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Hopf's ergodic theorem on Fuchsian groups.

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1. Hopf's theorem.

Let a non-euclidean metric in $|z|<1$ be defined by

$$ds=\frac{2|dz|}{1-|z|^2}, \qquad d\sigma=\frac{4dxdy}{(1-|z|^2)^2}\ (z=x+iy). \tag{1}$$

Let $\eta_1=e^{i\theta}$, $\eta_2=e^{i\varphi}$ be two points on $|z|=1$, then the pair (η_1, η_2) can be considered as a point of a torus $\Theta: 0\leqq\theta\leqq 2\pi$, $0\leqq\varphi\leqq 2\pi$. For a measurable set E on Θ, we define its measure by

$$\mu(E)=\iint_E d\theta d\varphi, \quad \text{so that} \quad \mu(\Theta)=4\pi^2. \tag{2}$$

Let G be a Fuchsian group of linear transformations, which make $|z|<1$ invariant and S_ν be any substitution of G and

$$T_\nu: \eta_1'=S_\nu(\eta_1), \quad \eta_2'=S_\nu(\eta_2),$$

then the totality of T_ν consititutes a group $\mathfrak{G}=G\times G$. Hopf[1] proved

THEOREM 1. *Let D_0 be the fundamental domain of G. If $\sigma(D_0)<\infty$, then there exists no measurable set E on Θ, which is invariant by $\mathfrak{G}$ and $0<\mu(E)<4\pi^2$, so that if $\mu(E)>0$, then $\mu(E)=4\pi^2$.*

In the former paper[2], I gave another proof of the theorem. I could simplify my former proof a little, which we shall show in § 3 and as an application of the theorem, we shall prove an analogue of Weyl's theorem on uniform distribution for Fuchsian groups in § 5.

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1) E. Hopf: Fuchsian groups and ergodic theory. Trans. Amer. Math. Soc. 39 (1936). Ergodentheorie. Berlin (1937).

2) M. Tsuji: On Hopf's ergodic theorem. Proc. Imp. Acad. (1944). Jap. Journ. Math. 19 (1945).

2. Some lemmas.

We use the following lemmas in the proof.

LEMMA 1.[3] *Let* $\Theta: 0 \leqq \theta \leqq 2\pi,\ 0 \leqq \varphi \leqq 2\pi$ *be a torus and* $f(\theta, \varphi)$ *be a bounded measurable function on* Θ *and*

$$u(z, w) = \frac{1}{4\pi^2} \iint_{\Theta} f(\theta, \varphi) \frac{(1-|z|^2)(1-|w|^2)}{|z-e^{i\theta}|^2\,|w-e^{i\varphi}|^2}\, d\theta d\varphi\,, \quad |z|<1,\ |w|<1.$$

Then for almost all (θ, φ) *on* Θ, *for a fixed* (θ, φ),

$$\lim_{z\to e^{i\theta},\, w\to e^{i\varphi}} u(z, w) = f(\theta, \varphi) \qquad \textit{uniformly,}$$

when $z \to e^{i\theta}$, $w \to e^{i\varphi}$ *from the inside of any Stolz domain, whose vertex is at* $e^{i\theta}$, $e^{i\varphi}$ *respectively.*

LEMMA 2. *Let* $S: z' = \dfrac{z-a}{1-az}$ $(0<a<1)$ *and* e *be a measurable set on* $|z|=1$, *which is contained in an arc*: $0<\eta \leqq |\arg z| \leqq \pi$ *and* $me \geqq \kappa > 0$. *Then* $S(e)$ *is contained in an arc* I: $|\arg z - \pi| \leqq \dfrac{2\pi(1-a)}{\sin^2\eta}$ *on* $|z|=1$ *and*

$$mS(e) \geqq k|I|\,,$$

where $|I| = \dfrac{4\pi(1-a)}{\sin^2\eta}$, $k = \kappa \dfrac{\sin^2\eta}{8\pi}$.

PROOF.
$$mS(e) = \int_e \frac{1-a^2}{|1-ae^{i\theta}|^2}\, d\theta\,.$$

Since on e

$$\frac{1-a}{2} \leqq \frac{1-a^2}{|1-ae^{i\theta}|^2} \leqq \frac{2(1-a)}{\sin^2\eta}\,,$$

we have

$$\frac{\kappa(1-a)}{2} \leqq mS(e) \leqq \frac{2(1-a)}{\sin^2\eta}\, me\,.$$

3) M. Tsuji, l.c. 2)

Especially, if $e=I_0$ is an arc: $0<\eta\leqq\arg z\leqq\pi$, then $mS(I_0)\leqq\frac{2\pi(1-a)}{\sin^2\eta}$, hence $S(e)$ is contained in an arc I: $|\arg z-\pi|\leqq\frac{2\pi(1-a)}{\sin^2\eta}$ and from $mS(e)\geqq\frac{\kappa(1-a)}{2}$, we have $mS(e)\geqq k|I|$, where $k=\frac{\kappa\sin^2\eta}{8\pi}$.

LEMMA 3. *Let G be a Fuchsian group, such that $\sigma(D_0)<\infty$ and K_0: $|z|\leqq\rho_0<1$ be a disc, contained in D_0. If $a\in K_0$, then*

$$\frac{\text{const.}}{1-r}\leqq n(r,a)\leqq\frac{\text{const.}}{1-r},$$

where $n(r,a)$ is the number of equivalents of a in $|z|\leqq r<1$.

PROOF. First we shall prove that for any $a\in D_0$,

$$n(r,a)\leqq\frac{\text{const.}}{1-r}, \tag{1}$$

where const. is independent of $a\in D_0$.

Let $[a,b]=\left|\frac{a-b}{1-\bar{a}b}\right|$ $(|a|<1, |b|<1)$, then $[S(a),S(b)]=[a,b]$, where S is any linear transformation, which makes $|z|<1$ invariant. Let a_n be an equivalent of a, contained in $|z|\leqq r$, so that $|a_n|\leqq r$, or $[a_n,0]\leqq r$. Hence if $a_n=S_n(a)$, $S_n\in G$, then $[S_n(a),0]=[a,S_n^{-1}(0)]\leqq r$, so that $n(r,a)$ is equal to to the number of equivalents z_n of $z=0$, contained in a disc $[z,a]\leqq r$. Since the non-euclidean area of the disc $[z,a]\leqq r$ is equal to that of $|z|\leqq r$, which is $\leqq\frac{\text{const.}}{1-r}$, the number of z_n, contained in $[z,a]\leqq r$ is $\leqq\frac{\text{const.}}{1-r}$, so that $n(r,a)\leqq\frac{\text{const.}}{1-r}$, where const. is independent of $a\in D_0$.

If D_0 has boundary points on $|z|=1$, let B_0 be the part of D_0, which lies outside the circle $|z|=\rho\ (>\rho_0)$. Since $\sigma(D_0)<\infty$, we take $1-\rho$ so small that $\sigma(B_0)<\varepsilon$. Let B_ν be equivalents of B_0 and $\tilde{B}=\sum_{\nu=0}^{\infty}B_\nu$ and $\tilde{B}(r)$ be its part, contained in $|z|\leqq r<1$. Then by (1),

$$\sigma(\tilde{B}(r))=\int_{B_0}n(r,a)d\sigma(a)\leqq\frac{\text{const.}}{1-r}\sigma(B_0)<\frac{\text{const.}\,\varepsilon}{1-r}. \tag{2}$$

We put $D_0'=D_0-B_0$ and D_ν' be its equivalents and $\widetilde{D}'=\sum_{\nu=0}^{\infty} D_\nu'$. Let $\varDelta(r)$: $|z|\leqq r$ be a disc, then $\widetilde{D}'(r)+\widetilde{B}(r)=\varDelta(r)$, so that

$$\sigma(\widetilde{D}'(r))+\sigma(\widetilde{B}(r))=\sigma(\varDelta(r))\geqq \frac{\text{const.}}{1-r}.$$

If ε is small, then by (2), $\sigma(\widetilde{D}'(r))\geqq \frac{\text{const.}}{1-r}$. From this, we can prove easily that $n(r,a)\geqq \frac{\text{const.}}{1-r}$, $a\in K_0$.

LEMMA 4. *Let G be a Fuchsian group, such that $\sigma(D_0)<\infty$ and K_0: $|z|\leqq \rho_0<1$ be a disc, contained in D_0 and K_n be its equivalents and $\widetilde{K}=\sum_{n=0}^{\infty} K_n$. Let $r_\nu=1-\lambda^\nu$ $(0<\lambda<1)$ $(\nu=1,2,\cdots)$. If λ is sufficiently small, then there exists ρ_ν $(r_\nu\leqq\rho_\nu\leqq r_{\nu+1})$, which satisfies the following condition. The circle $|z|=\rho_\nu$ intersects $\widetilde{K}$ in a set of arcs, among them there are such ones $\theta_j^{(\nu)}$ $(j=1,2,\cdots,s_\nu)$ of length $\rho_\nu|\theta_j^{(\nu)}|$, such that $|\theta_j^{(\nu)}|\geqq\kappa(1-\rho_\nu)$ $(j=1,2,\cdots,s_\nu)$, $\sum_{j=1}^{s_\nu}|\theta_j^{(\nu)}|\geqq\eta>0$ $(\nu=1,2,\cdots)$, where κ and η are constants, independent of ν.*

PROOF. By Lemma 3,

$$\frac{\text{const.}}{1-r}\leqq n(r,a)\leqq \frac{\text{const.}}{1-r}, \qquad a\in K_0. \tag{1}$$

Let $\widetilde{K}(r_\nu, r_{\nu+1})$ be the part of $\widetilde{K}$, contained in $r_\nu\leqq|z|\leqq r_{\nu+1}$, then by (1),

$$\sigma(\widetilde{K}(r_\nu,r_{\nu+1}))=\int_{K_0}(n(r_{\nu+1},a)-n(r_\nu,a))d\sigma(a)\geqq\int_{K_0}\left(\frac{\text{const.}}{1-r_{\nu+1}}-\frac{\text{const.}}{1-r_\nu}\right)d\sigma(a)$$

$$\geqq\frac{\text{const.}}{\lambda^{\nu+1}}-\frac{\text{const.}}{\lambda^\nu}\geqq\frac{\text{const.}}{\lambda^{\nu+1}}, \qquad \text{if } \lambda \text{ is small.} \tag{2}$$

Let $r\theta(r)$ be the linear measure of the part of $|z|=r$, contained in $\widetilde{K}$, then

$$\sigma(\widetilde{K}(r_\nu,r_{\nu+1}))=4\iint_{\widetilde{K}(r_\nu,r_{\nu+1})}\frac{rdrd\theta}{(1-r^2)^2}=4\int_{r_\nu}^{r_{\nu+1}}\frac{r\theta(r)dr}{(1-r^2)^2}<4\int_{r_\nu}^{r_{\nu+1}}\frac{\theta(r)dr}{(1-r)^2},$$

so that by (2),

$$\int_{r_\nu}^{r_{\nu+1}} \frac{\theta(r)dr}{(1-r)^2} \geqq \frac{\text{const.}}{\lambda^{\nu+1}}. \tag{3}$$

Let the maximum of $\theta(r)$ in $[r_\nu, r_{\nu+1}]$ be attained at $r=\rho_\nu$, then

$$\int_{r_\nu}^{r_{\nu+1}} \frac{\theta(r)dr}{(1-r)^2} \leqq \theta(\rho_\nu)\int_{r_\nu}^{r_{\nu+1}} \frac{dr}{(1-r)^2} < \frac{\theta(\rho_\nu)}{\lambda^{\nu+1}},$$

so that by (3),

$$\theta(\rho_\nu) \geqq 2\eta > 0 \quad (\nu=1, 2, \cdots), \tag{4}$$

where $\eta>0$ is a constant, independent of ν. Let $|z|=\rho_\nu$ intersect $\tilde{K}$ in a set of arcs $\theta_i^{(\nu)}$ $(i=1, 2, \cdots, N)$ of length $\rho_\nu|\theta_i^{(\nu)}|$, then

$$\theta(\rho_\nu) = \sum_{i=1}^{N} |\theta_i^{(\nu)}| \geqq 2\eta > 0. \tag{5}$$

Let $0<\kappa<1$. We divide $\{\theta_i^{(\nu)}\}$ into two classes $\{\theta_i^{(\nu)}\}=\{\theta_j^{(\nu)}\}+\{\theta_{j'}^{(\nu)}\}$, where $|\theta_j^{(\nu)}| \geqq \kappa(1-\rho_\nu)$ and $|\theta_{j'}^{(\nu)}| < \kappa(1-\rho_\nu)$. Since by (1), $N \leqq \dfrac{\text{const.}}{1-\rho_\nu}$, $\sum_{j'} |\theta_{j'}^{(\nu)}| \leqq \kappa N(1-\rho_\nu) \leqq \text{const.}\,\kappa$, hence if we take κ so small that $\text{const.}\,\kappa<\eta$, then by (5),

$$\sum_j |\theta_j^{(\nu)}| \geqq \eta > 0, \qquad |\theta_j^{(\nu)}| \geqq \kappa(1-\rho_\nu). \tag{6}$$

LEMMA 5. *Let G be a Fuchsian group, such that $\sigma(D_0)<\infty$. Then there exists no measurable set E on $|z|=1$, which is invariant by G and $0<mE<2\pi$, so that if $mE>0$, then $mE=2\pi$.*

PROOF. By lemma 3, $n(r, a) \geqq \dfrac{\text{const.}}{1-r}$, so that if a_n be equivalents of a, then $\sum_{n=0}^{\infty}(1-|a_n|)=\infty$. If we identify the equivalent sides of D_0, then D_0 can be considered as a Riemann surface F. Since $\sum_{n=0}^{\infty}(1-|a_n|)=\infty$, there exists no Green's function on F, so that F is of null boundary. We remark that if $\sigma(D_0)<\infty$, then D_0 lies entirely in $|z|=1$, with its boundary, or if D_0 has boundary points on $|z|=1$, then the number of sides of D_0 is finite and D_0 has only a finite number of parabolic vertices on $|z|=1$[4], hence F is a closed Riemann surface or an open Riemann surface, which is obtained from a closed surface by taking off a finite number of points.

Suppose that there exists a set E on $|z|=1$, which is invariant by G and $0<mE<2\pi$ and put

$$u(z)=\frac{1}{2\pi}\int_E \frac{1-|z|^2}{|z-e^{i\theta}|^2}\,d\theta\,,$$

then, since $0<mE<2\pi$, $u(z)\not\equiv$ const. and since $u(z)$ is invariant by G, $u(z)$ is a non-constant bounded harmonic function on F, which is of null boundary, which is absurd. Hence there exists no such a set E on $|z|=1$.

3. Proof of Theorem 1.

Suppose that there exists a measurable set E on the torus Θ: $0\leqq\theta\leqq2\pi$, $0\leqq\varphi\leqq2\pi$, which is invariany by $\mathfrak{G}=G\times G$ and $\mu(E)>0$ and we shall prove that $\mu(E)=4\pi^2$.

Let $f(\theta,\varphi)$ be the characteristic function of E and put

$$u(z,w)=\frac{1}{4\pi^2}\iint_\Theta f(\theta,\varphi)\,\frac{(1-|z|^2)(1-|w|^2)}{|z-e^{i\theta}|^2\,|w-e^{i\varphi}|^2}\,d\theta d\varphi\,,\quad |z|<1,\ |w|<1. \qquad (1)$$

Then $u(z,w)$ is invariant by $\mathfrak{G}$, such that $u(S(z),S(w))=u(z,w)$, $S\in G$. If we denote the Stolz domain: $|\arg(1-ze^{-i\theta})|\leqq\frac{\pi}{4}$ by $\varDelta(e^{i\theta})$, then by Lemma 1, for almost all (θ,φ) on Θ, for a fixed (θ,φ),

$$\lim_{z\to e^{i\theta},\, w\to e^{i\varphi}} u(z,w)=f(\theta,\varphi) \qquad \text{uniformly}, \qquad (2)$$

when $z\to e^{i\theta}$, $w\to e^{i\varphi}$ from the inside of $\varDelta(e^{i\theta})$, $\varDelta(e^{i\varphi})$ respectively.

Let $E(\theta)$ be the section of E by the line $\theta=$ const. $=\theta$ and $E(\varphi)$ be that by the line $\varphi=$ const. $=\varphi$, then

$$\mu(E)=\int_0^{2\pi} mE(\theta)d\theta>0\,, \qquad (3)$$

4) Siegel: Some remarks on discontinuous groups. Ann. Math. 46 (1945). M. Tsuji: Theory of Fuchsian groups. Jap. Journ. Math. 21 (1951).

where m denotes the linear measure.

If $mE(\theta)=0$ on a set e of positive measure, then since such a set e is invariant by G, by Lemma 5, $me=2\pi$, so that $\mu(E)=0$, which is absurd. Hence $mE(\theta)>0$ for almost all θ in $[0, 2\pi]$.

Hence by Egoroff's theorem, for any $\delta>0$, if η is sufficiently small, there exists a closed sub-set E_1 of E, which satisfies the following condition.

(i) E_1 lies outside the strip: $|\theta-\varphi|<\eta \pmod{2\pi}$.

(ii) Let e_1 be the projection of E_1 on the θ-axis, then $me_1>2\pi-\delta$ and if $\theta\in e_1$, then $mE_1(\theta)\geqq\eta>0$.

(iii) $\lim\limits_{z\to e^{i\theta}, w\to e^{i\varphi}} u(z, w)=1$ uniformly for $(\theta, \varphi)\in E_1$,

when $z\to e^{i\theta}$, $w\to e^{i\varphi}$ from the inside of $\Delta(e^{i\theta})$, $\Delta(e^{i\varphi})$ respectively, so that if $\theta\in e_1$, $z\in\Delta(e^{i\theta})$, $|z-e^{i\theta}|<\delta=\delta(\varepsilon)$, then

$$1-\varepsilon<u(z, e^{i\varphi})<1\,, \qquad \varphi\in E_1(\theta)\,, \tag{4}$$

where

$$u(z, e^{i\varphi})=\lim_{w\to e^{i\varphi}} u(z, w).$$

Let K_0: $|z|\leqq\rho_0<1$ be a disc, contained in D_0 and K_n be its equivalents, then by Lemma 4, there exists $\rho_1<\rho_2<\cdots<\rho_\nu\to 1$, such that $|z|=\rho_\nu$ intersects $\sum\limits_{n=0}^{\infty} K_n$ in a set of arcs $\theta_j^{(\nu)}$ $(j=1, 2, \cdots, s_\nu)$, such that

$$\left.\begin{aligned} |\theta_j^{(\nu)}| &\geqq \text{const.}\,(1-\rho_\nu) \quad (j=1, 2, \cdots, s_\nu)\,, \\ \sum_{j=1}^{s_\nu} |\theta_j^{(\nu)}| &\geqq \text{const.}>0 \qquad (\nu=1, 2, \cdots, \;)\,. \end{aligned}\right\} \tag{5}$$

Hence if we denote the projection of $\theta_j^{(\nu)}$ from $z=0$ on $|z|=1$ by $\alpha_j^{(\nu)}$, then

$$\left.\begin{aligned} |\alpha_j^{(\nu)}| &\geqq \text{const.}\,(1-\rho_\nu) \quad (j=1, 2, \cdots, s_\nu)\,, \\ \sum_{j=1}^{s_\nu} |\alpha_j^{(\nu)}| &\geqq \text{const.}>0 \qquad (\nu=1, 2, \cdots, \;)\,. \end{aligned}\right\} \tag{6}$$

In virtue of (6), if $\delta>0$ in (ii) is sufficiently small, then by taking a suitable sub-set from $\{j\}$, which we denote by $\{j\}$ again, we may assume that $\alpha_j^{(\nu)}$ contains a point $e^{i\omega_j^{(\nu)}}$, such that $\omega_j^{(\nu)}\in e_1$.

Let $K_j^{(\nu)}$ be the equivalent of K_0, which contains $\theta_j^{(\nu)}$ and let

$$K_j^{(\nu)}: \quad \left|\frac{z-z_j^{(\nu)}}{1-\bar{z}_j^{(\nu)}z}\right| \leqq \rho_0, \quad \text{where} \quad K_0 = S_j^{(\nu)}(K_j^{(\nu)}), \quad 0 = S_j^{(\nu)}(z_j^{(\nu)}), \quad S_j^{(\nu)} \in G. \tag{7}$$

If we put $\overset{*}{K}_j^{(\nu)} = S_j^{(\nu)}(K_0)$, then $\overset{*}{K}_j^{(\nu)}$ is obtained from $K_j^{(\nu)}$ by a rotation about $z=0$, so that the circle $|z|=\rho_\nu$ intersects $\overset{*}{K}_j^{(\nu)}$ in a arc, whose projection from $z=0$ on $|z|=1$ be denoted by $\overset{*}{\alpha}_j^{(\nu)}$, then $|\overset{*}{\alpha}_j^{(\nu)}| = |\alpha_j^{(\nu)}|$, so that

$$\left.\begin{aligned} &|\overset{*}{\alpha}_j^{(\nu)}| \geqq \text{const.}\,(1-\rho_\nu) \quad (j=1,2,\cdots,s_\nu), \\ &\sum_{j=1}^{s_\nu} |\overset{*}{\alpha}_j^{(\nu)}| \geqq \text{const.} > 0 \qquad (\nu=1,2,\cdots\ ). \end{aligned}\right\} \tag{8}$$

If the radius ρ_0 of K_0 is sufficiently small and $\nu \geqq \nu_0$, then $z_j^{(\nu)}$ lies in $\varDelta\left(e^{i\omega_j^{(\nu)}}\right)$, so that by (4),

$$1-\varepsilon_\nu < u(z_j^{(\nu)}, e^{i\varphi}) < 1, \qquad \varphi \in E_1(\omega_j^{(\nu)}), \tag{9}$$

where $\varepsilon_\nu \to 0$ with $\nu \to \infty$.

Since $u(z,w)$ is invariant by $\mathfrak{G}$,

$$1-\varepsilon_\nu < u(0, e^{i\varphi'}) < 1, \qquad e^{i\varphi'} = S_j^{(\nu)}(e^{i\varphi}) \subset S_j^{(\nu)}(E_1(\omega_j^{(\nu)})),$$

so that if we put $M_\nu = \sum_{j=1}^{s_\nu} S_j^{(\nu)}(E_1(\omega_j^{(\nu)}))$, then

$$1-\varepsilon_\nu < u(0, e^{i\varphi}) < 1, \qquad \varphi \in M_\nu. \tag{10}$$

Let

$$\overset{*}{K}_j^{(\nu)} = S_j^{(\nu)}(K_0): \quad \left|\frac{z-\zeta_j^{(\nu)}}{1-\bar{\zeta}_j^{(\nu)}z}\right| \leqq \rho_0, \qquad \arg \zeta_j^{(\nu)} = \psi_j^{(\nu)}, \tag{11}$$

then by the condition (i) and $mE_1(\omega_j^{(\nu)}) \geqq \eta > 0$ by the condition (ii), we see by Lemma 2 that $S_j^{(\nu)}(E_1(\omega_j^{(\nu)}))$ is contained in an arc $\overset{*}{I}_j^{(\nu)}$ on $|z|=1$, whose center is $e^{i\psi_j^{(\nu)}}$ and

$$mS_j^{(\nu)}(E_1(\omega_j^{(\nu)})) \geqq \text{const.}\,|\overset{*}{I}_j^{(\nu)}|, \quad \text{where} \quad |\overset{*}{I}_j^{(\nu)}| = \frac{4\pi(1-|\zeta_j^{(\nu)}|)}{\sin^2\eta}. \tag{12}$$

Since the radius of $\overset{*}{K}_j^{(\nu)}$ is $\leqq$const.$(1-\rho_\nu)$, $|\overset{*}{\alpha}_j^{(\nu)}| \leqq$ const.$(1-\rho_\nu)$

and since $|\overset{*}{I}{}_j^{(\nu)}| \geqq \text{const.}(1-\rho_\nu)$, we have $|\overset{*}{I}{}_j^{(\nu)}| \geqq \text{const.}|\overset{*}{\alpha}{}_j^{(\nu)}|$, so that by (8)

$$\sum_{j=1}^{s_\nu} |\overset{*}{I}{}_j^{(\nu)}| \geqq \text{const.} > 0 \quad (\nu=1, 2, \cdots,). \tag{13}$$

Since by (8), $|\overset{*}{\alpha}{}_j^{(\nu)}| \geqq \text{const.}(1-\rho_\nu)$ and $\overset{*}{\alpha}{}_j^{(\nu)}, \overset{*}{\alpha}{}_{j'}^{(\nu)}$ $(j \neq j')$ have no common points and $|\overset{*}{I}{}_j^{(\nu)}| \leqq \text{const.}(1-\rho_\nu)$, we see that $\{\overset{*}{I}{}_j^{(\nu)}\}$ overlap at most N-times, where N is independent of ν, so that since $S_j^{(\nu)}(E_1(\omega_j^{(\nu)})) \subset \overset{*}{I}{}_j^{(\nu)}$, we have by (12), (13),

$$mM_\nu \geqq \text{const.} \frac{1}{N} \sum_{j=1}^{s_\nu} |\overset{*}{I}{}_j^{(\nu)}| \geqq \text{const.} > 0 \quad (\nu=1, 2, \cdots,). \tag{14}$$

Hence if we put $M = \overline{\lim_\nu} M_\nu$, then $mM > 0$ and by (10),

$$u(0, e^{i\varphi}) = 1, \qquad \text{if} \qquad \varphi \in M. \tag{15}$$

Now

$$\left.\begin{aligned} u(0, w) &= \frac{1}{2\pi} \int_0^{2\pi} F(\varphi) \frac{1-|w|^2}{|w-e^{i\theta}|^2} d\varphi, \\ F(\varphi) &= \frac{1}{2\pi} \int_0^{2\pi} f(\theta, \varphi) d\theta = \frac{mE(\varphi)}{2\pi}, \end{aligned}\right\} \tag{16}$$

so that for almost all φ on M, $1 = u(0, e^{i\varphi}) = F(\varphi) = \dfrac{mE(\varphi)}{2\pi}$, or $mE(\varphi) = 2\pi$. Let M_0 be the set of φ, such that $mE(\varphi) = 2\pi$, then since $M \subset M_0$, except a null set, $mM_0 \geqq mM > 0$ and since such a set M_0 is invariant by G, by Lemma 5, $mM_0 = 2\pi$, so that

$$\mu(E) = \int_0^{2\pi} mE(\varphi) d\varphi = 4\pi^2.$$

4. Flow $T_t^{(\alpha)}$ $(-\infty < t < \infty)$.

We define the non-euclidean metric $ds = \dfrac{2|dz|}{1-|z|^2}$, $d\sigma = \dfrac{4dxdy}{(1-|z|^2)^2}$ $(z = x+iy)$ as in § 1. We suppose that $\sigma(D_0) < \infty$. Let z be any point of D_0 and we associate a direction φ at z, which makes an angle φ with the positive real axis, then the line elements (z, φ) $(z \in D_0, 0 \leqq \varphi \leqq 2\pi)$ constitute a space Ω. We define the volume element $d\mu$ in Ω by

$$d\mu = \frac{4dxdyd\varphi}{(1-|z|^2)^2}, \qquad z = x+iy,$$

then $\mu(\Omega)=2\pi\sigma(D_0)$. $d\mu$ is invariant for any linear transformation, which makes $|z|<1$ invariant.

Let $\alpha\,(0<\alpha<\pi)$ be fixed, then for any (z,φ) in $|z|<1$, there exists a unique circular arc $g_\alpha=g_\alpha(z,\varphi)$, which touches the direction φ at z and satisfies the following condition. Let $\eta_1=e^{i\theta_1}$, $\eta_2=e^{i\theta_2}$ be two end points of g_α on $|z|=1$, where η_2 is such that if we proceed on g_α in the direction φ, then g_α meets $|z|=1$ at η_2, where it makes an angle α with the positive tangent of $|z|=1$ at η_2.

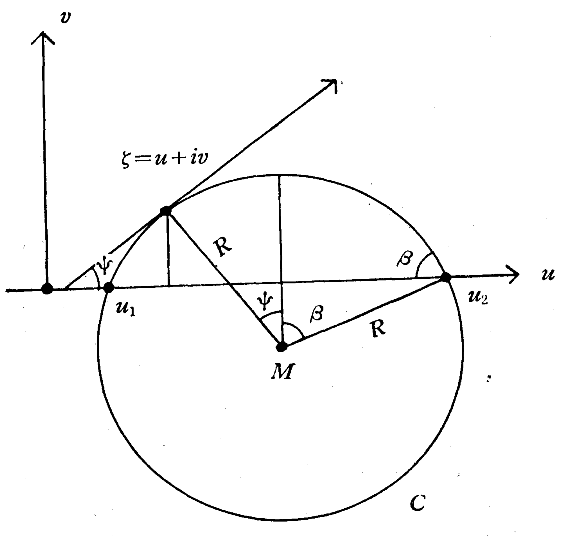
Let z_0 be the middle point of $\overset{\frown}{\eta_1\eta_2}$ and z be any point of g_α and s be the non-euclidean length of the arc $\overset{\frown}{z_0 z}$, where s is positive, if z lies on $\overset{\frown}{z_0\eta_2}$ and negative, if otherwise. Then we have a one-to-one correspondence between (z,φ) and (η_1,η_2,s), which we denote by $(z,\varphi)=(\eta_1,\eta_2,s)=(\theta_1,\theta_2,s)$. Then we shall prove

LEMMA 6. $$d\mu=\frac{4dxdyd\varphi}{(1-|z|^2)^2}=2\sin\alpha\,\frac{|d\eta_1||d\eta_2|ds}{|\eta_1-\eta_2|^2}.$$

PROOF. By $z=\dfrac{\zeta-i}{\zeta+i}$, we map $|z|<1$ on the upper half of the $\zeta=u+iv$-plane, then

$$ds=\frac{2|dz|}{1-|z|^2}=\frac{|d\zeta|}{v},\quad d\sigma=\frac{4dxdy}{(1-|z|^2)^2}=\frac{dudv}{v^2}. \tag{1}$$

Let $g_\alpha=g_\alpha(z,\varphi)$ become a circle C of radius R and of center M and η_1,η_2 become $u_1,u_2\,(u_1<u_2)$ respectively and let ψ be defined as in the figure, then



$$\left.\begin{aligned} R&=\frac{u_2-u_1}{2\sin\beta}\quad(\beta=\pi-\alpha),\\ u&=\frac{u_1+u_2}{2}-R\sin\psi\\ &=\frac{u_1+u_2}{2}-\frac{u_2-u_1}{2\sin\beta}\sin\psi,\\ v&=R\cos\psi-R\cos\beta\\ &=\frac{u_2-u_1}{2\sin\beta}(\cos\psi-\cos\beta),\end{aligned}\right\} \tag{2}$$

so that $\dfrac{\partial(u, v)}{\partial(u_1, u_2)} = \dfrac{1}{2\sin\beta}(\cos\psi - \cos\beta)$, hence

$$dudv = \frac{|\cos\psi - \cos\beta|}{2\sin\beta} du_1du_2 = \frac{v\, du_1du_2}{u_2 - u_1}. \tag{3}$$

By (1), $ds = \dfrac{Rd\psi}{v}$ and since $d\varphi = d\psi$, we have

$$\frac{4dxdyd\varphi}{(1-|z|^2)^2} = \frac{dudvd\psi}{v^2} = \frac{dudvds}{vR} = \frac{du_1du_2ds}{R(u_2-u_1)} = 2\sin\beta\frac{du_1du_2ds}{(u_2-u_1)^2}$$

$$= 2\sin\alpha\frac{du_1du_2ds}{(u_2-u_1)^2}. \tag{4}$$

Since the anharmonic ratio $[z_1, z_2, z_3, z_4] = \dfrac{z_3-z_1}{z_3-z_2} : \dfrac{z_4-z_1}{z_4-z_2}$ and hence $\dfrac{dz_1dz_2}{(z_1-z_2)^2} = -[z_1, z_2, z_1+dz_1, z_2+dz_2]$ is invariant by a linear transformation, we have $\dfrac{|d\eta_1||d\eta_2|}{|\eta_1-\eta_2|^2} = \dfrac{du_1du_2}{(u_1-u_2)^2}$, so that by (4)

$$\frac{4dxdyd\varphi}{(1-|z|^2)^2} = 2\sin\alpha\frac{|d\eta_1||d\eta_2|}{|\eta_1-\eta_2|^2}ds, \qquad \text{q.e.d.}$$

Now for a fixed α in $(0, \pi)$, we consider a flow $T_t^{(\alpha)}$ $(-\infty < t < \infty)$ in Ω

$$T_t^{(\alpha)}: \quad P = (\eta_1, \eta_2, s) \rightarrow P_t = (\eta_1, \eta_2, s+t),$$

where if the z-coordiante of P_t lies outside D_0, then we replace it by its equivalent in D_0. Then by Lemma 6, $T_t^{(\alpha)}$ is a mass-preserving transformation of Ω into itself. The flow $T_t^{(\alpha)}$ is said metric transitive, if a set M is invariant by $T_t^{(\alpha)}$, then $\mu(M)=0$, or $\mu(M)=\mu(\Omega)$.

THEOREM 2. *If $\sigma(D_0)<\infty$, then the flow $T_t^{(\alpha)}$ is metric transitive.*

PROOF. Let $(z, \varphi) = (\eta_1, \eta_2, s)$, $(z \in D_0, 0 \leq \varphi \leq 2\pi)$ and $g_\alpha = g_\alpha(z, \varphi) = g_\alpha(\eta_1, \eta_2, s)$, $(\eta_1 = e^{i\theta_1}, \eta_2 = e^{i\theta_2})$ be defined as before. Then the part of g_α contained in D_0 corresponds to $s_1(\theta_1, \theta_2) \leq s \leq s_2(\theta_1, \theta_2)$, so that if we denote the projection of Ω on the torus Θ: $0 \leq \theta_1 \leq 2\pi$, $0 \leq \theta_2 \leq 2\pi$ by A, then Ω consists of points (θ_1, θ_2, s): $(\theta_1, \theta_2) \in A$, $s_1(\theta_1, \theta_2) \leq s \leq s_2(\theta_1, \theta_2)$.

If a set M is invariant by the flow and $\mu(M)>0$, then we shall prove that $\mu(M)=\mu(\Omega)$.

Since M is invarinant by the flow, if we denote the projection of M on Θ by B, then M consists of points (θ_1, θ_2, s): $(\theta_1, \theta_2)\in B$, $s_1(\theta_1, \theta_2) \leqq s \leqq s_2(\theta_1, \theta_2)$. Let B_ν be equivalents of B by $\mathfrak{G}=G\times G$ and $\tilde{B}=\sum_{\nu=0}^{\infty}B_\nu$, then $\tilde{B}$ is invariant by $\mathfrak{G}$. Since $\mu(M)>0$, the measure of $\tilde{B}$ is positive, so that by Theorem 1, $\Theta-\tilde{B}$ is a null set, so that $A-A\tilde{B}$ is a null set. Since $A\tilde{B}=B$, $A-B$ is a null set, hence $\mu(M)=\mu(\Omega)$.

5. Analogue of Weyl's theorem on uniform distribution.

Since $T_t^{(\alpha)}$ is metric transitive and $\mu(\Omega)<\infty$, by Birkhoff's ergodic theorem, for any bounded measurable function $f(P)$ on Ω,

$$\lim_{L\to\infty}\frac{1}{L}\int_0^L f(P_t)dt=\frac{1}{\mu(\Omega)}\int_\Omega f(P)d\mu(P), \qquad (*)$$

for almost all points P in Ω.

By means of (*), we shall prove

THEOREM 3. *Let G be a Fuchsian group, such that $\sigma(D_0)<\infty$. Then there exists a set E on $|z|=1$ of measure 2π, which satisfies the following condition. Let M be a set in D_0, which is measurable in Jordan's sense and M_ν be its equivalents by G and $\tilde{M}=\sum_{\nu=0}^{\infty}M_\nu$.*

Let $e^{i\theta}\in E$ and l be a line through $e^{i\theta}$, directed inward of $|z|<1$, making an angle $\alpha\,(0<\alpha<\pi)$ with the positive tangent of $|z|=1$ at $e^{i\theta}$. Let l^L be its part of non-euclidean length L, measured from a fixed point on it and $L(M, e^{i\theta}, l)$ be the non-euclidean linear measure of the part of l^L, contained in $\tilde{M}$. Then

$$\lim_{L\to\infty}\frac{L(M, e^{i\theta}, l)}{L}=\frac{\sigma(M)}{\sigma(D_0)}, \quad e^{i\theta}\in E,$$

for any l and M.

PROOF. Let $\varDelta$ be a rectangle: $r_1\leqq x\leqq r_2$, $r_3\leqq y\leqq r_4$, contained in D_0, where r_1, r_2, r_3, r_4, are rational numbers and we call such $\varDelta$ a rational rectangle. Let $\varDelta_\nu$ be its equivalents and $\tilde{\varDelta}=\sum_{\nu=0}^{\infty}\varDelta_\nu$. We associate to every $z \in \varDelta$, directions φ $(0 \leqq \varphi \leqq 2\pi)$, then the line elements $P=(z, \varphi)$ $(z \in \varDelta, 0 \leqq \varphi \leqq 2\pi)$ constitue a sub-set Σ of Ω, whose volume is $2\pi\sigma(\varDelta)$. Let $f(P)$ be its chracteristic function, then by (*), if P does not belong to a null set $N(r_1, r_2, r_3, r_4)$ in Ω,

$$\lim_{L\to\infty} \frac{1}{L} \int_0^L f(P_t)dt = \frac{\mu(\Sigma)}{\mu(\Omega)} = \frac{\sigma(\varDelta)}{\sigma(D_0)}. \tag{1}$$

Let $N^{(\alpha)} = \sum N(r_1, r_2, r_3, r_4)$, added for all rationals r_1, r_2, r_3, r_4, then $N^{(\alpha)}$ is a null set, which depends on α. Hence if P does not belong to $N^{(\alpha)}$, then (1) holds for any rational rectangle $\varDelta$ in D_0.

Let $g_\alpha = g_\alpha(z, \varphi) = g_\alpha(\eta_1, \eta_2, s)$ $(\eta_1 = e^{i\theta_1}, \eta_2 = e^{i\theta_2})$ be a circular arc, defined before and g_α^L be its part of non-euclidean length L, measured from the middle point z_0 of $\widehat{\eta_1\eta_2}$, then $\int_0^L f(P_t)dt = L(\varDelta, g_\alpha) + O(1)$, where $L(\varDelta, g_\alpha)$ is the non-euclidean linear measure of the part of g_α^L, contained in $\tilde{\varDelta}$, so that if P does not belong to $N^{(\alpha)}$, then for any rational rectangle $\varDelta$ in D_0,

$$\lim_{L\to\infty} \frac{L(\varDelta, g_\alpha)}{L} = \frac{\sigma(\varDelta)}{\sigma(D_0)}. \tag{2}$$

Let $\varDelta$ be any rectangle in D_0, whose sides are parallel to the coordinates axes, then for any $\varepsilon > 0$, we choose two rational rectangles $\varDelta_1, \varDelta_2$, such that $\varDelta_1 \subset \varDelta \subset \varDelta_1$, $\sigma(\varDelta_2 - \varDelta_1) < \varepsilon$. Let l be a line through $\eta_2 = e^{i\theta_2}$, which touches g_α at η_2. If $z \in l$, $\zeta \in g_\alpha$ and $|z| = |\zeta|$, then if $1 - |z| < \delta = \delta(\varepsilon)$,

$$(1-\varepsilon)|dz| \leqq |d\zeta| \leqq (1+\varepsilon)|dz|, \tag{3}$$

where $\varepsilon \to 0$ with $\delta \to 0$. Since (2) holds for $\varDelta_1, \varDelta_2$, we have by (3),

$$(1-\eta)\frac{\sigma(\varDelta_1)}{\sigma(D_0)} \leqq \varliminf_{L\to\infty} \frac{L(\varDelta, e^{i\theta}, l)}{L} \leqq (1+\eta)\frac{\sigma(\varDelta_2)}{\sigma(D_0)}, \quad \theta = \theta_2,$$

where $\eta \to 0$ with $\varepsilon \to 0$. Hence for any rectangle $\varDelta$ in D_0, if P does not belong to $N^{(\alpha)}$,

$$\lim_{L\to\infty} \frac{L(\varDelta, e^{i\theta}, l)}{L} = \frac{\sigma(\varDelta)}{\sigma(D_0)}. \tag{4}$$

Let A be the projection of Ω on Θ and A_ν be its equivalents by $\mathfrak{G}$ and $\tilde{A}=\sum_{\nu=0}^{\infty} A_\nu$, then as before, $\Theta-\tilde{A}$ is a null set. Let $N_\Theta^{(\alpha)}$ be the projection of $N^{(\alpha)}$ on Θ and $N_{\Theta,\nu}^{(\alpha)}$ be its equivalents by $\mathfrak{G}$ and $\tilde{N}^{(\alpha)}=\sum_{\nu=0}^{\infty} N_{\Theta,\nu}^{(\alpha)}$, then $\tilde{N}^{(\alpha)}$ is a null set, so that $\tilde{N}=\sum_{\alpha}\tilde{N}^{(\alpha)}$, added for all rationals α in $(0,\pi)$ is a null set on Θ. Hence for a suitable θ_1^0, there exists a set E of measure 2π on the segment: $\theta_1=\theta_1^0$, $0\leqq\theta_2\leqq 2\pi$, such that E lies outside of $\tilde{N}$ and every point of which belongs to $\tilde{A}$. We denote the set of points $z=e^{i\theta}$, $\theta\in E$ by the same letter E. If $e^{i\theta}\in E$, then (4) holds for any line l through $e^{i\theta}$, making any rational angle α with the positive tangent of $|z|=1$ at $e^{i\theta}$ and for any rectangle Δ in D_0.

If α is an irrational number, let α' be a rational number, such that $|\alpha-\alpha'|<\varepsilon$ and l' be a line through $e^{i\theta}$, making an angle α' with the positive tangent of $|z|=1$ at $e^{i\theta}$. Let Δ be any rectangle in D_0, then we choose two rectangles Δ_1, Δ_2, such that $\Delta_1\subset\Delta\subset\Delta_2$, $\sigma(\Delta_2-\Delta_1)<\varepsilon$. Then since (4) holds for Δ_1, l' and Δ_2, l', we have

$$(1-\eta)\frac{\sigma(\Delta_1)}{\sigma(D_0)}\leqq\overline{\lim_{L\to\infty}}\frac{L(\Delta, e^{i\theta}, l)}{L}\leqq(1+\eta)\frac{\sigma(\Delta_2)}{\sigma(D_0)},$$

where $\eta\to 0$ with $\varepsilon\to 0$, hence if $e^{i\theta}\in E$, then

$$\lim_{L\to\infty}\frac{L(\Delta, e^{i\theta}, l)}{L}=\frac{\sigma(\Delta)}{\sigma(D_0)}, \tag{5}$$

which holds for any line l through $e^{i\theta}$ and for any rectangle Δ in D_0, so that for a set in D_0, which is a sum of a finite number of non-overlapping rectangles. Let M be a set in D_0, which is measurable in Jordan's sense, then for any $\varepsilon>0$, we choose M_1, M_2, which are sums of a finite number of non-overlapping rectangles, such that $M_1\subset M\subset M_2$, $\sigma(M_2-M_1)<\varepsilon$. Since the theorem holds for M_1, M_2, and $\varepsilon>0$ is arbitrary, the theorem holds for M.

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