On exact sequences of Hochschild and Serre.

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The cohomology theory of groups was enriched recently by the introduction by Hochschild and Serre of the theory of spectral sequences, a powerful tool in algebraic topology, into this domain. Hochschild and Serre introduced this method to study the cohomology of group extensions, and, as applications of the general theory, obtained two exact sequences, the one extending the fundamental propositions of Galois cohomology, and the other giving the cup product reduction theorem.[1] However, those who are interested in these concrete results mainly, may feel desirable to have a direct proof of these results, independent of the spectral sequence mechanism. We shall give such a proof in the present paper.

§ 1. We shall make use of the following notations. $\mathfrak{g}$ is a group, and $\mathfrak{h}$ its normal subgroup. $Z(\mathfrak{g})$ denotes the groupring of $\mathfrak{g}$ over the ring Z of rational integers; then every $\mathfrak{g}$-module ($\mathfrak{g}$-group) is canonically considered as a $Z(\mathfrak{g})$-module. If A and B are $\mathfrak{g}$-modules, the group $\mathrm{Hom}\,(B, A)$ of all additive homomorphisms of B into A is considered as a $\mathfrak{g}$-module, by

$$(\sigma\varphi)(b) = \sigma\varphi(\sigma^{-1}b)\,,$$

where $\sigma \in \mathfrak{g}$, $b \in B$ and $\varphi \in \mathrm{Hom}\,(B, A)$. We define the cohomology groups $H^p(\mathfrak{g}, A)$ of $\mathfrak{g}$ with respect to a $\mathfrak{g}$-module A as usual.[2] The notions such as cochains, cocycles and the coboundary operator will be used only with respect to the so-called non-homogeneous complex. The cohomology class of a cocycle f will be denoted by $<f>$. $^{\mathfrak{h}}A$ denotes the set of $\mathfrak{h}$-invariants of A, considered canonically as $\mathfrak{g}/\mathfrak{h}$-module. We denote by λ the lift:

1) G. Hochschild and J.-P. Serre, *Cohomology of group extensions*, Trans. Amer. Math. Soc. 74 (1953), pp. 110–134.

2) See, S. Eilenberg and S. MacLane, *Cohomology theory in abstract groups*. I, Ann. of Math. 48, (1947), pp. 51–78, and G. Hochschild and J.-P. Serre, loc. cit. in 1).

$$\lambda: \quad H^p(\mathfrak{g}/\mathfrak{h}, {}^{\mathfrak{h}}A) \to H^p(\mathfrak{g}, A),$$

and by ρ the restriction:

$$\rho: \quad H^p(\mathfrak{g}, A) \to H^p(\mathfrak{h}, A).$$

We use λ and ρ equally to denote the underlying transformations of non-homogeneous cochains, thus

$$\lambda f(\sigma_1, \cdots, \sigma_p) = f(\bar{\sigma}_1, \cdots, \bar{\sigma}_p) \quad \text{for} \quad f \in C^p(\mathfrak{g}/\mathfrak{h}, {}^{\mathfrak{h}}A),$$

$$\rho g(\eta_1, \cdots, \eta_p) = g(\eta_1, \cdots, \eta_p) \quad \text{for} \quad g \in C^p(\mathfrak{g}, A),$$

where $\sigma_i \in \mathfrak{g}$, $\eta_i \in \mathfrak{h}$, $\bar{\sigma}$ is the coset of σ by $\mathfrak{h}$, and C^p denotes the group of p-cochains. We have $\rho \cdot \lambda = 0$.

Our principal tool is the following *simultaneous reductions* of cohomology groups of $\mathfrak{g}$, $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$. B being any (additive) abelian group, let us denote by $B^{\mathfrak{g}}$ the group of all additive homomorphisms of $Z(\mathfrak{g})$ into B, and convert it into a $\mathfrak{g}$-module by setting

$$(\sigma\varphi)(r) = \varphi(r\sigma),$$

where $\sigma \in \mathfrak{g}$, $\varphi \in B^{\mathfrak{g}}$, $r \in Z(\mathfrak{g})$. A $\mathfrak{g}$-module is said to be $\mathfrak{g}$-*regular* if it is $\mathfrak{g}$-isomorphic with some $B^{\mathfrak{g}}$; any $\mathfrak{g}$-regular module is cohomologically trivial, i. e.

$$H^p(\mathfrak{g}, B^{\mathfrak{g}}) = 0, \qquad p = 1, 2, \cdots.$$

A $\mathfrak{g}$-module A may be considered as a submodule of $\mathfrak{g}$-regular module $A^{\mathfrak{g}}$, by identifying every element $a \in A$ with the homomorphism $r \to ra$, ($r \in Z(\mathfrak{g})$). Put $A^* = A^{\mathfrak{g}}/A$. Then, from the exact sequence of cohomology groups associated to the module extension

(1) $$0 \to A \to A^{\mathfrak{g}} \to A^* \to 0,$$

we have, in virtue of the regularity of $A^{\mathfrak{g}}$, the isomorphisms

(2) $$H^{p-1}(\mathfrak{g}, A^*) \cong H^p(\mathfrak{g}, A), \qquad p = 2, 3, \cdots.$$

The extension (1) is an *enlargement*; namely, $A^{\mathfrak{g}}$, as an abelian group, is a direct sum of A and a representative subgroup, say R. We denote by b' the representative in R of $b \in A^*$. Putting

$$s(\sigma)(b)=\sigma(\sigma^{-1}b)' - b'$$

for $\sigma \in \mathfrak{g}$, $b \in A^*$, we have a cocycle $s \in Z^1(\mathfrak{g}, \mathrm{Hom}\,(A^*, A))$, and the cohomology class $\langle s \rangle$ is determined uniquely, independent of the choice of R. It is natural and convenient to take as R the subgroup composed of all homomorphisms φ such that $\varphi(1)=0$. With these notations, the isomorphisms (2) are realized by taking the cup product with $\langle s \rangle$:

$$\langle g \rangle \to \langle s \rangle \cup \langle g \rangle \quad \text{for} \quad \langle g \rangle \in H^{p-1}(\mathfrak{g}, A^*) ,$$

relative to the natural pairing $\mathrm{Hom}\,(A^*, A) \cup A^* \to A$.

If we restrict our operator group to $\mathfrak{h}$, then $A^{\mathfrak{g}}$ remains regular as $\mathfrak{h}$-module, and we have, from (1) and with the same R as above, the exact sequence

$$(3) \qquad 0 \to {}^{\mathfrak{g}}A \to {}^{\mathfrak{g}}(A^{\mathfrak{g}}) \to {}^{\mathfrak{g}}(A^*) \to H^1(\mathfrak{h}, A) \to 0 ,$$

and the isomorphisms

$$(4) \qquad
\begin{aligned}
&H^{p-1}(\mathfrak{h}, A^*) \cong H^p(\mathfrak{h}, A) , \qquad p=2, 3, \cdots , \\
&\langle h \rangle \to \rho\langle s \rangle \cup \langle h \rangle .
\end{aligned}$$

Since ρ preserves the cup product, the following diagram is commutative:

$$(5) \qquad
\begin{array}{ccc}
H^{p-1}(\mathfrak{g}, A^*) & \xrightarrow{\ \rho\ } & H^{p-1}(\mathfrak{h}, A^*) \\
\Big\downarrow{\scriptstyle \langle s \rangle \cup} & & \Big\downarrow{\scriptstyle \rho\langle s \rangle \cup} \\
H^p(\mathfrak{g}, A) & \xrightarrow{\ \rho\ } & H^p(\mathfrak{h}, A) .
\end{array}$$

Now assume that $H^1(\mathfrak{h}, A)=0$; then (3) reduces to

$$0 \to {}^{\mathfrak{g}}A \to {}^{\mathfrak{g}}(A^{\mathfrak{g}}) \to {}^{\mathfrak{g}}(A^*) \to 0 ,$$

which may be considered as an enlargement of $\mathfrak{g}/\mathfrak{h}$-modules. Moreover ${}^{\mathfrak{g}}(A^{\mathfrak{g}})$ is $\mathfrak{g}/\mathfrak{h}$-regular. Hence we have third reduction

$$(6) \qquad
\begin{aligned}
&H^{p-1}(\mathfrak{g}/\mathfrak{h}, {}^{\mathfrak{g}}A^*) \cong H^p(\mathfrak{g}/\mathfrak{h}, {}^{\mathfrak{g}}A) , \qquad p=2, 3, \cdots , \\
&\langle f \rangle \to \langle \bar{s} \rangle \cup \langle f \rangle .
\end{aligned}$$

$\bar{s} \in Z^1(\mathfrak{g}/\mathfrak{h}, \mathrm{Hom}\,(^{\mathfrak{h}}A^*, {}^{\mathfrak{h}}A))$ satisfies

(7) $$\lambda(\bar{s} \cup f) = s \cup \lambda f,$$

and we see that the following diagram is commutative

(8)
$$
\begin{array}{ccc}
H^{p-1}(\mathfrak{g}/\mathfrak{h},\,{}^{\mathfrak{h}}A^*) & \xrightarrow{\;\lambda\;} & H^{p-1}(\mathfrak{g}, A^*) \\
\downarrow{\scriptstyle <\bar{s}> \cup} & & \downarrow{\scriptstyle <s> \cup} \\
H^p(\mathfrak{g}/\mathfrak{h},\,{}^{\mathfrak{h}}A) & \xrightarrow{\;\lambda\;} & H^p(\mathfrak{g}, A).
\end{array}
$$

§ 2.   The notations being as above, a cocycle $h \in Z^p(\mathfrak{h}, A)$ is called *transgressive* if there exist $g \in C^p(\mathfrak{g}, A)$ and $f \in Z^{p+1}(\mathfrak{g}/\mathfrak{h}, {}^{\mathfrak{h}}A)$ such that

$$h = \rho g, \qquad \delta g = \lambda f.$$

We may call the cohomology class $<h>$ *transgressive*, as a cocycle cohomologous to a transgressive cocycle is also transgressive. The class $<f>$ is not uniquely determined by $<h>$, but depends on the choice of g. The zero class of $H^p(\mathfrak{h}, A)$ is obviously transgressive, and it determines a subgroup T of $H^{p+1}(\mathfrak{g}/\mathfrak{h}, {}^{\mathfrak{h}}A)$ by the above process, and every transgressive class $<h>$ determines a coset of $H^{p+1}(\mathfrak{g}/\mathfrak{h}, {}^{\mathfrak{h}}A)$ by T. Hence, denoting by $H_t^p(\mathfrak{h}, A)$ the subgroup of transgressive classes of $H^p(\mathfrak{h}, A)$ and by $\bar{H}^{p+1}(\mathfrak{g}/\mathfrak{h}, {}^{\mathfrak{h}}A)$ the factor group of $H^{p+1}(\mathfrak{g}/\mathfrak{h}, {}^{\mathfrak{h}}A)$ by T, we can naturally define a homomorphism, called the *transgression*,

$$\tau: \quad H_t^p(\mathfrak{h}, A) \to \bar{H}^{p+1}(\mathfrak{g}/\mathfrak{h}, {}^{\mathfrak{h}}A).$$

It follows from the definition that $\rho(H^p(\mathfrak{g}, A)) \subset H_t^p(\mathfrak{h}, A)$ and that the sequence

$$H^p(\mathfrak{g}, A) \xrightarrow{\rho} H_t^p(\mathfrak{h}, A) \xrightarrow{\tau} \bar{H}^{p+1}(\mathfrak{g}/\mathfrak{h}, {}^{\mathfrak{h}}A) \xrightarrow{\lambda} H^{p+1}(\mathfrak{g}, A)$$

is exact.

As is well known, $\mathfrak{g}$ operates on $H^p(\mathfrak{h}, A)$: for a (non-homogeneous) p-cochain $h \in C^p(\mathfrak{h}, A)$, we define a new cochain σh, $\sigma \in \mathfrak{g}$, by

$$(\sigma h)(\eta_1, \cdots, \eta_p) = \sigma h(\eta_1^\sigma, \cdots, \eta_p^\sigma),$$

where $\eta_i \in \mathfrak{h}$, $\eta^\sigma = \sigma^{-1} \eta \sigma$; then the cochain transformation $h \to \sigma h$ induces in an obvious way an automorphism of $H^p(\mathfrak{h}, A)$ denoted also by σ.

We shall denote the subgroup of $\mathfrak{g}$-invariant classes of $H^p(\mathfrak{h}, A)$ by $\mathfrak{g}H^p(\mathfrak{h}, A)$, in accordance with our convention in §1. If $\mathfrak{h}$ coincides with $\mathfrak{g}$, this automorphism reduces to the identity, a fact verified usually by some chain homotopy. A slight modification of it will show that *any transgressive class is* $\mathfrak{g}$-*invariant*.[3] Namely, if $h = \rho g$, $\delta g = \lambda f$, we have $\sigma h - h = \delta k_\sigma$ with

$$k_\sigma(\eta_1, \cdots, \eta_{p-1}) = \sum_{i=1}^{p-1} (-1)^i g(\eta_1, \cdots, \eta_i, \sigma, \eta_{i+1}^\sigma, \cdots, \eta_{p-1}'') .$$

Hence $H_t^p(\mathfrak{h}, A)$ is a subgroup of $\mathfrak{g}H^p(\mathfrak{h}, A)$.

We notice also that by the reduction (4) we have in particular

$$(4') \qquad\qquad \mathfrak{g}H^{p-1}(\mathfrak{h}, A^*) \cong \mathfrak{g}H^p(\mathfrak{h}, A), \qquad p = 2, 3, \cdots,$$

as is seen from $\sigma(h_1 \cup h_2) = \sigma h_1 \cup \sigma h_2$ and $\sigma s \sim s$.

Now, we are ready to prove

THEOREM.[4] *Let* $p \geq 1$. *If* $H^i(\mathfrak{h}, A) = 0$, $i = 1, \cdots, p-1$, *then* $H_t^p(\mathfrak{h}, A)$ *coincides with* $\mathfrak{g}H^p(\mathfrak{h}, A)$, T *reduces to* 0, *and the sequence*

$$0 \to H^p(\mathfrak{g}/\mathfrak{h}, {}^\mathfrak{g}A) \xrightarrow{\lambda} H^p(\mathfrak{g}, A) \xrightarrow{\rho} \mathfrak{g}H^p(\mathfrak{h}. A)$$

$$\xrightarrow{\tau} H^{p+1}(\mathfrak{g}/\mathfrak{h}, {}^\mathfrak{g}A) \xrightarrow{\lambda} H^{p+1}(\mathfrak{g}, A)$$

is exact.

PROOF. The case $p = 1$ is easily proved. So we assume $p > 1$, and the theorem to be valid for $p-1$. We can utilize the simultaneous reductions mentioned in the preceding section, as the hypothesis of the theorem includes now $H^1(\mathfrak{h}, A) = 0$. Every class of $\mathfrak{g}H^p(\mathfrak{h}, A)$ has the form $\rho \langle s \rangle \cup \langle h \rangle$, $\langle h \rangle \in \mathfrak{g}H^{p+1}(\mathfrak{h}, A^*)$, by (4) and (4'). As we have $H^i(\mathfrak{h}, A^*) = 0$, $i = 1, \cdots, p-2$, h is transgressive by the induction assumption :

$$h = \rho g, \qquad \delta g = \lambda f,$$

and we have by (7)

$$\rho s \cup h = \rho(s \cup g),$$

$$\delta(s \cup g) = -s \cup \lambda f = -(\bar{s} \cup f);$$

3) See Appendix.
4) G. Hochschild and J.-P. Serre, loc. cit., Chap. III, § 4, Theorem 2.

hence every class of $^{\mathfrak{g}}H^p(\mathfrak{h}, A)$ is transgressive. Moreover, if we denote the mapping

$$\rho <s> \cup <h> \rightarrow -<\bar{s}> \cup \tau <h>$$

by τ_p for a moment, the diagram

$$H^{p-1}(\mathfrak{h}, A^*) \xrightarrow{\ (-1)^{p-1}\tau\ } H^p(\mathfrak{g}/\mathfrak{h}, {}^{\mathfrak{g}}A^*)$$

$$\downarrow{\rho<s>\cup} \qquad\qquad \downarrow{<\bar{s}>\cup}$$

$$H^p(\mathfrak{h}, A) \xrightarrow{\ (-1)^p\tau_p\ } H^{p+1}(\mathfrak{g}/\mathfrak{h}, {}^{\mathfrak{g}}A)$$

is commutative. Since the sequence

$$0 \rightarrow H^{p-1}(\mathfrak{g}/\mathfrak{h}, {}^{\mathfrak{g}}A^*) \xrightarrow{\lambda} H^{p-1}(\mathfrak{g}, A^*) \xrightarrow{\rho} {}^{\mathfrak{g}}H^{p-1}(\mathfrak{h}, A^*)$$

$$\xrightarrow{\tau} H^p(\mathfrak{g}/\mathfrak{h}, {}^{\mathfrak{g}}A^*) \xrightarrow{\lambda} H^p(\mathfrak{g}, A^*)$$

is exact by our induction assumption, it follows immediately from the commutativity of (5), (8), (9) that the sequence of the theorem is exact, τ replaced by τ_p. So it remains only to show that T reduces to 0. Thus, assume for $g \in C^p(\mathfrak{g}, A)$ that

$$\rho g = 0, \qquad \delta g = \lambda f, \qquad f \in Z^{p+1}(\mathfrak{g}/\mathfrak{h}, {}^{\mathfrak{g}}A).$$

As $\lambda <f> = 0$, there exists $<h> \in {}^{\mathfrak{g}}H^{p-1}(\mathfrak{h}, A)$ such that $<f> = \tau_p <h>$, by the exactness just proved. Hence, with a suitable $h \in <h>$, we have

$$h = \rho g', \qquad \delta g' = \lambda f', \qquad f' \sim f.$$

Put $f' - f = \delta k$, and let $h_1 = g' - g - \lambda k$, then we have

$$\delta h_1 = 0, \qquad \rho h_1 = h,$$

which implies $<h> \in \rho H^{p-1}(\mathfrak{g}, A)$. Hence we have $<f> = \tau_p <h> = 0$, which completes our proof.

THEOREM.[5] *Let $p \geq 1$. If $H^i(\mathfrak{h}, A) = 0$, $i = 2, \cdots, p$, then the following sequence, with suitably defined homomorphisms μ, κ, is exact:*

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5) G. Hochschild and J.-P. Serre, loc. cit., Chap. III, 6, Theorem 3.

$$H^p(\mathfrak{g}/\mathfrak{h}, \, {}^\mathfrak{g}A) \xrightarrow{\lambda} H^p(\mathfrak{g}, A) \xrightarrow{\mu} H^{p-1}(\mathfrak{g}/\mathfrak{h}, H^1(\mathfrak{h}, A))$$

$$\xrightarrow{\kappa} H^{p+1}(\mathfrak{g}/\mathfrak{h}, \, {}^\mathfrak{g}A) \xrightarrow{\lambda} H^{p+1}(\mathfrak{g}, A).$$

PROOF. If $p=1$, it suffices to take the exact sequence of the preceding theorem with $\mu=\rho$, $\kappa=\tau$. Let $p>1$, then, the reduction made to A as above, the hypothesis of the theorem takes the form

$$H^i(\mathfrak{h}, A^*)=0, \qquad i=1, \cdots, p-1.$$

Hence, by (an elementary part of) the preceding theorem,

$$\lambda: \quad H^{p-1}(\mathfrak{g}/\mathfrak{h}, \, {}^\mathfrak{g}A^*) \rightarrow H^{p-1}(\mathfrak{g}, A^*)$$

is an isomorphism onto, and we can define μ by the commutativity of the diagram

(10)

$$
\begin{array}{ccc}
H^{p-1}(\mathfrak{g}/\mathfrak{h}, \, {}^\mathfrak{q}A^*) & \xrightarrow{\beta^*} & H^{p-1}(\mathfrak{g}/\mathfrak{h}, H^1(\mathfrak{h}, A)) \\
\downarrow{\lambda} & & \\
H^{p-1}(\mathfrak{g}, A^*) & & \Big|1 \\
\downarrow{<s>\cup} & & \\
H^p(\mathfrak{g}, A) & \xrightarrow{\mu} & H^{p-1}(\mathfrak{g}/\mathfrak{h}, H^1(\mathfrak{h}, A)),
\end{array}
$$

where the upper horizontal line is a section of the exact sequence of cohomology groups

(11)

$$\rightarrow H^{p-1}(\mathfrak{g}/\mathfrak{h}, \, {}^\mathfrak{g}A^\mathfrak{q}/{}^\mathfrak{g}A) \xrightarrow{\alpha^*} H^{p-1}(\mathfrak{g}/\mathfrak{h}, \, {}^\mathfrak{g}A^*) \xrightarrow{\beta^*} H^{p-1}(\mathfrak{g}/\mathfrak{h}, H^1(\mathfrak{h}, A))$$

$$\xrightarrow{\delta^*} H^p(\mathfrak{g}/\mathfrak{h}, \, {}^\mathfrak{g}A^\mathfrak{q}/{}^\mathfrak{g}A) \xrightarrow{\alpha^*} H^p(\mathfrak{g}/\mathfrak{h}, \, {}^\mathfrak{g}A^*)$$

associated to the $\mathfrak{g}/\mathfrak{h}$-module extension

(3') $$0 \rightarrow ({}^\mathfrak{g}A^\mathfrak{q})/{}^\mathfrak{g}A \xrightarrow{\alpha} {}^\mathfrak{g}A^* \xrightarrow{\beta} H^1(\mathfrak{h}, A) \rightarrow 0.$$

On the other hand, 1-cohomology class $<s^*>$ determined by the enlargement

$$0 \rightarrow {}^\mathfrak{g}A \rightarrow {}^\mathfrak{g}(A^\mathfrak{q}) \rightarrow ({}^\mathfrak{g}A^\mathfrak{q})/{}^\mathfrak{g}A \rightarrow 0$$

gives us the following reduction

$$H^p(\mathfrak{g}/\mathfrak{h}, {}^{\mathfrak{g}}A^{\mathfrak{g}}/{}^{\mathfrak{h}}A) \cong H^{p+1}(\mathfrak{g}/\mathfrak{h}, {}^{\mathfrak{h}}A),$$

$$\langle k \rangle \rightarrow \langle s^* \rangle \cup \langle k \rangle,$$

and the diagram

(12)

$$
\begin{array}{ccc}
H^p(\mathfrak{g}/\mathfrak{h}, {}^{\mathfrak{g}}A^{\mathfrak{g}}/{}^{\mathfrak{h}}A) & \xrightarrow{\ \omega^*\ } & H^p(\mathfrak{g}/\mathfrak{h}, {}^{\mathfrak{h}}A^*) \\
\Big\downarrow & & \Big\downarrow \lambda \\
\langle s^* \rangle \cup & & H^p(\mathfrak{g}, A^*) \\
\Big\downarrow & & \Big\downarrow \langle s \rangle \cup \\
H^{p+1}(\mathfrak{g}/\mathfrak{h}, {}^{\mathfrak{h}}A) & \xrightarrow{\ \lambda\ } & H^{p+1}(\mathfrak{g}, A)
\end{array}
$$

is commutative. If we put $\kappa(x) = \langle s^* \rangle \cup \delta^* x$ for $x \in H^{p-1}(\mathfrak{g}/\mathfrak{h}, H^1(\mathfrak{h}, A))$, we can map the exact sequence (11) isomorphically to the sequence of the theorem, in virtue of the commutativity of (10) and (12).[6]

Appendix.

To see that every transgressive class is $\mathfrak{g}$-invariant, we utilized in §2 some chain homotopy. This mechanism is generalized by Hochschild and Serre to "a general identity",[7] which is necessary for interpretation of the spectral sequence. Here we shall briefly indicate how this identity is treated from the axiomatic point of view.

Thus, the cohomology theory of $\mathfrak{g}$ with respect to a $\mathfrak{g}$-module A is defined to be that of a $\mathfrak{g}$-*free* $\mathfrak{g}$-*complex acyclic with respect to the augmentation by* Z, the ring of integers; any two such complexes give rise to canonically equivalent theories. The usual non-homogeneous theory makes use of the *non-homogeneous complex* $\mathfrak{C}(\mathfrak{g})$, whose group of p-chains $C^p(\mathfrak{g})$ has a $\mathfrak{g}$-basis denoted by $[\sigma_1, \cdots, \sigma_p]$, σ's running over $\mathfrak{g}$. We consider $\mathfrak{C}(\mathfrak{g})$ as a *double* $\mathfrak{g}$-*complex*, by defining the right operation by

$$[\sigma_1, \cdots, \sigma_p]\tau = \tau[\sigma_1^\tau, \cdots, \sigma_p^\tau],$$

where $\tau \in \mathfrak{g}$ and $\sigma^\tau = \tau^{-1}\sigma\tau$. This right operation turns out to be a chain

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6) The case $p=1$ can be deduced from (3) in a similar way as this proof.
7) Chap. II, § 3.

equivalence of $\mathfrak{C}(\mathfrak{g})$ with itself, which implies the triviality of $\mathfrak{g}$-operations on $H(\mathfrak{g}, A)$.

We have a canonical[8] isomorphism

$$H(\mathfrak{C}(\mathfrak{h}), A) \cong H(Z(\mathfrak{g}) \otimes_{\mathfrak{h}} \mathfrak{C}(\mathfrak{h}), A),$$

where $Z(\mathfrak{g}) \otimes_{\mathfrak{h}} \mathfrak{C}(\mathfrak{h})$[9] is a $\mathfrak{g}$-free complex acyclic with respect to the augmentation by $Z(\mathfrak{g}) \otimes_{\mathfrak{h}} Z$; it is also a subcomplex of $\mathfrak{C}(\mathfrak{g})$, invariant by the right operations by $\mathfrak{g}$, so that we may regard $Z(\mathfrak{g}) \otimes_{\mathfrak{h}} \mathfrak{C}(\mathfrak{h})$ as a double $\mathfrak{g}$-complex. Now, consider the *tensor product complex*

$$\mathfrak{P} = (Z(\mathfrak{g}) \otimes_{\mathfrak{h}} \mathfrak{C}(\mathfrak{h})) \otimes_{\mathfrak{g}} \mathfrak{C}(\mathfrak{g}).$$

We see that $\mathfrak{P}$ is a $\mathfrak{g}$-free complex (having as $\mathfrak{g}$-basis the product of $\mathfrak{h}$-basis of $\mathfrak{C}(\mathfrak{h})$ and $\mathfrak{g}$-basis of $\mathfrak{C}(\mathfrak{g})$) with the augmentation $Z \cong (Z(\mathfrak{g}) \otimes_{\mathfrak{h}} Z) \otimes_{\mathfrak{g}} Z$. Hence there exists a chain transformation

$$\gamma : \quad \mathfrak{P} \to \mathfrak{C}(\mathfrak{g})$$

by Cartan's Lemma, which in turn induces a homomorphism

$$\gamma^* : \quad H^p(\mathfrak{g}, A) \to H^p(\mathfrak{P}, A).$$

On the other hand, if we denote the canonical isomorphism

$$\text{Hom}^{\mathfrak{g}}\left((Z(\mathfrak{g}) \otimes_{\mathfrak{h}} \mathfrak{C}(\mathfrak{h})) \otimes_{\mathfrak{g}} \mathfrak{C}(\mathfrak{g}), A\right) \cong \text{Hom}^{\mathfrak{g}}\left(\mathfrak{C}(\mathfrak{g}), \text{Hom}^{\mathfrak{g}}(Z(\mathfrak{g}) \otimes_{\mathfrak{h}} \mathfrak{C}(\mathfrak{h}), A)\right)$$

by $g \to g'$, we have by definitions

$$(\delta g)'(\zeta_p) = \delta(g'(\zeta_p)) + (-1)^p (\delta g')(\zeta_p)$$

for $\zeta_p \in C^p(\mathfrak{g})$. We define the homomorphism

$$\gamma^p : \quad C^{p+q}(\mathfrak{g}, A) \to C^p(\mathfrak{g}, C^q(\mathfrak{h}, A))$$

by

$$f \to (\gamma^p f)(\zeta_p)(\eta_q) = f(\gamma(\eta_q \otimes \zeta_p)),$$

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8) The homology and the cohomology theory can be built up based on a few fundamental properties of tensor product and the group of homomorphisms, as is shown e.g. in S. Eilenberg and N. Steenrod, *Foundations of algebraic topology*. Princeton University Press, 1952. $\text{Hom}^{\mathfrak{g}}(B \otimes_{\mathfrak{h}} A, C) \cong \text{Hom}^{\mathfrak{h}}(A, \text{Hom}^{\mathfrak{g}}(B, C))$ is one of them, where A is an $\mathfrak{h}$-module, C a $\mathfrak{g}$-module and B a $(\mathfrak{g}, \mathfrak{h})$-double module.

9) $Z(\mathfrak{g})$ is considered as a right $\mathfrak{g}$-module by the right multiplication.

where $\zeta_p \in C^p(\mathfrak{g})$, $\eta_q \in C^q(\mathfrak{h})$. Then we have

(*) $\gamma^p(\delta f)(\zeta_p) = \delta(\gamma^p f(\zeta_p)) + (-1)^{q+1} \delta(\gamma^{p-1} f)(\zeta_p)$.

A concrete form of γ^p is supplied by $f \to f_j$ of Hochschild and Serre,[7] and then (*) is essentially equivalent to their *Proposition 2*. The chain homotopy used in § 2 is the case $p=1$ of (*). The chain transformation γ that leads to this γ^p may be called the *shuffling product* of $\mathfrak{C}(\mathfrak{h})$ and $\mathfrak{C}(\mathfrak{g})$ into $\mathfrak{C}(\mathfrak{g})$; though we do not give here its explicit form in view of its somewhat complicated nature, this will be clear from the form of γ^p, or will be easily found in generalizing the above mentioned chain homotopy.[10]

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10) Cf. S. Eilenberg and S. MacLane, *On the groups $H(\Pi, n)$*, I. Ann. of Math. 58 (1953), pp. 55-106, where the case of abelian groups is treated.

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