamma \to \Theta, \mathfrak{F}(\mathfrak{t})}{\Gamma \to \Theta, \exists \mathfrak{x} \mathfrak{F}(\mathfrak{x})}$

where $\mathfrak{A}, \mathfrak{B}, \mathfrak{D}, \mathfrak{E}$ are arbitrary formulae; $\forall \mathfrak{x} \mathfrak{F}(\mathfrak{x})$ or $\exists \mathfrak{x} \mathfrak{F}(\mathfrak{x})$ is an arbitrary formula of such form, and at this time $\mathfrak{F}(\mathfrak{a})$ or $\mathfrak{F}(\mathfrak{t})$ is the result of substituting a free variable $\mathfrak{a}$ or a term $\mathfrak{t}$ for all free occurrences of the bound variable $\mathfrak{x}$ in $\mathfrak{F}(\mathfrak{x})$, respectively; and Γ, Δ, Θ, Λ are arbitrary finite sequences of zero or more formulae.

Restriction on variable: The free variable $\mathfrak{a}$ of the $\forall$-succedent[10] or the $\exists$-antecedent[10] shall not occur in its conclusion.

In each of the logical rules, the formula in which the logical symbol is introduced is called the *principal formula,* and the one or two formulae shown explicitly in the premises the *side formulae.*

1.5. 'Proof'.

As the (*formal*) *proof,* we use one *in tree form*: it is a finite occurrences of one or more sequents in a partial ordering, which has one lowermost sequent—the *endsequent*—and logical axioms as the uppermost sequents, and in which the premises for each inference are written immediately over the conclusion, as in the statement of the rules of inference, and no occurrence of a sequent serves as premise for more than one inference.

A *proof of* a sequent is a proof which has the sequent as the endsequent. A sequent is said to be *provable* (*without cut*), when and only when there exists a proof of the sequent (which contains no cut).

1.6. 'Axiom system'.

A formula containing no free variable is said to be *closed.* A finite or infinite set of closed formulae is called an *axiom system.* When an axiom system is fixed to be considered, each closed formula of the system may be called an *axiom.* A finite sequence of formulae containing only axioms of an axiom system $\boldsymbol{A}$ is called an *axiom sequence* of $\boldsymbol{A}$.

1.61. Let $\boldsymbol{A}$ be an axiom system, and $\mathfrak{A}$ be a formula. When and only when there exists an axiom sequence Γ_0—it may be empty—of $\boldsymbol{A}$, and the sequent

$$\Gamma_0 \rightarrow \mathfrak{A}$$

is provable, $\mathfrak{A}$ is said to be *provable from* $\boldsymbol{A}$.

1.62. An axiom system $\boldsymbol{A}$ is said to be *contradictory*, when and only when there exists a formula $\mathfrak{A}$ and when $\mathfrak{A}$ and $\neg\mathfrak{A}$ are both provable from $\boldsymbol{A}$.

If an axiom system $\boldsymbol{A}$ is contradictory, there exist a formula $\mathfrak{A}$ and axiom sequences Γ_1 and Γ_2 of $\boldsymbol{A}$ and the sequents

$$\Gamma_1 \rightarrow \mathfrak{A} \quad \text{and} \quad \Gamma_2 \rightarrow \neg\mathfrak{A}$$

are both provable, accordingly

$$\Gamma_2 \rightarrow \neg\mathfrak{A} \quad \text{and} \quad \neg\mathfrak{A}, \Gamma_1 \rightarrow$$

are both provable (by a $\neg$-antecedent), hence

$$\Gamma_2, \Gamma_1 \rightarrow$$

is provable (by a cut), and the sequence Γ_2, Γ_1 is an axiom sequence of $\boldsymbol{A}$. Conversely, if there exists an axiom sequence Γ_0 of an axiom system $\boldsymbol{A}$ and if

$$\Gamma_0 \rightarrow$$

is provable, for any formula $\mathfrak{A}$,

$$\Gamma_0 \rightarrow \mathfrak{A} \quad \text{and} \quad \Gamma_0 \rightarrow \neg\mathfrak{A}$$

are both provable (by thinnings), then $\mathfrak{A}$ and $\neg\mathfrak{A}$ are both provable from $\boldsymbol{A}$, hence $\boldsymbol{A}$ is contradictory.

An axiom system which is not contradictory is said to be *consistent.*

1.63. Let $\boldsymbol{A}$ be an axiom system, and $\mathfrak{A}$ be a fomula. When and only when each of the functions and the predicates which occur in $\mathfrak{A}$ occurs at least in one axiom of $\boldsymbol{A}$, $\mathfrak{A}$ is said to be *dependent on* $\boldsymbol{A}$.

1.64. Let $\boldsymbol{A}$ be an axiom system which has at least one axiom containing the 2-place predicate '$*=*$'. When and only when each formula which is dependent on $\boldsymbol{A}$ and has the form

$$V\mathfrak{x}(\mathfrak{x}=\mathfrak{x})$$

or

$$V\mathfrak{x}V\mathfrak{y}\,[\mathfrak{x}=\mathfrak{y}\supset(\mathfrak{F}(\mathfrak{x})\supset\mathfrak{F}(\mathfrak{y}))\,]$$

is provable from $\boldsymbol{A}$, $\boldsymbol{A}$ is called an *axiom system with equality.*

1.7. THE PREDICATE CALCULUS '*LK*'.

Gentzen's predicate calculus *LK* is obtained from the present logical system by omitting the logical symbol ε. When we wish to express that a conception is in *LK*, we shall use the corresponding terminology to it with the phrase 'in *LK*' or the prefix '*LK*-', as in the following

EXAMPLE 3. An *LK*-formula is a formula which does not contain the logical symbol ε.

EXAMPLE 4. A logical axiom in *LK* has the form $\mathfrak{D}\rightarrow\mathfrak{D}$, where $\mathfrak{D}$ is an arbitrary *LK*-formula.

EXAMPLE 5. A sequent is *LK*-provable, if and only if there exists an *LK*-proof of the sequent.

EXAMPLE 6. An axiom system in *LK* with equality is an axiom system with equality in *LK*.

Gentzen [1] proved the following important theorem:

HAUPTSATZ ON *LK*. *If a sequent is LK-provable, then it is LK-provable without cut.*

§ 2. Preparative considerations on *LK*.

This paragraph is devoted to the proof of two following theorems:

THEOREM 1. *Let A be an axiom system in LK,*

$$\forall \xi \exists \mathfrak{y} \mathfrak{A}(\mathfrak{y}, \xi)$$

*be a closed LK-formula which is LK-provable from **A**, $\mathfrak{f}$ be a function symbol contained in none of the axioms of **A** nor $\mathfrak{A}(\mathfrak{y}, \xi)$, and **A**$'$ be the axiom system obtained from **A** by adjoining the axiom*

$$\forall \xi \mathfrak{A}(\mathfrak{f}(\xi), \xi).$$

*If **A** is LK-consistent, then **A**$'$ is also LK-consistent.*

THEOREM 2. *Let **A** be an axiom system with equality in LK,*

$$\forall \xi \exists \mathfrak{y} \mathfrak{A}(\mathfrak{y}, \xi)$$

*be a closed LK-formula which is LK-provable from **A** and is dependent on **A**, $\mathfrak{f}$ be a function symbol contained in none of the axioms of **A**, and **A**$'$ be the axiom system obtained from **A** by adjoining the axioms*

$$\forall \xi \mathfrak{A}(\mathfrak{f}(\xi), \xi)$$

and

$$\forall \xi \forall \eta (\xi = \eta \supset \mathfrak{f}(\xi) = \mathfrak{f}(\eta)).$$

Then:

1) ***A**$'$ is an axiom system with equality in LK.*

2) *If **A** is LK-consistent, then **A**$'$ is LK-consistent.*

In the following, we shall show the proof of Theorem 1 and 2) of Theorem 2, because 1) of Theorem 2 is almost clear.

Let **A**$'$ be *LK*-cotradictory. Then there exists an axiom sequence Γ_0 of **A** and the sequent

$$\forall \xi \mathfrak{A}(\mathfrak{f}(\xi), \xi), \Gamma_0 \rightarrow \qquad \text{(case of Theorem 1)}$$

or

$$\forall \xi \mathfrak{A}(\mathfrak{f}(\xi), \xi), \ \forall \xi \forall \eta (\xi = \eta \supset \mathfrak{f}(\xi) = \mathfrak{f}(\eta)), \ \Gamma_0 \rightarrow \qquad \text{(case of Theorem 2)}$$

is *LK*-provable, hence there exists an *LK*-proof ***P*** of that sequent which contains no cut (Hauptsatz on *LK*). For our purpose, it is sufficient to prove under the hypothesis of the existence of ***P*** that the sequent

$$\Gamma_0 \rightarrow$$

is *LK*-provable, because if so **A** is *LK*-contradictory. Then, in the following, we shall consider the fixed formal proof $\boldsymbol{P}$.

2.1. A formula containing occurrences of the function symbol $\mathfrak{f}$ which have bound variables as some arguments is called an $\mathfrak{f}$-*formula.*

2.11. An $\mathfrak{f}$-formula in $\boldsymbol{P}$ has the form which is the result of substituting terms for all free occurrences of bound variables in

$$\mathfrak{A}(\mathfrak{f}(\xi), \xi)$$

or

$$\xi = \eta \supset \mathfrak{f}(\xi) = \mathfrak{f}(\eta) \qquad \text{(case of Theorem 2)}$$

with some universal quantifiers standing at the front.

2.12. Each occurrence of the $\mathfrak{f}$-formulae in $\boldsymbol{P}$ occurs only in the antecedent of an occurrence of a sequent in $\boldsymbol{P}$.

2.2. A term of the form

$$\mathfrak{f}(\tau)$$

is called an $\mathfrak{f}$-*term,* where τ is a sequence of arbitrary terms.

2.21. Let

$$\mathfrak{f}(\tau_1), \mathfrak{f}(\tau_2), \ldots, \mathfrak{f}(\tau_n)$$

be all of the distinct $\mathfrak{f}$-terms contained in $\boldsymbol{P}$, and $\mathfrak{f}(\tau_i)$ be contained in none of $\mathfrak{f}(\tau_{i+1}), \ldots, \mathfrak{f}(\tau_n)$ as subterm $(i = 1, 2, \ldots, n-1)$.[11)]

2.22. Let

$$\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_n$$

be n distinct free variables not contained in $\boldsymbol{P}$.

2.23. The formula or term obtained from a formula or term $\mathfrak{B}$ by the following substitution is described as $\mathfrak{B}^*$: firstly, we substitute $\mathfrak{a}_1$ for $\mathfrak{f}(\tau_1)$ throughout $\mathfrak{B}$; secondly, $\mathfrak{a}_2$ for $\mathfrak{f}(\tau_2)$ throughout the above result; thirdly, $\mathfrak{a}_3$ for $\mathfrak{f}(\tau_3)$ throughout the second result; and so on.

When Γ stands for a finite sequence of formulae:

$$\mathfrak{A}_1, \mathfrak{A}_2, \ldots, \mathfrak{A}_m,$$

Γ^* means the sequence

$$\mathfrak{A}_1^*, \mathfrak{A}_2^*, \ldots, \mathfrak{A}_m^*.$$

When τ stands for a finite sequence of terms:

$$\mathfrak{t}_1, \mathfrak{t}_2, \ldots, \mathfrak{t}_m,$$

τ^* means the sequence

$$\mathfrak{t}_1^*, \mathfrak{t}_2^*, \ldots, \mathfrak{t}_m^* .$$

2.231. If $\mathfrak{B}$ is $\mathfrak{f}(\tau_i)$, $\mathfrak{B}^*$ is $\mathfrak{a}_i\,(i=1,2,\ldots,n)$.

2.232. If and only if $\mathfrak{A}$ is an $\mathfrak{f}$-formula, $\mathfrak{A}^*$ contains the function symbol $\mathfrak{f}$, where $\mathfrak{A}$ is a formula contained in $\boldsymbol{P}$.

2.233. If $\mathfrak{a}$ is a free variable other than $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$, and is contained in τ_i^*, it is contained in τ_i.

2.234. If $\mathfrak{a}_j$ is contained in τ_i^*, $\mathfrak{f}(\tau_j)$ is contained in τ_i, accordingly $i<j$ (cf. 2.21).

2.235. $(\mathfrak{A}(\mathfrak{f}(\tau_i), \tau_i))^*$ is $\mathfrak{A}(\mathfrak{a}_i, \tau_i^*)$.

2.3. For any sequence $i_1, \ldots, i_k$ of arbitrary suffixes of $\tau_1, \ldots, \tau_n$, we define the sequence $\Phi(i_1, \ldots, i_k)$[12] of the formulae as follows:

2.31. Case of Theorem 1: $\Phi(i_1, \ldots, i_k)$ is the finite sequence constituted by all of the formulae

$$\mathfrak{A}(\mathfrak{a}_{i_r}, \tau_{i_r}^*) \qquad (r=1,2,\ldots,k).$$

2.32. Case of Theorem 2: $\Phi(i_1, \ldots, i_k)$ is the finite sequence constituted by all of the formulae

$$\mathfrak{A}(\mathfrak{a}_{i_r}, \tau_{i_r}^*) \qquad (r=1,2,\ldots,k)$$

and

$$\tau_{i_r}^* = \tau_{i_s}^* \supset \mathfrak{a}_{i_r} = \mathfrak{a}_{i_s} \qquad (r, s=1,2,\ldots,k).$$

2.4. In the following, we assume, without loss of generality, that the sequent

$$\Gamma_0 \rightarrow \forall \mathfrak{x} \exists \mathfrak{y} \mathfrak{A}(\mathfrak{y}, \mathfrak{x})$$

is LK-provable, and, but in the case of Theorem 2, Γ_0 is an axiom system with equality in LK itself, and $\forall \mathfrak{x} \exists \mathfrak{y} \mathfrak{A}(\mathfrak{y}, \mathfrak{x})$ is dependent on Γ_0.

2.5. LEMMA 1. *Let τ be a sequence of arbitrary terms, and $\mathfrak{a}$ be a free variable not contained in the sequences Γ, Θ, $\Phi(i_1, \ldots, i_k)$ and τ. If the sequent*

$$\tau = \tau_{i_1}^* \supset \mathfrak{a} = \mathfrak{a}_{i_1}, \ldots, \tau = \tau_{i_k}^* \supset \mathfrak{a} = \mathfrak{a}_{i_k},$$

$$\mathfrak{A}(\mathfrak{a}, \tau), \Gamma, \Phi(i_1, \ldots, i_k), \Gamma_0 \rightarrow \Theta$$

is LK-provable, then the sequent

$$\mathfrak{A}(\mathfrak{a}, \tau), \Gamma, \Phi(i_1,\ldots, i_k), \ \Gamma_0 \rightarrow \Theta$$

is LK-provable (case of Theorem 2).

PROOF. From that Γ_0 is the axiom system with equality in *LK*,

$$\tau = \tau^*_{i_r}, \tau^*_{i_r} = \tau^*_{i_1} \supset \mathfrak{a} = \mathfrak{a}_{i_1}, \ldots, \tau^*_{i_r} = \tau^*_{i_k} \supset \mathfrak{a} = \mathfrak{a}_{i_k},$$

$$\mathfrak{A}(\mathfrak{a}, \tau^*_{i_r}), \Gamma, \Phi(i_1,\ldots, i_k), \ \Gamma_0 \rightarrow \Theta$$

is *LK*-provable, and from that $\mathfrak{a}$ is not contained in Γ, Θ, $\Phi(i_1,\ldots, i_k)$ and τ,

$$\tau = \tau^*_{i_r}, \ \tau^*_{i_r} = \tau^*_{i_1} \supset \mathfrak{a}_{i_r} = \mathfrak{a}_{i_1}, \ldots, \tau^*_{i_r} = \tau^*_{i_k} \supset \mathfrak{a}_{i_r} = \mathfrak{a}_{i_k},$$

$$\mathfrak{A}(\mathfrak{a}_{i_r}, \tau^*_{i_r}), \Gamma, \Phi(i_1,\ldots, i_k), \Gamma_0 \rightarrow \Theta$$

consequently

$$\tau = \tau^*_{i_r}, \Gamma, \Phi(i_1,\ldots, i_k), \Gamma_0 \rightarrow \Theta$$

are *LK*-provable, hence

$$\Gamma, \Phi(i_1,\ldots, i_k), \ \Gamma_0 \rightarrow \Theta, \ \tau = \tau^*_{i_r} \supset \mathfrak{a} = \mathfrak{a}_{i_r}$$

is *LK*-provable ($r = 1, 2,\ldots, k$).

Accordingly, from those k sequents and the first sequent, the sequent

$$\mathfrak{A}(\mathfrak{a}, \tau), \Gamma, \Phi(i_1,\ldots, i_k), \Gamma_0 \rightarrow \Theta$$

is *LK*-provable by k cuts with the helps of some interchanges and contractions, q. e. d.

2.6. LEMMA 2. *If a sequent of the form*

$$\Gamma, \Phi(i_1,\ldots, i_k, j_1,\ldots, j_l), \Gamma_0 \rightarrow \Theta$$

is LK-provable, and the free variables $\mathfrak{a}_{j_1},\ldots, \mathfrak{a}_{j_l}$ *are not contained in* $\Gamma, \Theta, \Phi(i_1,\ldots, i_k)$, *then*

$$\Gamma, \Phi(i_1,\ldots, i_k), \Gamma_0 \rightarrow \Theta$$

is LK-provable.

PROOF. The mathematical induction on l.

2.61. When $l = 0$: The lemma evidently holds.

2.62. When $l > 0$: We can assume without loss of generality that $j_1 > j_2 > \ldots > j_l$. Then the sequent

$$\mathfrak{A}(\mathfrak{a}_{j_l}, \tau^*_{j_l}), \Gamma, \Phi(i_1,\ldots, i_k, j_1,\ldots, j_{l-1}), \Gamma_0 \to \Theta$$

is *LK*-provable, where, in the case of Theorem 2, Lemma 1 has been used. From that the free variable $\mathfrak{a}_{j_l}$ is not contained in Γ, Θ, $\Phi(i_1,\ldots, i_k, j_1,\ldots, j_{l-1})$ and $\tau^*_{j_l}$ (cf. 2.234 and 2.3), the sequent

$$\exists \mathfrak{y}\mathfrak{A}(\mathfrak{y}, \tau^*_{j_l}), \Gamma, \Phi(i_1,\ldots, i_k, j_1,\ldots, j_{l-1}), \Gamma_0 \to \Theta$$

is *LK*-provable, hence

$$\Gamma, \Phi(i_1,\ldots, i_k, j_1,\ldots, j_{l-1}), \Gamma_0 \to \Theta$$

is so (cf. 2.4). Accordingly, by the hypothesis of the induction,

$$\Gamma, \Phi(i_1,\ldots, i_k), \Gamma_0 \to \Theta$$

is *LK*-provable, q. e. d.

2.7. When $\mathfrak{S}$ is a sequent

$$\Gamma \to \Theta$$

in $\boldsymbol{P}$, $\mathfrak{S}^*$ means the sequent[13]

$$\Gamma^{\#*}, \Phi(i_1,\ldots, i_k), \Gamma_0 \to \Theta^*$$

where $\Gamma^{\#}$ is the result of suppressing all occurrences of the $\mathfrak{f}$-formula in Γ, and the $\mathfrak{f}$-terms

$$\mathfrak{f}(\tau_{i_1}), \mathfrak{f}(\tau_{i_2}),\ldots, \mathfrak{f}(\tau_{i_k})$$

are all of the distinct $\mathfrak{f}$-terms contained in $\Gamma^{\#}$ or Θ (cf. 2.12 and 2.232).

For the sake of our proof of Theorem 1 and 2) of Theorem 2, it is sufficient to prove the following

LEMMA 3. *If $\mathfrak{S}$ is a sequent in $\boldsymbol{P}$, then $\mathfrak{S}^*$ is LK-provable.*

Because, let $\mathfrak{S}$ be the endsequent of $\boldsymbol{P}$, then $\mathfrak{S}^*$ is the sequent

$$\Gamma_0, \Gamma_0 \to,$$

accordingly the sequent

$$\Gamma_0 \to$$

is *LK*-provable.

PROOF OF LEMMA 3.

2.71. When $\mathfrak{S}$ is a logical axiom $\mathfrak{D} \to \mathfrak{D}$, $\mathfrak{D}$ is not an $\mathfrak{f}$-formula (cf. 2.12), then $\mathfrak{S}^*$ has the form

$$\mathfrak{D}^*, \Phi(i_1,\ldots, i_k), \Gamma_0 \to \mathfrak{D}^*$$

and it is *LK*-provable.

2.72. When $\mathfrak{S}$ is the conclusion of an inference $\mathfrak{J}$ in ***P***, we shall show that the concerned property of $\mathfrak{S}$ holds under the hypothesis that each of one or two premises of $\mathfrak{J}$ has such property.

2.721. When the principal formula of $\mathfrak{J}$ is an $\mathfrak{f}$-formula[14]:

2.721.1. When the side formula of $\mathfrak{J}$ is an $\mathfrak{f}$-formula:

Let $\mathfrak{S}'$ be the premise of $\mathfrak{J}$, then $\mathfrak{S}^*$ is identical with $\mathfrak{S}'^*$ and is *LK*-provable.

2.721.2 When the side formula of $\mathfrak{J}$ is not an $\mathfrak{f}$-formula:

The side formula of $\mathfrak{J}$ is one of the formulae

$$\mathfrak{A}(\mathfrak{f}(\tau_i), \tau_i) \qquad (i=1, 2,\ldots, n)$$

and

$$\tau_i=\tau_j \supset \mathfrak{f}(\tau_i)=\mathfrak{f}(\tau_j) \qquad (i, j=1, 2,\ldots, n)$$

(cf. 2.11).

2.721.21. If the premise $\mathfrak{S}'$ of $\mathfrak{J}$ is

$$\mathfrak{A}(\mathfrak{f}(\tau_i), \tau_i), \Gamma \rightarrow \Theta\,,$$

$\mathfrak{S}'^*$ is

$$\mathfrak{A}(\mathfrak{a}_i, \tau_i^*), \Gamma^{\#*}, \Phi(i_1,\ldots, i_k, j_1,\ldots, j_l), \Gamma_0 \rightarrow \Theta^*$$

where

$$\mathfrak{f}(\tau_{i_1}),\ldots, \mathfrak{f}(\tau_{i_k})$$

are all distinct $\mathfrak{f}$-terms in $\Gamma^\#$ and Θ, and

$$\mathfrak{f}(\tau_{j_1}),\ldots, \mathfrak{f}(\tau_{j_l})$$

are all distinct $\mathfrak{f}$-terms which are contained in $\mathfrak{A}(\mathfrak{f}(\tau_i), \tau_i)$ and are not in $\Gamma^\#$ and Θ. By the hypothesis of the induction, that sequent is *LK*-provable, accordingly the sequent

$$\Gamma^{\#*}, \Phi(i_1,\ldots, i_k, j_1,\ldots, j_l), \Gamma_0 \rightarrow \Theta^*$$

is so, hence

$$\Gamma^{\#*}, \Phi(i_1,\ldots, i_k), \Gamma_0 \rightarrow \Theta^*$$

is *LK*-provable (cf. Lemma 2 and 2.234), and it is $\mathfrak{S}^*$.

2.721.22. If the premise $\mathfrak{S}'$ of $\mathfrak{J}$ is

$$\tau_i=\tau_j \supset \mathfrak{f}(\tau_i)=\mathfrak{f}(\tau_j), \Gamma \rightarrow \Theta\,,$$

$\mathfrak{S}^*$ has the form

$$\Gamma^{\#*}, \Phi(i_1,\dots,i_k), \Gamma_0 \rightarrow \Theta^*,$$

and $\mathfrak{S}'^*$

$$\tau_i^* = \tau_j^* \supset \mathfrak{a}_i = \mathfrak{a}_j, \Gamma^{\#*}, \Phi(i_1,\dots,i_k, j_1,\dots,j_l), \Gamma_0 \rightarrow \Theta^*.$$

By the hypothesis of the induction, $\mathfrak{S}'^*$ is LK-provable, then

$$\Gamma^{\#*}, \Phi(i_1,\dots,i_k, j_1,\dots,j_l), \Gamma_0 \rightarrow \Theta^*$$

is so, hence $\mathfrak{S}^*$ is LK-provable.

2.722. When the principal formula of $\mathfrak{J}$ is not an $\mathfrak{f}$-formula:

2.722.1. When $\mathfrak{J}$ is an V-antecedent:

Let $\mathfrak{S}$ be

$$V\mathfrak{x}\mathfrak{F}(\mathfrak{x}), \Gamma \rightarrow \Theta,$$

then the premise $\mathfrak{S}'$ of $\mathfrak{J}$ has the form

$$\mathfrak{F}(\mathfrak{t}), \Gamma \rightarrow \Theta,$$

and $\mathfrak{S}^*$ and $\mathfrak{S}'^*$ have the forms

$$V\mathfrak{x}\mathfrak{F}^*(\mathfrak{x}), \Gamma^{\#*}, \Phi(i_1,\dots,i_k), \Gamma_0 \rightarrow \Theta^*$$

and

$$\mathfrak{F}^*(\mathfrak{t}^*), \Gamma^{\#*}, \Phi(i_1,\dots,i_k, j_1,\dots,j_l), \Gamma_0 \rightarrow \Theta^*,$$

respectively. By the hypothesis of the induction, $\mathfrak{S}'^*$ is LK-provable, then

$$V\mathfrak{x}\mathfrak{F}^*(\mathfrak{x}), \Gamma^{\#*}, \Phi(i_1,\dots,i_k, j_1,\dots,j_l), \Gamma_0 \rightarrow \Theta^*$$

is so, hence $\mathfrak{S}^*$ is LK-provable.

2.722.2. When $\mathfrak{J}$ is an V-succedent:

Let $\mathfrak{S}$ be

$$\Gamma \rightarrow \Theta, V\mathfrak{x}\mathfrak{F}(\mathfrak{x}),$$

then the premise $\mathfrak{S}'$ of $\mathfrak{J}$ has the form

$$\Gamma \rightarrow \Theta, \mathfrak{F}(\mathfrak{a})$$

and $\mathfrak{S}^*$ and $\mathfrak{S}'^*$ have the forms

$$\Gamma^{\#*}, \Phi(i_1,\dots,i_k), \Gamma_0 \rightarrow \Theta^*, V\mathfrak{x}\mathfrak{F}^*(\mathfrak{x})$$

and

$$\Gamma^{\#*}, \Phi(i_1,\ldots, i_k), \Gamma_0 \rightarrow \Theta^*, \mathfrak{F}^*(\mathfrak{a}),$$

respectively. By the hypothesis of the induction, $\mathfrak{S}'^*$ is *LK*-provable, then $\mathfrak{S}^*$ is *LK*-provable (cf. 2.22 and 2.233).

2.722.3. When $\mathfrak{F}$ is an inference of other kind: The treatment is similar to the above, q. e. d.

§ 3. Extension theorems.

3.1. '$<\mathfrak{F}(\mathfrak{f}(\xi), \xi)>$-extension'.

Let A be an axiom system in *LK*, and $\mathfrak{f}$ be a function symbol contained in none of the axioms of A. Then we shall call the axiom system in *LK* obtained from A by adjoining the new *LK*-axiom

$$V\xi V\mathfrak{x}(\mathfrak{F}(\mathfrak{x}, \xi) \supset \mathfrak{F}(\mathfrak{f}(\xi), \xi))$$

the $<\mathfrak{F}(\mathfrak{f}(\xi), \xi)>$-*extension of* A, where $\mathfrak{F}(\mathfrak{x}, \xi)$ does not contain $\mathfrak{f}$.

THEOREM 3. *Let* A *be an axiom system in LK. If* A *is LK-consistent, then the* $<\mathfrak{F}(\mathfrak{f}(\xi), \xi)>$-*extension of* A *is also LK-consistent.*

PROOF is obtained immediately from Theorem 1 (§ 2) and the *LK*-provability of the sequent

$$\rightarrow V\xi \exists \mathfrak{y} V\mathfrak{x}(\mathfrak{F}(\mathfrak{x}, \xi) \supset \mathfrak{F}(\mathfrak{y}, \xi)).$$

3.2. '$<\mathfrak{F}_1(\mathfrak{f}_1(\xi_1), \xi_1),\ldots, \mathfrak{F}_n(\mathfrak{f}_n(\xi_n), \xi_n)>$-extension'.

The $<\mathfrak{F}_1(\mathfrak{f}_1(\xi_1), \xi_1),\ldots, \mathfrak{F}_{n-1}(\mathfrak{f}_{n-1}(\xi_{n-1}), \xi_{n-1}), \mathfrak{F}_n(\mathfrak{f}_n(\xi_n), \xi_n)>$-*extension of* A, which is an axiom system in *LK*, is the $<\mathfrak{F}_n(\mathfrak{f}_n(\xi_n), \xi_n)>$-extension of the $<\mathfrak{F}_1(\mathfrak{f}_1(\xi_1), \xi_1),\ldots, \mathfrak{F}_{n-1}(\mathfrak{f}_{n-1}(\xi_{n-1}), \xi_{n-1})>$-extension of A ($n=2, 3,\ldots$).

THEOREM 4. *Let* A *be an axiom system in LK. If* A *is LK-consistent, then the* $<\mathfrak{F}_1(\mathfrak{f}_1(\xi_1), \xi_1),\ldots, \mathfrak{F}_n(\mathfrak{f}_n(\xi_n), \xi_n)>$-*extension of* A *is LK-consistent.*

PROOF is obtained by the mathematical induction on n (cf. Theorem 3).

3.3. '$<\mathfrak{F}(\mathfrak{f}(\xi), \xi); =>$-extension'.

Let A be an axiom system with equality in *LK*, and $\mathfrak{f}$ be a function symbol contained in none of the axioms of A. Then we shall call the axiom system obtained from A by adjoining the new *LK*-axioms

$$V\xi V\mathfrak{x}(\mathfrak{F}(\mathfrak{x}, \xi) \supset \mathfrak{F}(\mathfrak{f}(\xi), \xi))$$

and

$$\forall \xi \forall \eta (\xi = \eta \supset \mathfrak{f}(\xi) = \mathfrak{f}(\eta))$$

the $<\mathfrak{F}(\mathfrak{f}(\xi), \xi); =>$*-extension of* $\boldsymbol{A}$, where $\mathfrak{F}(\mathfrak{x}, \xi)$ does not contain $\mathfrak{f}$, and each of the functions and the predicates contained in $\mathfrak{F}(\mathfrak{x}, \xi)$ is contained in some axiom of $\boldsymbol{A}$.

THEOREM 5. *Let* $\boldsymbol{A}$ *be an axiom system with equality in LK, and* $\boldsymbol{A}'$ *be the* $<\mathfrak{F}(\mathfrak{f}(\xi), \xi); =>$*-extension of* $\boldsymbol{A}$. *Then:*

1) $\boldsymbol{A}'$ *is an axiom system with equality in LK.*

2) *If* $\boldsymbol{A}$ *is LK-consistent, then* $\boldsymbol{A}'$ *is LK-consistent.*

PROOF is obtained from Theorem 2 (§ 2).

3.4. '$<\mathfrak{F}_1(\mathfrak{f}_1(\xi_1), \xi_1), \ldots, \mathfrak{F}_n(\mathfrak{f}_n(\xi_n), \xi_n); =>$-extension'.

The $<\mathfrak{F}_1(\mathfrak{f}_1(\xi_1), \xi_1), \ldots, \mathfrak{F}_{n-1}(\mathfrak{f}_{n-1}(\xi_{n-1}), \xi_{n-1}), \mathfrak{F}_n(\mathfrak{f}_n(\xi_n), \xi_n); =>$*-extension of* $\boldsymbol{A}$, which is an axiom system with equality in *LK*, is the $<\mathfrak{F}_n(\mathfrak{f}_n(\xi_n), \xi_n); =>$-extension of the $<\mathfrak{F}_1(\mathfrak{f}_1(\xi_1), \xi_1), \ldots, \mathfrak{F}_{n-1}(\mathfrak{f}_{n-1}(\xi_{n-1}), \xi_{n-1}); =>$-extension of $\boldsymbol{A}$ ($n = 2, 3, \ldots$).

THEOREM 6. *Let* $\boldsymbol{A}$ *be an axiom system with equality in LK, and* $\boldsymbol{A}'$ *be the* $<\mathfrak{F}_1(\mathfrak{f}_1(\xi_1), \xi_1), \ldots, \mathfrak{F}_n(\mathfrak{f}_n(\xi_n), \xi_n); =>$*-extension of* $\boldsymbol{A}$. *Then:*

1) $\boldsymbol{A}'$ *is an axiom system with equality in LK.*

2) *If* $\boldsymbol{A}$ *is LK-consistent, then* $\boldsymbol{A}'$ *is LK-consistent.*

PROOF is obtained by the mathematical induction on n (cf. Theorem 5).

§ 4. Main theorems.

4.1. MAIN THEOREM 1. *Let* $\boldsymbol{A}$ *be an axiom system in LK. If* $\boldsymbol{A}$ *is LK-consistent, then* $\boldsymbol{A}$ *is consistent* (*in the predicate calculus with* ε-*symbol*).

4.11. PROOF. We assume that $\boldsymbol{A}$ is contradictory. Then there exists an axiom sequence Γ_0 of $\boldsymbol{A}$ and the sequent

$$\Gamma_0 \rightarrow$$

is provable. Now, let $\boldsymbol{P}$ be a proof of this sequent.

4.111. Let

$$\varepsilon \mathfrak{x}_1 \mathfrak{F}_1(\mathfrak{x}_1, \alpha_1), \ldots, \varepsilon \mathfrak{x}_n \mathfrak{F}_n(\mathfrak{x}_n, \alpha_n)$$

be types of all of the ε-terms contained in $\boldsymbol{P}$ which are not congruent with each other, the grade of $\varepsilon \mathfrak{x}_i \mathfrak{F}_i(\mathfrak{x}_i, \alpha_i)$ be not greater than those of $\varepsilon \mathfrak{x}_{i+1}\mathfrak{F}_{i+1}(\mathfrak{x}_{i+1}, \alpha_{i+1}), \dots, \varepsilon \mathfrak{x}_n \mathfrak{F}_n(\mathfrak{x}_n, \alpha_n)$ $(i=1, 2, \dots, n-1)$, and α's indicate all of the arguments of the types, respectively.

4.112. Let

$$\mathfrak{f}_0, \mathfrak{f}_1, \dots, \mathfrak{f}_n$$

be $n+1$ distinct function symbols not contained in $\boldsymbol{P}$.

4.113. For any formula or term $\mathfrak{B}$ containing no function symbols such as $\mathfrak{f}_0, \mathfrak{f}_1, \dots, \mathfrak{f}_n$, now we inductively define the *LK*-formula or *LK*-term described as $\overline{\mathfrak{B}}$ (4.113.1–4.113.62). When Γ or τ stands for a finite sequence of formulae: $\mathfrak{A}_1, \dots, \mathfrak{A}_m$ or of terms: $\mathfrak{t}_1, \dots, \mathfrak{t}_m$, $\overline{\Gamma}$ or $\overline{\tau}$ means the sequence $\overline{\mathfrak{A}}_1, \dots, \overline{\mathfrak{A}}_m$ or $\overline{\mathfrak{t}}_1, \dots, \overline{\mathfrak{t}}_m$, respectively.

4.113.1. If $\mathfrak{a}$ is a free variable, $\overline{\mathfrak{a}}$ is $\mathfrak{a}$.

4.113.2. If τ is a sequence of terms, and $\mathfrak{f}$ is a function, $\overline{\mathfrak{f}(\tau)}$ is $\mathfrak{f}(\overline{\tau})$.

4.113.3. If τ is a sequence of terms, and $\mathfrak{P}$ is a predicate, $\overline{\mathfrak{P}(\tau)}$ is $\mathfrak{P}(\overline{\tau})$.

4.113.4. If $\mathfrak{A}$ is a formula, $\overline{\neg \mathfrak{A}}$ is $\neg \overline{\mathfrak{A}}$. If $\mathfrak{A}$ and $\mathfrak{B}$ are formulae, $\overline{\mathfrak{A} \& \mathfrak{B}}$, $\overline{\mathfrak{A} \vee \mathfrak{B}}$ and $\overline{\mathfrak{A} \supset \mathfrak{B}}$ are $\overline{\mathfrak{A}} \& \overline{\mathfrak{B}}$, $\overline{\mathfrak{A}} \vee \overline{\mathfrak{B}}$ and $\overline{\mathfrak{A}} \supset \overline{\mathfrak{B}}$, respectively.

4.113.5. If $\forall \mathfrak{x} \mathfrak{F}(\mathfrak{x})$ or $\exists \mathfrak{x} \mathfrak{F}(\mathfrak{x})$ is a formula of such form, $\mathfrak{a}$ is a free variable not contained in $\mathfrak{F}(\mathfrak{x})$, and $\overline{\mathfrak{F}}(\mathfrak{x})$ is the result of substituting $\mathfrak{x}$ for $\mathfrak{a}$ throughout $\overline{\mathfrak{F}(\mathfrak{a})}$ (cf. 1.121), $\overline{\forall \mathfrak{x} \mathfrak{F}(\mathfrak{x})}$ or $\overline{\exists \mathfrak{x} \mathfrak{F}(\mathfrak{x})}$ is $\forall \mathfrak{x} \overline{\mathfrak{F}}(\mathfrak{x})$ or $\exists \mathfrak{x} \overline{\mathfrak{F}}(\mathfrak{x})$, respectively.

4.113.61. If τ is a sequence of terms, $\overline{\varepsilon \mathfrak{x}_i \mathfrak{F}_i(\mathfrak{x}_i, \tau)}$ is $\mathfrak{f}_i(\overline{\tau})$ $(i=1, 2, \dots, n)$.

4.113.62. If none of $\varepsilon \mathfrak{x}_i \mathfrak{F}_i(\mathfrak{x}_i, \alpha_i)$ $(i=1, 2, \dots, n)$ is a type of an ε-term $\varepsilon \mathfrak{x} \mathfrak{F}(\mathfrak{x})$, $\overline{\varepsilon \mathfrak{x} \mathfrak{F}(\mathfrak{x})}$ is $\mathfrak{f}_0$.

4.114. Let Ψ be the sequence of the formulae:

$$\forall \xi_1 \forall \mathfrak{x} (\overline{\mathfrak{F}}_1(\mathfrak{x}, \xi_1) \supset \overline{\mathfrak{F}}_1(\mathfrak{f}_1(\xi_1), \xi_1)),$$

$$\dots, \forall \xi_n \forall \mathfrak{x} (\overline{\mathfrak{F}}_n(\mathfrak{x}, \xi_n) \supset \overline{\mathfrak{F}}_n(\mathfrak{f}_n(\xi_n), \xi_n)).$$

If the sequent $\Gamma \to \Theta$ occurs in $\boldsymbol{P}$, the sequent

$$\overline{\Gamma}, \Psi \to \overline{\Theta}$$

is *LK*-provable. It can be proved by the mathematical induction on the number of occurrences of sequents contained above $\Gamma\rightarrow\Theta$ in $\boldsymbol{P}$. Especially,

$$\Gamma_0, \Psi\rightarrow$$

is *LK*-provable.

4.115. If $\mathfrak{B}$ is a formula or term containing no function symbols $\mathfrak{f}_0, \mathfrak{f}_1, \ldots, \mathfrak{f}_n$, and $\overline{\mathfrak{B}}$ contains $\mathfrak{f}_i$, then the grade of $\mathfrak{B}$ is not less than that of the ε-type $\varepsilon\mathfrak{x}_i\mathfrak{F}_i(\mathfrak{x}_i, \alpha_i)$ $(i=1, 2, \ldots, n)$. Accordingly, $\mathfrak{f}_i$ does not occur in $\overline{\mathfrak{F}_1}(\mathfrak{x}, \xi_1), \ldots, \overline{\mathfrak{F}_i}(\mathfrak{x}, \xi_i)$ (cf. 4.111).

4.116. Let $\boldsymbol{A}'$ be the $<\overline{\mathfrak{F}}_1(\mathfrak{f}_1(\xi_1), \xi_1), \ldots, \overline{\mathfrak{F}}_n(\mathfrak{f}_n(\xi_n), \xi_n)>$-extension of $\boldsymbol{A}$ (cf. 4.115), then the sequence Γ_0, Ψ is an axiom sequence of $\boldsymbol{A}'$, consequently $\boldsymbol{A}'$ is *LK*-contradictory (cf. 4.114). Hence, $\boldsymbol{A}$ is *LK*-contradictory (cf. Theorem 4 3.2), q. e. d.

4.12. COROLLARY. *Let* $\boldsymbol{A}$ *be an axiom system in LK, and* $\mathfrak{A}$ *be an LK-formula. If* $\mathfrak{A}$ *is provable from* $\boldsymbol{A}$*, then* $\mathfrak{A}$ *is LK-provable from* $\boldsymbol{A}$*.*

PROOF. Let us regard the free variables contained in $\mathfrak{A}$ as fixed individuals, then the axiom system $\boldsymbol{A}'$ in *LK* obtained from $\boldsymbol{A}$ by adjoining the new axiom $\neg\mathfrak{A}$ is contradictory, accordingly $\boldsymbol{A}'$ is *LK*-contradictory. Hence, there exists an axiom sequence Γ_0 of $\boldsymbol{A}$ and

$$\neg\mathfrak{A}, \Gamma_0\rightarrow$$

is *LK*-provable, i. e.

$$\Gamma_0\rightarrow\mathfrak{A}$$

is *LK*-provable. Hence, $\mathfrak{A}$ is *LK*-provable from $\boldsymbol{A}$, q. e. d.

4.2. MAIN THEOREM 2. *Let* $\boldsymbol{A}$ *be an axiom system with equality in LK, and* $\boldsymbol{A}^e$ *be the axiom system obtained from* $\boldsymbol{A}$ *by adjoining all of the axioms of the form*

$$V\xi V\eta(\xi=\eta\supset\varepsilon\mathfrak{x}\mathfrak{F}(\mathfrak{x}, \xi)=\varepsilon\mathfrak{x}\mathfrak{F}(\mathfrak{x}, \eta))$$

which are dependent on $\boldsymbol{A}$*. Then:*

1) $\boldsymbol{A}^e$ *is an axiom system with equality (in the predicate calculus with* ε*-symbol).*

2) *If* $\boldsymbol{A}$ *is LK-consistent, then* $\boldsymbol{A}^e$ *is consistent (in the predicate calculus with* ε*-symbol).*

4.21. PROOF. 1) is almost clear, then, in the following, we shall prove 2) only.

We assume that $\boldsymbol{A}^e$ is contradictory. Then there exist an axiom sequence Γ_0 of $\boldsymbol{A}$ and a finite sequence Ω of axioms of the form

$$V\xi V\eta(\xi=\eta\supset\varepsilon\mathfrak{x}\mathfrak{F}(\mathfrak{x},\xi)=\varepsilon\mathfrak{x}\mathfrak{F}(\mathfrak{x},\eta))$$

which are dependent on $\boldsymbol{A}$, and the sequent

$$\Gamma_0, \Omega\rightarrow$$

is provable. We can assume without loss of generality that the sequent

$$\Gamma_0\rightarrow V\mathfrak{x}(\mathfrak{x}=\mathfrak{x})$$

is *LK*-provable. Now, let $\boldsymbol{P}$ be a proof of the sequent $\Gamma_0, \Omega\rightarrow$.

We use the symbols $\mathfrak{f}_1,\dots,\mathfrak{f}_n$, $\overline{\mathfrak{F}}_1(\mathfrak{f}_1(\xi_1),\xi_1),\dots,\overline{\mathfrak{F}}_n(\mathfrak{f}_n(\xi_n),\xi_n)$ and Ψ in the similar sense to the sense of those in the proof (4.11) of Main theorem 1, and Σ stands for the sequence

$$V\xi_1V\eta_1(\xi_1=\eta_1\supset\mathfrak{f}_1(\xi_1)=\mathfrak{f}_1(\eta_1)),$$
$$\dots, V\xi_nV\eta_n(\xi_n=\eta_n\supset\mathfrak{f}_n(\xi_n)=\mathfrak{f}_n(\eta_n)).$$

Then the sequent

$$\Gamma_0, \Psi, \Sigma\rightarrow$$

is *LK*-provable. Let $\boldsymbol{A}'$ be the $<\overline{\mathfrak{F}}_1(\mathfrak{f}_1(\xi_1),\xi_1),\dots,\overline{\mathfrak{F}}_n(\mathfrak{f}_n(\xi_n),\xi_n);=>$-extension of $\boldsymbol{A}$, then the sequence Γ_0, Ψ, Σ is an axiom sequence of $\boldsymbol{A}'$, accordingly $\boldsymbol{A}'$ is *LK*-contradictory. Hence, $\boldsymbol{A}$ is *LK*-contradictory (cf. Theorem 6 3.4), q. e. d.

4.22. COROLLARY. *Let* $\boldsymbol{A}$ *be an axiom system with equality in LK,* $\boldsymbol{A}^e$ *be the axiom system with equality obtained from* $\boldsymbol{A}$ *by adjoining the axioms of the form*

$$V\xi V\eta(\xi=\eta\supset\varepsilon\mathfrak{x}\mathfrak{F}(\mathfrak{x},\xi)=\varepsilon\mathfrak{x}\mathfrak{F}(\mathfrak{x},\eta))$$

which are dependent on $\boldsymbol{A}$*, and* $\mathfrak{A}$ *be an LK-formula depending on* $\boldsymbol{A}$*. If* $\mathfrak{A}$ *is provable from* $\boldsymbol{A}^e$*, then* $\mathfrak{A}$ *is LK-provable from* $\boldsymbol{A}$*.*

PROOF. Let us regard the free variables contained in $\mathfrak{A}$ as fixed individuals, $\boldsymbol{A}'$ be the axiom system obtained from $\boldsymbol{A}$ by adjoining the new axiom $\neg\mathfrak{A}$, and $\boldsymbol{A}'^e$ be the axiom system obtained from $\boldsymbol{A}'$ by adjoining all of the axioms of the form

$$V\xi V\eta(\xi=\eta\supset\varepsilon\mathfrak{x}\mathfrak{F}(\mathfrak{x},\xi)=\varepsilon\mathfrak{x}\mathfrak{F}(\mathfrak{x},\eta))$$

which are dependent on $\boldsymbol{A}'$, then $\boldsymbol{A}'$ is an axiom system with equality in LK, $\boldsymbol{A}'^{e}$ is an axiom system with equality and is obtained from $\boldsymbol{A}^{e}$ by adjoining the axiom $\neg\mathfrak{A}$, and $\boldsymbol{A}'^{e}$ is contradictory, accordingly $\boldsymbol{A}'$ is LK-contradictory. Hence, there exists an axiom sequence Γ_0 of $\boldsymbol{A}$ and

$$\neg\mathfrak{A}, \Gamma_0 \rightarrow$$

is LK-provable, i. e.

$$\Gamma_0 \rightarrow \mathfrak{A}$$

is LK-provable. Hence, $\mathfrak{A}$ is LK-provable from $\boldsymbol{A}$, q. e. d.

Appendix.

In this appendix, we shall consider the ε-symbol on proposition. For this purpose, we use the extended predicate calculus named L_0K in my previous paper [4].

A.1. THE PREDICATE CALCULUS 'L_0K'.

The predicate calculus L_0K is obtained from Gentzen's LK (cf. 1.7) by extending the conception of formula and rules of inference, as follows:

A.11. To the definition of formula in LK (cf. 1.11 and Example 3, 1.7), we adjoin the two following items:

A.111. If $\mathfrak{a}^0$ is a free propositional variable, then $\mathfrak{a}^0$ is a *formula.*

A.112. If $\mathfrak{F}(\mathfrak{a}^0)$ is a formula, $\mathfrak{a}^0$ is a free propositional variable, $\mathfrak{x}$ is a bound variable, and $\mathfrak{F}(\mathfrak{x})$ is the result of substituting $\mathfrak{x}$ for $\mathfrak{a}^0$ throughout $\mathfrak{F}(\mathfrak{a}^0)$, then $\forall^0\mathfrak{x}\mathfrak{F}(\mathfrak{x})$ and $\exists^0\mathfrak{x}\mathfrak{F}(\mathfrak{x})$ are *formulae.*

A.12. Additional logical rules of inference for L_0K.

Introduction of	in antecedent.	in succedent.
$\forall^0$:	$\dfrac{\mathfrak{F}(\mathfrak{A}), \Gamma \rightarrow \Theta}{\forall^0\mathfrak{x}\mathfrak{F}(\mathfrak{x}), \Gamma \rightarrow \Theta}$	$\dfrac{\Gamma \rightarrow \Theta, \mathfrak{F}(\mathfrak{a}^0)}{\Gamma \rightarrow \Theta, \forall^0\mathfrak{x}\mathfrak{F}(\mathfrak{x})}$
$\exists^0$:	$\dfrac{\mathfrak{F}(\mathfrak{a}^0), \Gamma \rightarrow \Theta}{\exists^0\mathfrak{x}\mathfrak{F}(\mathfrak{x}), \Gamma \rightarrow \Theta}$	$\dfrac{\Gamma \rightarrow \Theta, \mathfrak{F}(\mathfrak{A})}{\Gamma \rightarrow \Theta, \exists^0\mathfrak{x}\mathfrak{F}(\mathfrak{x})}$

where $\forall^0\mathfrak{x}\mathfrak{F}(\mathfrak{x})$ or $\exists^0\mathfrak{x}\mathfrak{F}(\mathfrak{x})$ is an arbitrary formula of such form, and at this time $\mathfrak{F}(\mathfrak{A})$ or $\mathfrak{F}(\mathfrak{a}^0)$ is the result of substituting an arbitrary formula $\mathfrak{A}$ or a free propositional variable $\mathfrak{a}^0$ for all of the free occurrences of the bound variable $\mathfrak{x}$ in $\mathfrak{F}(\mathfrak{x})$, respectively; and Γ and Θ are arbitrary finite sequences of formulae.

Restriction on variable: The free propositional variable $\mathfrak{a}^0$ of the $\forall^0$-succedent or the $\exists^0$-antecedent shall not occur in its conclusion.

A.2. THEOREM. *Let* $\varepsilon^0 \mathfrak{x} \mathfrak{F}(\mathfrak{x})$ *be the abbreviation of*

$$\forall^0 \mathfrak{x} (\mathfrak{x} \supset \mathfrak{F}(\mathfrak{x}))\,.$$

Then any sequent of the form

$$\mathfrak{F}(\mathfrak{A}) \rightarrow \mathfrak{F}(\varepsilon^0 \mathfrak{x} \mathfrak{F}(\mathfrak{x}))$$

is L_0K-provable.

If we regard $\varepsilon^0 \mathfrak{x} \mathfrak{F}(\mathfrak{x})$ as the abbreviation of

$$\exists^0 \mathfrak{x} (\mathfrak{x} \& \mathfrak{F}(\mathfrak{x}))\,,$$

then the similar theorem is proved, too.

References.

[1] G. Gentzen, Untersuchungen über das logische Schliessen. Math. Zeitschr., 39, 176–210, 405–431 (1935).

[2] D. Hilbert, Die logischen Grundlagen der Mathematik. Math. Ann., 88, 151–165 (1923).

[3] D. Hilbert and W. Ackermann, Grundzüge der theoretischen Logik. Berlin (1928).

[4] S. Maehara, Gentzen's theorem on an extended predicate calculus. Proc. Jap. Acad., 30, 923–926 (1954).

Notes.

1) Main theorems 1 (4.1) and 2 (4.2).

2) Cf. Gentzen [1].

3) Cf., for example, Hilbert [2]. But in it, he used the 'τ-symbol', instead of the ε-symbol, that makes $\tau x F(x)$ mean $\varepsilon x \neg F(x)$, where $\neg F(x)$ means 'not $F(x)$'.

4) For example, 'engerer Funktionenkalkül' in Hilbert and Ackermann [3].

5) An individual is regarded as a function of zero argument place.

6) A proposition is regarded as a predicate of zero argument place.

7) We call each of them a *free occurrence of* the bound variable $\mathfrak{x}$ in $\mathfrak{F}(\mathfrak{x})$.

8) For a formula $\forall \mathfrak{x} \mathfrak{F}(\mathfrak{x})$, $\exists \mathfrak{x} \mathfrak{F}(\mathfrak{x})$ or term $\varepsilon \mathfrak{x} \mathfrak{F}(\mathfrak{x})$, there may be the freedom of choosing the formula $\mathfrak{F}(\mathfrak{a})$, but the totality of the subterms of the latter which are those of the former is independent of the way how choose the latter.

9) Cf. Gentzen [1] and Example 4 (1.7).

10) They are the abbreviations of 'introduction of $\forall$ in succedent' and 'introduction of $\exists$ in antecedent', respectively.

11) For the sake of this, for example, it suffices that the grade of $\mathfrak{f}(\tau_i)$ is not less than those of $\mathfrak{f}(\tau_{i+1}), \ldots, \mathfrak{f}(\tau_n)$.

12) This sequence is in general not uniquely determined.

13) There exist in general more than one sequent which are described as $\mathfrak{S}^*$ (cf. note 12)). Yet, if one of them is LK-provable, then any one of them is so.

14) In such a case, $\mathfrak{J}$ is an $\mathfrak{V}$-antecedent (cf. 2.11 and 2.12).