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Borel's direction of a meromorphic function in a unit circle.

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1. Analogue of Biernacki-Rauch's theorem.

Let $f(z)$ be a meromorphic function of finite order $\rho>0$ for $|z|<\infty$, then Valiron[1] proved that there exists a Borel's direction J: $\arg z=\theta_0$, which satisfies the following condition. Let ω: $|\arg z-\theta_0|<\delta$ be any small angular domain, which contains J and $z_\nu(a,\omega)$ be zero points of $f(z)-a$ in ω, multiple zeros being counted only once, then for any $\varepsilon>0$,

$$\sum_\nu \frac{1}{|z_\nu(a,\omega)|^{\rho-\varepsilon}}=\infty$$

with two possible exceptions for a.

If $f(z)$ is of divergence type, then

$$\sum_\nu \frac{1}{|z_\nu(a,\omega)|^{\rho}}=\infty$$

with two possible exceptions for a.

This Valiron's theorem is generalized by Biernacki and Rauch as follows.

Let $g(z)$ be a meromorphic function of order $<\rho$ for $|z|<\infty$, and $z_\nu(f=g,\omega)$ be zero points of $f(z)-g(z)$ in ω, multiple zeros being counted only once, then for any $\varepsilon>0$,

$$\sum_\nu \frac{1}{|z_\nu(f=g,\omega)|^{\rho-\varepsilon}}=\infty$$

1) G. Valiron: Recherches sur le théorème de M. Borel dans la théorie des fonctions méromorphes. Acta Math. 52 (1928). M. Tsuji: On Borel's directions of moromorphic functions of finite order. Tohoku Math. Journ. 2 (1950).

with two possible exceptions for g. (Biernacki).[2]

If $f(z)$ is of divergence type and $g(z)$ be such that $\int^{\infty}\frac{T(r, g)dr}{r^{\rho+1}}<\infty$, where $T(r, g)$ is the Nevanlinna's characteristic function of g, then

$$\sum_{\nu}\frac{1}{|z_{\nu}(f=g, \omega)|^{\rho}}=\infty$$

with two possible exceptions for g. (Rauch).[3]

We shall prove the following analogue of Biernacki-Rauch's theorem for a meromorphic function in a unit circle.

THEOREM 1. *Let $f(z)$ be a meromorphic function of finite order $\rho>0$ in $|z|<1$. Then there exists a point z_0 on $|z|=1$ and a line J through z_0, directed inward of $|z|<1$, which may coincide with the tangent of $|z|=1$ at z_0, which satisfies the following condition.*

Let ω be any small angular domain, which contains J and is bounded by two lines through z_0 and $g(z)$ be a meromorphic function of order $<\rho$ in $|z|<1$ and $z_{\nu}(f=g, \omega)$ be zero points of $f(z)-g(z)$ in ω, multiple zeros being counted only once, then for any $\varepsilon>0$,

$$\sum_{\nu}(1-|z_{\nu}(f=g, \omega)|)^{\rho+1-\varepsilon}=\infty$$

with two possible exceptions for g.

If $f(z)$ is of divergence type and $g(z)$ be such that $\int_0^1 T(r, g)(1-r)^{\rho-1}dr<\infty$, then

$$\sum_{\nu}(1-|z_{\nu}(f=g, \omega)|)^{\rho+1}=\infty$$

with two possible exceptions for g.

2. Some lemmas.

For the proof, we shall use the following lemmas.

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2) M. Biernacki: Sur les directions de Borel des fonctions méromorphes. Acta Math. 56 (1930). M. Tsuji: On Borel's directions of meromorphic functions of finite order, III. Kōdai Math. Seminar Reports. (1950),

3) A. Rauch: Extensions de théorème relatifs aux directions de Borel des fonctions méromorphes. Journ. de Math. 12 (1933). M. Tsuji. l.c. 2).

LEMMA 1.[4] *Let $w(z)$ be meromorphic in $|z|<1$ and*

$$S(r)=\frac{1}{\pi}\iint_{|z|\leq r}\left(\frac{|w'(z)|}{1+|w(z)|^2}\right)^2 rdrd\theta\,, \quad z=re^{i\theta}\,.$$

If the number of zero points of $\prod_{i=1}^{3}(w(z)-a_i)$ in $|z|<1$ be $\leqq n$, where multiple zeros are counted only once, then

$$S(r)\leqq n+\frac{A}{1-r}\,, \qquad 0\leqq r<1\,,$$

where $A>0$ is a constant, which depends on a_1, a_2, a_2 only.

LEMMA 2. *Let $w(z)$ be meromorphic in $|z|<1$ and $\Delta\subset\Delta_0$ be two angular domains, each of which is bounded by two lines through $z=1$, directed inward of $|z|<1$ and $\Delta(r)$, $\Delta_0(r)$ be the part of Δ, Δ_0, which lies in $0<r_0\leqq|z|\leqq r<1$ $\left(r_0\geqq\frac{1}{2}\right)$, where r_0 is so chosen, that the circle $|z|=r_0$ meets the both sides of Δ, Δ_0.*

We put

$$S(r,\Delta)=\frac{1}{\pi}\iint_{\Delta(r)}\left(\frac{|w'(z)|}{1+|w(z)|^2}\right)^2 rdrd\theta\,, \quad z=re^{i\theta}\,,$$

$$T(r,\Delta)=\int_{r_0}^{r}\frac{S(r,\Delta)}{r}\,dr\,.$$

Let $n(r,a;\Delta_0)$ be the number of zero points of $w(z)-a$ in $\Delta_0(r)$, multiple zeros being counted only once and

$$N(r,a;\Delta_0)=\int_{r_0}^{r}\frac{n(r,a;\Delta_0)}{r}\,dr\,.$$

Then

$$S(r,\Delta)\leqq 3\sum_{i=1}^{3}n\left(\frac{r+3}{4},\, a_i;\, \Delta_0\right)+O\left(\log\frac{1}{1-r}\right),$$

$$T(r,\Delta)\leqq 21\sum_{i=1}^{3}N\left(\frac{r+3}{4},\, a_i;\, \Delta_0\right)+O(1)\,.$$

4) J. Dufresnoy: Sur les domains couverts par des valeurs d'une fonction méromorphe ou algébroïde Ann. Ecole Norm. sup. (3), 58 (1941). M. Tsuji. l.c. 1).

PROOF. Let $r_\nu = 1 - \frac{1-r_0}{2^\nu}$ $(\nu = 0, 1, 2, \cdots)$ and for $\nu \geqq 2$, Δ_ν be the part of Δ, which lies in $r_{\nu-1} \leqq |z| \leqq r_\nu$ and Δ_ν^0 be that of Δ_0, which lies in $r_{\nu-2} \leqq |z| \leqq r_{\nu+1}$. We put

$$S_\nu = \frac{1}{\pi} \iint_{\Delta_\nu} \left(\frac{|w'(z)|}{1+|w(z)|^2} \right)^2 r dr d\theta \tag{1}$$

and n_ν^0 be the number of zero points of $\prod_{i=1}^{3} (w(z)-a_i)$ in Δ_ν^0, multiple zeros being counted only once.

Let L be the bisector of the two bounding lines of Δ and z_ν be the common point of L with the circle $|z| = \frac{r_{\nu-1}+r_\nu}{2}$.

We map Δ_ν^0 conformally on $|\zeta| < 1$, such that z_ν becomes $\zeta = 0$, then the image of Δ_ν is contained in $|\zeta| \leqq \lambda < 1$, where λ is a constant, independent of ν.

Hence if we apply Lemma 1, then $S_\nu \leqq n_\nu^0 + K$ (K=const.), so that

$$\sum_{\nu=2}^{n} S_\nu \leqq \sum_{\nu=2}^{n} n_\nu^0 + Kn = \sum_{\nu=2}^{n} n_\nu^0 + O\left(\log \frac{1}{1-r_n}\right).$$

$\sum_{\nu=2}^{n} S_\nu = S(r_n, \Delta) - S(r_1, \Delta)$ and since Δ_ν^0 overlap at most 3-times, we have $\sum_{i=1}^{n} n_\nu^0 \leqq 3 \sum_{i=1}^{3} n(r_{n+1}, a_i; \Delta_0)$, so that

$$S(r_n, \Delta) \leqq 3 \sum_{i=1}^{3} n(r_{n+1}, a_i; \Delta_0) + O\left(\log \frac{1}{1-r_n}\right). \tag{2}$$

If $r_{n-1} \leqq r \leqq r_n$, then $S(r, \Delta) \leqq S(r_n, \Delta)$ and $r_{n+1} = \frac{r_{n-1}+3}{4} \leqq \frac{r+3}{4}$, hence

$$S(r, \Delta) \leqq 3 \sum_{i=1}^{3} n\left(\frac{r+3}{4}, a_i; \Delta_0\right) + O\left(\log \frac{1}{1-r}\right), \tag{3}$$

so that

$$T(r, \Delta) \leqq 3 \sum_{i=1}^{3} \int_{r_0}^{r} \frac{n\left(\frac{r+3}{4}, a_i; \Delta_0\right)}{r} dr + O(1) = 12 \sum_{i=1}^{3} \int_{t_0}^{\frac{r+3}{4}} \frac{n(t, a_i; \Delta_0)}{4t-3} dt + O(1),$$

where $t_0=\frac{r_0+3}{4}\geqq\frac{\frac{1}{2}+3}{4}=\frac{7}{8}$.

Since $4t-3\geqq\frac{4t}{7}$ for $t\geqq\frac{7}{8}$, we have

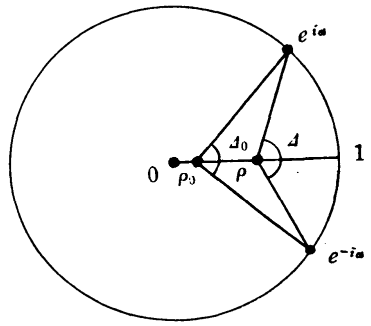
$$T(r,\Delta)\leqq 21\sum_{i=1}^{3}N\Big(\frac{r+3}{4},a_i;\Delta_0\Big)+O(1). \qquad (4)$$

LEMMA 3. *Let $w(z)$ be meromorphic in $|z|<1$ and $\Delta\subset\Delta_0$ be two sectors*

Δ : $|z|\leqq 1$, $|\arg(z-\rho)|\leqq\alpha$, $0<\rho<1$,

Δ_0: $|z|\leqq 1$, $|\arg(z-\rho_0)|\leqq\alpha$, $0<\rho_0<\rho<1$

and $\Delta(r)$, $\Delta_0(r)$ be the part of Δ,Δ_0, which lies in $\frac{1+\rho}{2}=r_0\leqq|z|\leqq r<1$.



Let $S(r,\Delta)$, $T(r,\Delta)$, $n(r,a;\Delta_0)$, $N(r,a;\Delta_0)$ be defined as Lemma 2. Then

$$S(r,\Delta)\leqq 9\sum_{i=1}^{3}n\Big(\frac{r+3}{4},a_i;\Delta_0\Big)+O\Big(\frac{1}{1-r}\Big),$$

$$T(r,\Delta)\leqq 63\sum_{i=1}^{3}N\Big(\frac{r+3}{4},a_i;\Delta_0\Big)+O\Big(\log\frac{1}{1-r}\Big).$$

PROOF. Let $r_\nu=1-\frac{1-r_0}{2^\nu}$ $(\nu=0,1,2,\cdots)$ and for $\nu\geqq 2$,

$$\left.\begin{aligned}\Delta_{\nu,s}:\ &\frac{(s-1)\delta}{2^\nu}\leqq\arg z\leqq\frac{s\delta}{2^\nu},\quad r_{\nu-1}\leqq|z|\leqq r_\nu,\\ &\qquad\qquad (s=0,\pm1,\pm2,\cdots)\\ \Delta^0_{\nu,s}:\ &\frac{(s-2)\delta}{2^\nu}\leqq\arg z\leqq\frac{(s+1)\delta}{2^\nu},\quad r_{\nu-2}\leqq|z|\leqq r_{\nu+1},\end{aligned}\right\} \qquad (1)$$

where we choose $\delta>0$ so small that if $\Delta_{\nu,s}$ has common points with Δ, then $\Delta^0_{\nu,s}$ is contained in Δ_0 and the range of s is such that $\Delta_{\nu,s}$ has common points with Δ, so that the number of such s is $O(2^\nu)$.

Let

$$S_{\nu,s}=\frac{1}{\pi}\iint_{\Delta_{\nu,s}}\left(\frac{|w'(z)|}{1+|w(z)|^2}\right)^2 r\,dr\,d\theta \tag{2}$$

and $n^0_{\nu,s}$ be the number of zero points of $\prod_{i=1}^{3}(w(z)-a_i)$ in $\Delta^0_{\nu s}$, multiple zeros being counted only once.

If we map $\Delta^0_{\nu,s}$ conformally on $|\zeta|<1$, such that the center of $\Delta^0_{\nu,s}$ becomes $\zeta=0$, then the image of $\Delta_{\nu,s}$ is contained in $|\zeta|\leqq\lambda<1$, where λ is a constant, independent of ν and s. Hence if we apply Lemma 1, $S_{\nu,s}\leqq n^0_{\nu,s}+\text{const.}$, so that

$$\sum_{\nu=2}^{n}\sum_{s}S_{\nu,s}\leqq\sum_{\nu=2}^{n}\sum_{s}n^0_{\nu,s}+O(2^n)=\sum_{\nu=2}^{n}\sum_{s}n^0_{\nu,s}+O\left(\frac{1}{1-r_n}\right).$$

$\sum_{\nu=2}^{n}\sum_{s}S_{\nu,s}\geqq S(r_n,\Delta)-S(r_1,\Delta)$ and since $\Delta^0_{\nu,s}$ overlap at most 9-times, $\sum_{\nu=2}^{n}\sum_{s}n^0_{\nu,s}\leqq 9\sum_{i=1}^{3}n(r_{n+1},a_i;\Delta_0)$, so that

$$S(r_n,\Delta)\leqq 9\sum_{i=1}^{3}n(r_{n+1},a_i;\Delta_0)+O\left(\frac{1}{1-r_n}\right). \tag{3}$$

If $r_{n-1}\leqq r\leqq r_n$, then $S(r,\Delta)\leqq S(r_n,\Delta)$ and $r_{n+1}\leqq\frac{r+3}{4}$, hence

$$S(r,\Delta)\leqq 9\sum_{i=1}^{3}n\left(\frac{r+3}{4},a_i;\Delta_0\right)+O\left(\frac{1}{1-r}\right). \tag{4}$$

From this we have as Lemma 2,

$$T(r,\Delta)\leqq 63\sum_{i=1}^{3}N\left(\frac{r+3}{4},a_i;\Delta_0\right)+O\left(\log\frac{1}{1-r}\right).$$

LEMMA 4. *Let* $f(z)=\dfrac{g_1(z)w(z)+g_2(z)}{g_3(z)w(z)+g_4(z)}$, *where* $w(z)$, $g_i(z)$ $(i=1,2,3,4)$ *are meromorphic in* $|z|<1$.

Let $\Delta\subset\Delta_0$ *be two angular domains of Lemma 2. Then*

$$S(r,f;\Delta)\leqq 27S\left(\frac{r+63}{64},w;\Delta_0\right)+O\left(\int_0^{\frac{r+127}{128}}\frac{T(r,g)}{(1-r)^2}dr\right),$$

where

$$T(r,g)=\sum_{i=1}^{4}T(r,g_i).$$

The same relation holds, if $\Delta \subset \Delta_0$ are sectors of Lemma 3, where 27 *is replaced by* 729.

PROOF. Let $\Delta \subset \Delta_0$ be two angular domains of Lemma 2. We define Δ_ν, Δ_ν^0 as before. If we map Δ_ν^0 conformally on $|\zeta| < 1$, such that z_ν becomes $\zeta = 0$, then the image of Δ_ν is contained in $|\zeta| \leqq \rho_0 < 1$, where ρ_0 is a constant, independent of ν.

First we consider two special cases: $f(z) = w(z) + g(z)$, $f(z) = w(z) g(z)$ and first consider the case $f(z) = w(z) + g(z)$.

Let D_ν be the image of $|\zeta| \leqq \dfrac{1+\rho_0}{2}$ in Δ_ν^0 and put

$$\frac{1}{|D_\nu|} \iint_{D_\nu} \log \sqrt{1+|g|^2}\, r dr d\theta = M_\nu, \qquad z = re^{i\theta}, \tag{1}$$

where $|D_\nu|$ denotes the area of D_ν. Then we see easily that

$$\iint_{|\zeta| \leq \frac{1+\rho_0}{2}} \log \sqrt{1+|g|^2}\, \rho d\rho d\varphi \leqq A M_\nu, \quad A = \text{const.}, \quad \zeta = \rho e^{i\varphi}. \tag{2}$$

Let $B > 0$ and E be the set of ρ in $\left[\rho_0, \dfrac{1+\rho_0}{2}\right]$, such that

$$\int_{|\zeta| = \rho} \log \sqrt{1+|g|^2}\, d\varphi > B M_\nu,$$

then

$$B M_\nu \int_E d\rho \leqq \int_E d\rho \int_{|\zeta| = \rho} \log \sqrt{1+|g|^2}\, d\varphi \leqq \frac{1}{\rho_0} \int_E \rho d\rho \int_{|\zeta| = \rho} \log \sqrt{1+|g|^2}\, d\varphi$$

$$\leqq \frac{1}{\rho_0} \iint_{|\zeta| \leq \frac{1+\rho_0}{2}} \log \sqrt{1+|g|^2}\, \rho d\rho d\varphi \leqq \frac{A M_\nu}{\rho_0},$$

so that

$$\int_E d\rho \leqq \frac{A}{B\rho_0},$$

hence if $B > 0$ is sufficiently large, there exists $\rho_1 < \rho_2$, such that $\rho_0 \leqq \rho_1 \leqq \rho_0 + \dfrac{1-\rho_0}{6}$, $\rho_0 + \dfrac{2(1-\rho_0)}{6} \leqq \rho_2 \leqq \dfrac{1+\rho_0}{2}$ and

$$\int_{|\zeta|=\rho_1} \log \sqrt{1+|g|^2}\, d\varphi \leqq BM_\nu\,, \quad \int_{|\zeta|=\rho_2} \log \sqrt{1+|g|^2}\, d\varphi \leqq BM_\nu\,. \tag{3}$$

Let

$$\left.\begin{aligned} S(f, \varDelta_\nu) &= \frac{1}{\pi}\iint_{\varDelta_\nu}\left(\frac{|f'(z)|}{1+|f(z)|^2}\right)^2 r dr d\theta\,, \\ S(f, \rho) &= \frac{1}{\pi}\iint_{|\zeta|\leqq\rho}\left(\frac{|f'|}{1+|f|^2}\right)^2 \rho d\rho d\varphi\,, \quad f' = \frac{df}{d\zeta}\,, \end{aligned}\right\} \tag{4}$$

then

$$S(f, \varDelta_\nu) \leqq S(f, \rho_1) \leqq S(f, \rho_2) \leqq S(f, 1) = S(f, \varDelta_\nu^0)\,.$$

By Nevanlinna's fundamental theorem,

$$\int_{\rho_1}^{\rho_2} \frac{S(f, \rho)}{\rho}\, d\rho = \frac{1}{2\pi}\int_{|\zeta|=\rho_2} \log \sqrt{1+|w+g|^2}\, d\varphi$$

$$-\frac{1}{2\pi}\int_{|\zeta|=\rho_1} \log \sqrt{1+|w+g|^2}\, d\varphi + \int_{\rho_1}^{\rho_2} \frac{n(\rho, w+g, \infty)}{\rho}\, d\rho, \tag{5}$$

$$\int_{\rho_1}^{\rho_2} \frac{S(w, \rho)}{\rho}\, d\rho = \frac{1}{2\pi}\int_{|\zeta|=\rho_2} \log \sqrt{1+|w|^2}\, d\varphi$$

$$-\frac{1}{2\pi}\int_{|\zeta|=\rho_1} \log \sqrt{1+|w|^2}\, d\varphi + \int_{\rho_1}^{\rho_2} \frac{n(\rho, w, \infty)}{\rho}\, d\rho\,. \tag{6}$$

Now by (3),

$$\frac{1}{2\pi}\int_{|\zeta|=\rho_2} \log \sqrt{1+|w+g|^2}\, d\varphi \leqq \frac{1}{2\pi}\int_{|\zeta|=\rho_2} \log \sqrt{1+|w|^2}\, d\varphi$$

$$+\frac{1}{2\pi}\int_{|\zeta|=\rho_2} \log \sqrt{1+|g|^2}\, d\varphi + 1 \leqq \frac{1}{2\pi}\int_{|\zeta|=\rho_2} \log \sqrt{1+|w|^2}\, d\varphi$$

$$+O(M_\nu)+1\,,$$

$$\frac{1}{2\pi}\int_{|\zeta|=\rho_1} \log \sqrt{1+|w|^2}\, d\varphi = \frac{1}{2\pi}\int_{|\zeta|=\rho_1} \log \sqrt{1+|w+g-g|^2}\, d\varphi$$

$$\leqq \frac{1}{2\pi}\int_{|\zeta|=\rho_1} \log \sqrt{1+|w+g|^2}\, d\varphi + \frac{1}{2\pi}\int_{|\zeta|=\rho_1} \log \sqrt{1+|g|^2}\, d\varphi + 1$$

$$\leqq \frac{1}{2\pi}\int_{|\zeta|=\rho_1} \log \sqrt{1+|w+g|^2}\, d\varphi + O(M_\nu) + 1\,,$$

so that

$$\frac{1}{2\pi}\int_{|\zeta|=\rho_2}\log\sqrt{1+|w+g|^2}\,d\varphi-\frac{1}{2\pi}\int_{|\zeta|=\rho_1}\log\sqrt{1+|w+g|^2}\,d\varphi$$

$$\leqq\frac{1}{2\pi}\int_{|\zeta|=\rho_2}\log\sqrt{1+|w|^2}\,d\varphi-\frac{1}{2\pi}\int_{|\zeta|=\rho_1}\log\sqrt{1+|w|^2}\,d\varphi$$

$$+O(M_\nu)+2\,. \tag{7}$$

$$\int_{\rho_1}^{\rho_2}\frac{n(\rho,w+g,\infty)}{\rho}\,d\rho\leqq\int_{\rho_1}^{\rho_2}\frac{n(\rho,w,\infty)}{\rho}\,d\rho+\int_{\rho_1}^{\rho_2}\frac{n(\rho,g,\infty)}{\rho}\,d\rho$$

$$\leqq\int_{\rho_1}^{\rho_2}\frac{n(\rho,w,\infty)}{\rho}\,d\rho+O(n(\rho_2,g,\infty))\,,$$

where $n(\rho, g, \infty)$ is the number of poles of g in $|\zeta|\leqq\rho$.

Now the image of $|\zeta|\leqq\rho_2$ in Δ_ν^0 is contained in $|z|\leqq\tau_\nu\,(<r_{\nu+1})$, such that $r_{\nu+1}-\tau_\nu\geqq\dfrac{\text{const.}}{2^\nu}$, so that if we denote the number of poles of $g(z)$ in $|z|\leqq r$ by $n(r, g, \infty)$, then $n(\rho_2, g, \infty)\leqq n(\tau_\nu, g, \infty)$, so that

$$\int_{\rho_1}^{\rho_2}\frac{n(\rho,w+g,\infty)}{\rho}\,d\rho\leqq\int_{\rho_1}^{\rho_2}\frac{n(\rho,w,\infty)}{\rho}\,d\rho+O(n(\tau_\nu,g,\infty))\,. \tag{8}$$

Hence by (5), (6), (7), (8),

$$\int_{\rho_1}^{\rho_2}\frac{S(f,\rho)}{\rho}\,d\rho\leqq\int_{\rho_1}^{\rho_2}\frac{S(w,\rho)}{\rho}\,d\rho+O(M_\nu)+O(n(\tau_\nu,g,\infty))+2\,.$$

Since

$$S(f,\rho_1)\log\frac{\rho_2}{\rho_1}\leqq\int_{\rho_1}^{\rho_2}\frac{S(f,\rho)}{\rho}\,d\rho\,,\quad\int_{\rho_1}^{\rho_2}\frac{S(w,\rho)}{\rho}\,d\rho\leqq S(w,\rho_2)\log\frac{\rho_2}{\rho_1}\,,$$

we have

$$S(f,\rho_1)\leqq S(w,\rho_2)+O(M_\nu)+O(n(\tau_\nu,g,\infty))+O(1)\,.$$

Since $S(f,\Delta_\nu)\leqq S(f,\rho_1)$ and $S(w,\rho_2)\leqq S(w,\Delta_\nu^0)$,

$$S(f,\Delta_\nu)\leqq S(w,\Delta_\nu^0)+O(M_\nu)+O(n(\tau_\nu,g,\infty))+O(1)\,. \tag{9}$$

Since $\varDelta_\nu \subset D_\nu \subset \varDelta_\nu^0$,

$$M_\nu = \frac{1}{|D_\nu|}\iint_{D_\nu} \log \sqrt{1+|g|^2}\, rdrd\theta \leqq \frac{1}{|\varDelta_\nu|}\iint_{\varDelta_\nu^0} \log \sqrt{1+|g|^2}\, rdrd\theta$$

$$\leqq \text{const. } 2^{2\nu}\int_{r_{\nu-2}}^{r_{\nu+1}} rdr \int_{|z|=r} \log \sqrt{1+|g|^2}\, d\theta \leqq \text{const. } 2^\nu\, T(r_{\nu+1}, g). \quad (10)$$

$$T(r_{\nu+1}, g)+O(1) \geqq \int_{\tau_\nu}^{r_{\nu+1}} \frac{n(r, g, \infty)}{r}\, dr \geqq n(\tau_\nu, g, \infty)(r_{\nu+1}-\tau_\nu)$$

$$\geqq \text{const. } \frac{n(\tau_\nu, g, \infty)}{2^\nu}, \quad \text{or}$$

$$n(\tau_\nu, g, \infty) \leqq \text{const. } 2^\nu T(r_{\nu+1}, g) \quad (11)$$

and

$$\int_{r_{\nu+1}}^{r_{\nu+2}} \frac{T(r, g)}{(1-r)^2}\, dr \geqq \text{const. } 2^\nu T(r_{\nu+1}, g), \quad (12)$$

so that from (9), (10), (11), (12),

$$S(f, \varDelta_\nu) \leqq S(w, \varDelta_\nu^0) + O\left(\int_{r_{\nu+1}}^{r_{\nu+2}} \frac{T(r, g)}{(1-r)^2}\, dr\right), \quad f=w+g. \quad (13)$$

Next consider the case $f(z)=w(z)g(z)$.

We choose ρ_1 in $\left[\rho_0, \rho_0+\frac{1-\rho_0}{6}\right]$ and ρ_2 in $\left[\rho_0+\frac{2(1-\rho_0)}{6}, \frac{1+\rho_0}{2}\right]$, such that

$$\int_{|\zeta|=\rho_2} \log \sqrt{1+|g|^2}\, d\varphi = O(M_\nu), \quad \int_{|\zeta|=\rho_1} \log \sqrt{1+\frac{1}{|g|^2}}\, d\varphi = O(M_\nu^*), \quad (14)$$

where

$$M_\nu^* = \frac{1}{|D_\nu|}\iint_{D_\nu} \log \sqrt{1+\frac{1}{|g|^2}}\, rdrd\theta. \quad (15)$$

Then

$$\frac{1}{2\pi}\int_{|\zeta|=\rho_2} \log \sqrt{1+|wg|^2}\, d\varphi \leqq \frac{1}{2\pi}\int_{|\zeta|=\rho_2} \log \sqrt{1+|w|^2}\, d\varphi$$

$$+\frac{1}{2\pi}\int_{|\zeta|=\rho_2}\log\sqrt{1+|g|^2}\,d\varphi\leqq\frac{1}{2\pi}\int_{|\zeta|=\rho_2}\log\sqrt{1+|w|^2}\,d\varphi+O(M_\nu),$$

$$\frac{1}{2\pi}\int_{|\zeta|=\rho_1}\log\sqrt{1+|w|^2}\,d\varphi=\frac{1}{2\pi}\int_{|\zeta|=\rho_1}\log\sqrt{1+\left|\frac{wg}{g}\right|^2}\,d\varphi$$

$$\leqq\frac{1}{2\pi}\int_{|\zeta|=\rho_1}\log\sqrt{1+|wg|^2}\,d\varphi+\frac{1}{2\varphi}\int_{|\zeta|=\rho_1}\log\sqrt{1+\frac{1}{|g|^2}}\,d\varphi$$

$$\leqq\frac{1}{2\pi}\int_{|\zeta|=\rho_1}\log\sqrt{1+|wg|^2}\,d\varphi+O(M_\nu^*),$$

so that

$$\frac{1}{2\pi}\int_{|\zeta|=\rho_2}\log\sqrt{1+|wg|^2}\,d\varphi-\frac{1}{2\varphi}\int_{|\zeta|=\rho_1}\log\sqrt{1+|wg|^2}\,d\varphi$$

$$\leqq\frac{1}{2\pi}\int_{|\zeta|=\rho_2}\log\sqrt{1+|w|^2}\,d\varphi-\frac{1}{2\varphi}\int_{|\zeta|=\rho_1}\log\sqrt{1+|w|^2}\,d\varphi$$

$$+O(M_\nu)+O(M_\nu^*). \tag{16}$$

Since as (10), $M_\nu^*\leqq\text{const.}\,2^\nu T\left(r_{\nu+1},\frac{1}{g}\right)=\text{const.}\,2^\nu T(r_{\nu+1},g)$, we have (13) for $f=wg$, so that

$$S(f,\Delta_\nu)\leqq S(w,\Delta_\nu^0)+O\left(\int_{r_{\nu+1}}^{r_{\nu+2}}\frac{T(r,g)}{(1-r)^2}\,dr\right),\ f=w+g,\ f=wg. \tag{17}$$

If we sum up (17) for $\nu=2,3,\cdots,n$, then since Δ_ν^0 overlap at most 3-times,

$$S(r_n,f;\Delta)\leqq 3S(r_{n+1},w;\Delta_0)+O\left(\int_0^{r_{n+2}}\frac{T(r,g)}{(1-r)^2}\,dr\right).$$

If $r_{n-1}\leqq r\leqq r_n$, then $S(r,\Delta)\leqq S(r_n,\Delta)$ and $r_{n+1}=\frac{r_{n-1}+3}{4}\leqq\frac{r+3}{4}$, $r_{n+2}=\frac{1+r_{n+1}}{2}\leqq\frac{r+7}{8}$, so that

$$S(r,f;\Delta)\leqq 3S\left(\frac{r+3}{4},w;\Delta_0\right)+O\left(\int_0^{\frac{r+7}{8}}\frac{T(r,g)}{(1-r)^2}\,dr\right), \tag{18}$$

$$f=w+g,\qquad f=wg.$$

Now we consider the general case $f=\dfrac{g_1w+g_2}{g_3w+g_4}$, then

$f=h_1+\dfrac{h_2}{w+h_3}$, where $h_1=\dfrac{g_1}{g_3}$, $h_2=\dfrac{g_2g_3-g_1g_4}{g_3^2}$, $h_3=\dfrac{g_4}{g_3}$, so that

$$f=w_1+h_1,\quad w_1=h_2w_2,\quad w_2=\frac{1}{w_3},\quad w_3=w+h_3. \tag{19}$$

Let $\varDelta\subset\varDelta_1\subset\varDelta_2\subset\varDelta_0$ be four angular domains, each of which is bounded by two lines through $z=1$, directed inward of $|z|<1$. Then by (18),

$$S(r,f;\varDelta)\leqq 3S(r_1,w_1;\varDelta_1)+O\Big(\int_0^{r_1'}\frac{T(r,h_1)}{(1-r)^2}dr\Big),\quad r_1=\frac{r+3}{4},\quad r_1'=\frac{r+7}{8},$$

$$S(r_1,w_1;\varDelta_1)\leqq 3S(r_2,w_2;\varDelta_2)+O\Big(\int_0^{r_2'}\frac{T(r,h_2)}{(1-r)^2}dr\Big),\quad r_2=\frac{r_1+3}{4},\quad r_2'=\frac{r_1+7}{8},$$

$$S(r_2,w_2;\varDelta_2)=S(r_2,w_3;\varDelta_2),$$

$$S(r_2,w_3;\varDelta_2)\leqq 3S(r_3,w;\varDelta_0)+O\Big(\int_0^{r_3'}\frac{T(r,h_3)}{(1-r)^2}dr\Big),$$

$$r_3=\frac{r_2+3}{4}=\frac{r+63}{64},\qquad r_3'=\frac{r_2+7}{8}=\frac{r+127}{128}.$$

Since $T(r,h_i)=O(T(r,g))$ $(i=1,2,3)$, we have

$$S(r,f;\varDelta)\leqq 27S\Big(\frac{r+63}{64},w;\varDelta_0\Big)+O\Big(\int_0^{\frac{r+127}{128}}\frac{T(r,g)}{(1-r)^2}dr\Big). \tag{20}$$

Hence the case, where $\varDelta\subset\varDelta_0$ are angular domains of Lemma 2, is proved. Similarly we can prove the case, where $\varDelta\subset\varDelta_0$ are sectors of Lemma 3, by taking $\varDelta_{\nu,s}$, $\varDelta_{\nu,s}^0$ instead of $\varDelta_\nu$, $\varDelta_\nu^0$, then since $\varDelta_{\nu,s}^0$ overlap at most 9-times, we have the similar relations as (18), where 3 is replaced be 9, so that in (20), 27 is replaced by 729. Hence the lemma is proved.

3. Proof of Theorem 1.

Now we shall prove Theorem 1. Let $k>0$ be such that

$$\int_0^1 T(r,f)(1-r)^{k-1}dr=\infty\,, \tag{1}$$

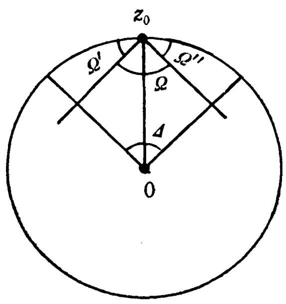
where $k=\rho-\varepsilon\ (\varepsilon>0)$ in general and $k=\rho$, if $f(z)$ is of divergence type. By dividing the unit circle $|z|=1$ into 2^n equal parts, we see that there exists an angular domain $\varDelta_n$ of maginitude $\frac{2\pi}{2^n}$, bounded by two lines through $z=0$, such that $\varDelta_1\supset\varDelta_2\supset\cdots\supset\varDelta_n\supset\cdots$

$$\int_0^1 T(r,f;\varDelta_n)(1-r)^{k-1}dr=\infty\,. \tag{2}$$

Let $\varDelta_n$ converge to a direction L: $\arg z=\theta_0$, $z_0=e^{i\theta_0}$, then for any small angular domain $\varDelta$: $|\arg z-\theta_0|<\delta$, which contains L,

$$\int_0^1 T(r,f;\varDelta)(1-r)^{k-1}dr=\infty\,. \tag{3}$$

We denote the sector: $|\arg z-\theta_0|\leqq\delta$, $|z|<1$ by the same letter $\varDelta$.



Let Ω be an angular domain of maginitude $<\pi$, bounded by two lines through z_0, which lie symmetric to L. We denote the common part of $\varDelta$ and Ω by the same letter Ω. Then $\varDelta$ consists of three parts: $\varDelta=\Omega+\Omega'+\Omega''$, as shown in the figure, so that
$T(r,f;\varDelta)=T(r,f;\Omega)+T(r,f;\Omega')+T(r,f;\Omega'')$,
hence

$$\infty=\int_0^1 T(r,f;\varDelta)(1-r)^{k-1}dr=\int_0^1 T(r,f;\Omega)(1-r)^{k-1}dr$$

$$+\int_0^1 T(r,f;\Omega')(1-r)^{k-1}dr+\int_0^1 T(r,f;\Omega')(1-r)^{k-1}dr\,. \tag{4}$$

First suppose that

$$\int_0^1 T(r,f;\Omega)(1-r)^{k-1}dr=\infty\,. \tag{5}$$

By dividing Ω into 2^n equal parts by lines through z_0, we see that there exists a line J in Ω through z_0, such that for any small angular domain ω, which contains J and is bounded by two lines through z_0,

$$\int_0^1 T(r, f; \omega)(1-r)^{k-1}dr=\infty . \tag{6}$$

Let $\omega \subset \omega_1 \subset \omega_0$ be three small angular domains, each of which is bounded by two lines through z_0.

Let $g_i(z)$ $(i=1, 2, 3)$ be three meromorphic functions in $|z|<1$, such that

$$\int_0^1 T(r, g_i)(1-r)^{k-1}dr<\infty \quad (i=1, 2, 3) . \tag{7}$$

Hence if we put $T(r, g)=\sum_{i=1}^{3} T(r, g_i)$, then

$$\int_0^1 T(r, g)(1-r)^{k-1}dr<\infty . \tag{8}$$

We shall prove that for one of g_i,

$$\sum_{\nu} (1-|z_\nu(f=g_i, \omega_0)|)^{k+1}=\infty .$$

If we put

$$w=\frac{f-g_1}{f-g_3}\cdot\frac{g_2-g_3}{g_2-g_1}, \quad \text{then} \quad f=\frac{h_1w+h_2}{h_3w+h_4}, \tag{9}$$

where $h_1=g_3(g_2-g_1)$, $h_2=g_1(g_3-g_2)$, $h_3=g_2-g_1$, $h_4=g_3-g_2$.

Hence $T(r, h_i)=O(T(r, g))$ $(i=1, 2, 3, 4)$, so that by Lemma 4,

$$S(r, f; \omega)\leqq \text{const.}\ S\left(\frac{r+63}{64}, w; \omega_1\right)+O(\Phi(r)), \tag{10}$$

where

$$\Phi(r)=\int_0^{\frac{r+127}{128}} \frac{T(r, g)}{(1-r)^2}\, dr .$$

Hence

$$T(r, f; \omega)\leqq \text{const.}\ T\left(\frac{r+63}{64}, w; \omega_1\right)+O(\Psi(r)), \tag{11}$$

where

$$\Psi(r)=\int_0^r \Phi(r)dr\,,$$

so that by (6),

$$\infty=\int_0^1 T(r,f;\omega)(1-r)^{k-1}\,dr \leqq \text{const.}\int_0^1 T(r,w;\omega_1)(1-r)^{k-1}\,dr$$
$$+O\Big(\int_0^1 \Psi(r)(1-r)^{k-1}\,dr\Big)\,. \tag{12}$$

We shall prove that $\int_0^1 \Psi(r)(1-r)^{k-1}dr<\infty$.

$$\int_0^r \Psi(r)(1-r)^{k-1}\,dr=\Big[-\frac{(1-r)^k\,\Psi(r)}{k}\Big]^r+\frac{1}{k}\int_0^r (1-r)^k\,\Phi(r)dr$$
$$\leqq \frac{1}{k}\int_0^r (1-r)^k\,\Phi(r)dr \leqq \frac{1}{k(k+1)}\int^r (1-r)^{k+1}\Phi'(r)dr$$
$$\leqq \text{const.}\int_0^r T\Big(\frac{r+127}{128},g\Big)(1-r)^{k-1}dr=O(1)\,.$$

Hence $\int_0^1 \Psi(r)(1-r)^{k-1}dr<\infty$, so that from (12),

$$\int_0^1 T(r,w;\omega_1)(1-r)^{k-1}dr=\infty\,. \tag{13}$$

If we put $a_1=0$, $a_2=1$, $a_3=\infty$ and apply Lemma 2 for $\omega_1\subset\omega_0$, then from (13), we see that one of

$$\int_0^1 N(r,w=0;\omega_0)(1-r)^{k-1}\,dr\,,\qquad \int_0^1 N(r,w=1;\omega_0)(1-r)^{k-1}dr\,,$$
$$\int_0^1 N(r,w=\infty;\omega_0)(1-r)^{k-1}dr$$

is ∞. Without loss of generality, we assume that

$$\int_0^1 N(r,w=0;\omega_0)(1-r)^{k-1}dr=\infty\,. \tag{14}$$

Since

$$n(r, w=0;\, \omega_0) \leqq n(r, f=g_1;\, \omega_0) + n(r, g_2=g_3;\, \omega_0) + \sum_{i=1}^{3} n(r, g_i=\infty;\, \omega_0)\,,$$

$$N(r, w=0;\, \omega_0) \leqq N(r, f=g_1;\, \omega_0) + N(r, g_2=g_3;\, \omega_0 + \sum_{i=1}^{3} N(r, g_i=\infty;\, \omega_0)$$

$$\leqq N(r, f=g_1;\, \omega_0) + O(T(r, g))\,, \tag{15}$$

we have from (14), (8),

$$\int_0^1 N(r, f=g_1;\, \omega_0)(1-r)^{k-1}dr = \infty\,, \quad \text{or} \quad \sum_\nu (1-|z_\nu(f=g_1;\, \omega_0)|)^{k+1} = \infty\,.$$

Hence

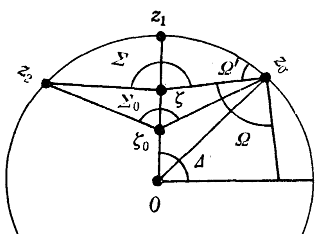
$$\sum_\nu (1-|z_\nu(f=g;\, \omega_0)|)^{k+1} = \infty \tag{16}$$

with two possible exceptions for g.

If $\int_0^1 T(r, f;\, \Omega)(1-r)^{k-1}dr < \infty$, then from (4), one of $\int_0^1 T(r, f;\, \Omega')(1-r)^{k-1}dr$, $\int_0^1 T(r, f;\, \Omega'')(1-r)^{k-1}dr$ is ∞. Suppose that

$$\int_0^1 T(r, f;\, \Omega')(1-r)^{k-1}\, dr = \infty\,. \tag{17}$$

Let ζ be the common point of the boundaries of Δ and Ω, which lies on the boundary of Ω' and z_1 be the point on $|z|=1$, which lies on the line: $\arg z = \arg \zeta$. Let ζ_0 be a point on the segment $\overline{0\zeta}$, such that $\zeta = \dfrac{\zeta_0 + z_1}{2}$, and z_2 be the point on $|z|=1$, which lies symmetric to z_0 with respect to the line $\overline{oz_1}$. Let Σ be the angular domain, bounded by two lines $\overline{\zeta z_0}$ and $\overline{\zeta z_2}$ and Σ_0 be that, bounded by two lines $\overline{\zeta_0 z_0}$ and $\overline{\zeta_0 z_2}$. Then by (17),



$$\int_0^1 T(r, f;\, \Sigma)(1-r)^{k-1}dr = \infty\,. \tag{18}$$

By means (18) and Lemma 3, we can prove similarly as above that $\sum_\nu (1-|z_\nu(f=g;\, \Sigma_0)|)^{k+1} = \infty$, with two possible exceptions for g. Hence if $\int_0^1 T(r, f; \Omega)(1-r)^{k-1}dr < \infty$ for any Ω, then if we denote the tangent of $|z|=1$ at z_0, directed toward z_1, by J, then

$$\sum_\nu (1-|z_\nu(f=g, \omega)|)^{k+1} = \infty , \tag{19}$$

with two possible exceptions for g for any angular domain ω, which contains J.

Hence by (16), (19), in any case, there exists a line J through z_0, such that $\sum_\nu (1-|z_\nu(f=g; \omega)|)^{k+1} = \infty$, with two possible exceptions for g, for any angular domain ω, which contains J.

If $f(z)$ is of divergence type, then $k=\rho$, so that J satisfies the condition of the theorem.

If $f(z)$ is of convergence type, let $k=\rho-\varepsilon_n$ $(\varepsilon_1>\varepsilon_2>\cdots>\varepsilon_n\to 0)$, then we see easily that z_0 can be chosen independently of n. Let J_n be the corresponding line through z_0 and J be one of limiting postitions of J_n, then J satisfies the condition of the theorem. Hence the theorem is proved.

4. Meromorphic functions of the class: $\overline{\lim\limits_{r\to 1}}\, T(r, f)/\log \dfrac{1}{1-r} = \infty$.

Let $f(z)$ be meromorphic in $|z|<1$, such that $\overline{\lim\limits_{r\to 1}}\, T(r, f)/\log \dfrac{1}{1-r} = \infty$, then as well known, $f(z)$ takes any value infinitely often in $|z|<1$, with two possible exceptions. We shall prove

Theorem 2. *Let $f(z)$ be meromorphic in $|z|<1$, such that*

$$\overline{\lim_{r\to 1}}\, T(r, f)/\log \frac{1}{1-r} = \infty ,$$

and $g(z)$ be meromorphic in $|z|<1$, such that $T(r, g)=O(1)$. Then there exists a point z_0 on $|z|=1$ and a line J through z_0, directed inward of $|z|<1$, which may coincide with the tangent of $|z|=1$ at z_0, such that in any small angular domain ω, which contains J and is bounded by two lines through z_0, $f(z)-g(z)$ has infinitely many zero points, with two possible exceptions for g. More precisely

$$\varlimsup_{r\to1} N(r, f=g;\omega)/\log\frac{1}{1-r}=\infty$$

with two possible exceptions for g.

PROOF. Similarly as the proof of Theorem 1, there exists a line L: $\arg z=\theta_0$, $z_0=e^{i\theta_0}$, such that for any small angular domain Δ : $|\arg z-\theta_0|<\delta$, which contains L,

$$\varlimsup_{r\to1} T(r, f;\Delta)/\log\frac{1}{1-r}=\infty\,. \tag{1}$$

We define Ω, Ω' Ω'' as before, then one of

$$\varlimsup_{r\to1} T(r, f;\Omega)/\log\frac{1}{1-r}\,,\qquad \varlimsup_{r\to1} T(r, f;\Omega')/\log\frac{1}{1-r}\,,$$

$$\varlimsup_{r\to1} T(r, f;\Omega'')/\log\frac{1}{1-r}$$

is ∞.

Suppose that

$$\varlimsup_{r\to1} T(r, f;\Omega)/\log\frac{1}{1-r}=\infty\,. \tag{2}$$

Then as before, there exists a line J in Ω, through z_0, such that for any small angular domain ω, which contains J and is bounded by two lines through z_0,

$$\varlimsup_{r\to1} T(r, f;\omega)/\log\frac{1}{1-r}=\infty\,. \tag{3}$$

Let $\omega\subset\omega_1\subset\omega_0$ be three angular domains of any small magnitude, each of which is bounded by two lines through z_0 and $g_i(z)$ $(i=1, 2, 3)$ be three meromorphic functions in $|z|<1$, such that $T(r, g_i)=O(1)$ $(i=1, 2, 3)$ and put

$$w=\frac{f-g_1}{f-g_3}\cdot\frac{g_2-g_3}{g_2-g_1}\,,\quad f=\frac{h_1w+h_2}{h_3w+h_4}\,, \tag{4}$$

$$T(r, g)=\sum_{i=1}^{3} T(r, g_i)\,,$$

Then $T(r, h_i)=O(T(r, g))=O(1)$ $(i=1, 2, 3, 4)$.

By Lemma 4,

$$S(r,f;\omega)\leqq \text{const. } S\Big(\frac{r+63}{64},\ w;\ \omega_1\Big)+O(\Phi(r)), \tag{5}$$

where

$$\Phi(r)=\int_0^{\frac{r+127}{128}} \frac{T(r,g)}{(1-r)^2}\,dr=O\Big(\frac{1}{1-r}\Big),$$

so that

$$T(r,f;\omega)\leqq \text{const. } T\Big(\frac{r+63}{64},\ w;\ \omega_1\Big)+O\Big(\log\frac{1}{1-r}\Big).$$

Hence by (3),

$$\overline{\lim_{r\to1}}\, T(r,w;\omega_1)/\log\frac{1}{1-r}=\infty, \tag{6}$$

so that by Lemma 2, for one of $N(r,w=0;\omega_0)$, $N(r,w=1;\omega_0)$, $N(r,w=\infty;\omega_0)$, say, $N(r,w=0;\omega_0)$,

$$\overline{\lim_{r\to1}}\, N(r,w=0;\omega_0)/\log\frac{1}{1-r}=\infty. \tag{7}$$

Similarly as the proof of Theorem 1, we have

$$N(r,w=0;\omega_0)\leqq N(r,f=g_1;\omega_0)+O(T(r,g))=N(r,f=g_1;\omega_0)+O(1),$$

so that

$$\overline{\lim_{r\to1}}\, N(r,f=g_1;\omega_0)/\log\frac{1}{1-r}=\infty. \tag{8}$$

Hence J satisfies the condition of the theorem.

If $T(r,f;\Omega)=O\Big(\log\frac{1}{1-r}\Big)$ for any Ω, then we can prove as Theorem 1, that the tangent J of $|z|=1$ at z_0 satisfies the condition of the theorem.

5. Meromorphic functions of the class: $\lim_{r\to1} T(r,f)=\infty$.

Let $f(z)$ be meromorphc in $|z|<1$, such that $\lim_{r\to1} T(r,f)=\infty$, then as well known, $f(z)$ takes any value infinitely often in $|z|<1$, except a set of logarithmic capacity zero.

THEOREM 3.[5] *Let $f(z)$ be meronorphic in $|z|<1$, such that*

$$\lim_{r\to 1} T(r,f)=\infty .$$

Then there exists a direction J: $\arg z=\theta_0$, such that in any small angular domain ω: $|\arg z-\theta_0|<\delta$, $f(z)$ takes any value infinitely often, except a set of logarithmic capacity zero.

In the former paper, I map the sector: $|\arg z-\theta_0|<\delta$, $|z|<1$ on $|\zeta|<1$ conformally, and reduce the theorem to the case of a meromorphic function in $|\zeta|<1$, such that $\lim_{\rho\to 1} T(\rho,f)=\infty$, $\zeta=\rho e^{i\varphi}$. We shall give a direct proof as follows.

PROOF. By dividing the unit circle $|z|=1$ into 2^n equal parts, we see that there exists a direction J: $\arg z=\theta_0$, such that for any small angular domain ω: $|\arg z-\theta_0|<\delta$, which contains J,

$$\lim_{r\to 1} T(r,f;\omega)=\infty . \tag{1}$$

We suppose that $\theta_0=0$ and put

$$\omega: |\arg z|<\delta, \qquad \omega_0: |\arg z|<2\delta \tag{2}$$

and $\omega(r)$, $\omega_0(r)$ be the part of ω, ω_0, which lies in $0<r_0\leqq|z|\leqq r<1$ and Γ_r be the boundary of $\omega_0(r)$.

Let $g(z,\zeta_0)\left(\zeta_0=\frac{r_0}{2}\right)$ be the Green's function of the sector: $|\arg z|<2\delta$, $|z|<r$, with ζ_0 as its pole and $[a,b]=\frac{|a-b|}{\sqrt{(1+|a|^2)(1+|b|^2)}}$.

We put

$$u(z)=\log\frac{[f(z),b]}{[f(z),a]}, \quad v(z)=g(z,\zeta_0) \tag{3}$$

and apply Green's formula: $\int_C\left(u\frac{\partial v}{\partial\nu}-v\frac{\partial u}{\partial\nu}\right)ds=0$ for the domain, which is obtained from $\omega_0(r)$, by taking off small discs about zero points $z_\nu(a)$, $z_\nu(b)$ of $f(z)-a$, $f(z)-b$ in $\omega_0(r)$ and then making the radii of

5) G. Valiron: Points de Picard et points de Borel des fonctions méromorphes dans un cercle. Bull. des Sci. Math. (1932). M. Tsuji: On Borel's directions of meromorphic functions of finite order, II. Kōdai Math. Seminar Reports. (1950).

these discs tend to zero, we have

$$\frac{1}{2\pi}\int_{\Gamma_\nu} u\frac{\partial g}{\partial \nu}\,ds-\frac{1}{2\pi}\int_{\gamma_0} g\frac{\partial u}{\partial \nu}\,ds+\sum_{z_\nu(a)\in\omega_0(r)} g(z_\nu(a),\zeta_0)-\sum_{z_\nu(b)\in\omega_0(r)} g(z_\nu(b),\zeta_0)=0, \quad (4)$$

where ν is the inner normal of Γ_r and γ_0 is the arc of the circle $|z|=r_0$, such that $|\arg z|\leqq 2\delta$.

Hence if we put

$$T^*(r,a)=\frac{1}{2\pi}\int_{\Gamma_r}\log\frac{1}{[f(z),a]}\frac{\partial g(z,\zeta_0)}{\partial\nu}\,ds$$
$$-\frac{1}{2\pi}\int_{\gamma_0} g(z,\zeta_0)\frac{\partial}{\partial\nu}\log\frac{1}{[f(z),a]}\,ds+\sum_{z_\nu(a)\in\omega_0(r)} g(z_\nu(a),\zeta_0), \quad (5)$$

then $T^*(r,a)=T^*(r,b)$, so that $T^*(r,a)$ is independent of a, hence if we put

$$T^*(r)=T^*(r,a), \quad (6)$$

then

$$T^*(r)\geqq\sum_{z_\nu(a)\in\omega_0(r)} g(z_\nu(a),\zeta_0)-O(1)\geqq\sum_{z_\nu(a)\in\omega(r)} g(_\nu(a),\zeta_0)-O(1).$$

If $z_\nu(a)\in\omega(r)$, then

$$g(z_\nu(a),\zeta_0)\geqq \text{const.}\,(r-|z_\nu(a)|),$$

so that if we denote the number of zero points of $f(z)-a$ in $\omega(r)$ by $n(r,a;\omega)$, then

$$T^*(r)\geqq \text{const.}\int_{r_0}^{r}(r-t)dn(t,a;\omega)-O(1)=\text{const.}\int_{r_0}^{r}n(t,a;\omega)dt-O(1).$$

Let $d\omega(a)$ be the surface element on the $w=f(z)$-sphere K, then multiplying $\frac{d\omega(a)}{\pi}$ and integrating on K, we have by (1),

$$T^*(r)\geqq \text{const.}\int_{r_0}^{r}S(t,\omega)dt-O(1)\geqq \text{const.}\,T(r,f;\omega)-O(1)\to\infty,\ r\to 1. \quad (8)$$

Suppose that $f(z)$ takes any value a of a set E of positive logarithmic capacity finite times in ω_0, then we may assume that E is a closed set, so that by taking r_0 sufficiently near to 1, we assume that $f(z)\neq a$, $a\in E$, in $\omega_0(r)$, so that if $a\in E$,

$$T^*(r)=T^*(r,a)=\frac{1}{2\pi}\int_{\Gamma_r}\log\frac{1}{[f(z),a]}\frac{\partial g(z,\zeta_0)}{\partial\nu}\,ds-O(1)\,. \tag{9}$$

Let

$$u(w)=\int_E\log\frac{1}{[w,a]}\,d\mu(a)\,,\qquad \int_E d\mu(a)=1$$

be the conductor potential of E, such that $u(w)\leqq M<\infty$ for any w, then from (9),

$$T^*(r)=\int_E T^*(r,a)d\mu(a)=O(1)\,,$$

which contradicts (8). Since $\delta>0$ is arbitrary, $f(z)$ takes any value infinitely often in any small angular domain, which contains J, except a set of logarithmic capacity zero.

REMARK. It is probable that there exists a point z_0 on $|z|=1$ and a line J through z_0, such that in any small angular domain, which contains J and is bounded by two lines through z_0, $f(z)$ takes any value infinitely often, except a set of logarithmic capacity zero, but I have no proof for it.

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