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A simple proof of Dirichlet principle.

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1. In the usual proof of the Dirichlet principle, the solvabilty of the Dirichlet problem is not assumed. But since this can be proved simply by Perron's method, if we assume this, then we can prove the Dirichlet principle simply, which we shall show in the following lines. First we shall prove two lemmas.

LEMMA 1. *Let $w(z)$ be continuous in a ring domain $\Delta: 0<\rho\leqq|z|\leqq 1$ and have piece-wise continuous partial derivatives of the first order and its Dirichlet integral $D_{\Delta}[w]=D[w]$ be finite. Let $u(z)$ be harmonic in $\rho<|z|<1$ and continuous in $\rho\leqq|z|\leqq 1$, such that $u(z)=w(z)$ on $|z|=\rho$ and $|z|=1$. Then*

$$D[u]\leqq D[w].$$

PROOF. Let in $\rho<|z|<1$,

$$u(z)=u(re^{i\theta})=A\log r+a_0+\sum_{k=1}^{\infty}(a_k r^k+a_{-k}r^{-k})\cos k\theta$$
$$+\sum_{k=1}^{\infty}(b_k r^k+b_{-k}r^{-k})\sin k\theta, \tag{1}$$

where

$$\left.\begin{aligned}
a_0&=\frac{1}{2\pi}\int_0^{2\pi}u(e^{i\theta})d\theta=\frac{1}{2\pi}\int_0^{2\pi}w(e^{i\theta})d\theta,\\
a_k+a_{-k}&=\frac{1}{\pi}\int_0^{2\pi}u(e^{i\theta})\cos k\theta\,d\theta=\frac{1}{\pi}\int_0^{2\pi}w(e^{i\theta})\cos k\theta\,d\theta,\\
b_k+b_{-k}&=\frac{1}{\pi}\int_0^{2\pi}u(e^{i\theta})\sin k\theta\,d\theta=\frac{1}{\pi}\int_0^{2\pi}w(e^{i\theta})\sin k\theta\,d\theta,
\end{aligned}\right\}\tag{2}$$

$$\left.\begin{aligned} A\log\rho+a_0&=\frac{1}{2\pi}\int_0^{2\pi}u(\rho e^{i\theta})d\theta=\frac{1}{2\pi}\int_0^{2\pi}w(\rho e^{i\theta})d\theta\,,\\ a_k\rho^k+a_{-k}\rho^{-k}&=\frac{1}{\pi}\int_0^{2\pi}u(\rho e^{i\theta})\cos k\theta\,d\theta=\frac{1}{\pi}\int_0^{2\pi}w(\rho e^{i\theta})\cos k\theta\,d\theta\,,\\ b_k\rho^k+b_{-k}\rho^{-k}&=\frac{1}{\pi}\int_0^{2\pi}u(\rho e^{i\theta})\sin k\theta\,d\theta=\frac{1}{\pi}\int_0^{2\pi}w(\rho e^{i\theta})\sin k\theta\,d\theta\,. \end{aligned}\right\} \qquad (2')$$

We put

$$u_n(z)=A\log r+a_0+\sum_{k=1}^{n}(a_kr^k+a_{-k}r^{-k})\cos k\theta+\sum_{k=1}^{n}(b_kr^k+b_{-k}r^{-k})\sin k\theta\,, \qquad (3)$$

then $\Delta u_n=0$ and

$$\frac{\partial u_n}{\partial r}=\frac{A}{r}+\sum_{k=1}^{n}k(a_kr^{k-1}-a_{-k}r^{-k-1})\cos k\theta+\sum_{k=1}^{n}k(b_kr^{k-1}-b_{-k}r^{-k-1})\sin k\theta\,,$$

so that

$$\frac{1}{\pi}\int_0^{2\pi}u_n(e^{i\theta})\frac{\partial u_n(e^{i\theta})}{\partial r}d\theta=2a_0A+\sum_{k=1}^{n}k(a_k^2-a_{-k}^2)+\sum_{k=1}^{n}k(b_k^2-b_{-k}^2)\,,$$

and by (2)

$$\frac{1}{\pi}\int_0^{2\pi}w(e^{i\theta})\frac{\partial u_n(e^{i\theta})}{\partial r}d\theta=\frac{A}{\pi}\int_0^{2\pi}w(e^{i\theta})d\theta+\sum_{k=1}^{n}k(a_k-a_{-k})\frac{1}{\pi}\int_0^{2\pi}w(e^{i\theta})\cos k\theta\,d\theta$$

$$+\sum_{k=1}^{n}k(b_k-b_{-k})\frac{1}{\pi}\int_0^{2\pi}w(e^{i\theta})\sin k\theta\,d\theta=2a_0A+\sum_{k=1}^{n}k(a_k^2-a_{-k}^2)+\sum_{k=1}^{n}k(b_k^2-b_{-k}^2)\,,$$

whence follows

$$\int_0^{2\pi}(w(e^{i\theta})-u_n(e^{i\theta}))\frac{\partial u_n(e^{i\theta})}{\partial r}d\theta=0\,.$$

Similarly we have

$$\int_0^{2\pi}(w(\rho e^{i\theta})-u_n(\rho e^{i\theta}))\frac{\partial u_n(\rho e^{i\theta})}{\partial \rho}\rho\,d\theta=0\,.$$

Hence by the Green's formula, we have $D_\Delta[w-u_n,u_n]=0$, so that

$$D_\Delta[w]=D_\Delta[u_n]+D_\Delta[w-u_n]\geqq D_\Delta[u_n]\geqq D_{\Delta'}[u_n]\,,$$

where Δ': $\rho<\rho_1\leqq|z|\leqq\rho_2<1$, hence if we make $n\to\infty$ and then $\rho_1\to\rho$, $\rho_2\to 1$, we have $D[u]\leqq D[w]$.

Similarly we can prove

LEMMA 2. *Let $w(z)$ be continuous in $|z|\leqq 1$ and have piece-wise* *continuous partial derivatives of the first order and its Dirichlet integral $D[w]$ be finite. Let $u(z)$ be harmonic in $|z|<1$ and continuous in $|z|\leq 1$, such that $u(z)=w(z)$ on $|z|=1$. If $w(z)$ is not harmonic in $|z|<1$, then $D[u]<D[w]$.*[1]

2. Now we shall prove the following Dirichlet principle.

THEOREM 1.[2] *Let F be a compact Riemann surface, whose boundary Γ consists of a finite number of disjoint Jordan curves $\{\Gamma_i\}$, $\Gamma=\sum_{i=1}^{k}\Gamma_i$. Suppose that there exists a function $w_0(z)$ on F, which is continuous on F and has piece-wise continuous partial derivatives of the first order and its Dirichlet integral $D_F[w_0]=D[w_0]$ is finite.*

Let $\mathfrak{F}$ be the family of functions $w(z)$, which have the same properties as $w_0(z)$ and $w(z)=w_0(z)$ on Γ, and put

$$d=\inf_{w\in\mathfrak{F}} D[w].$$

Then there exists a harmonic function $u(z)\in\mathfrak{F}$ on F, such that

$$D[u]=d.$$

PROOF. By the definition of d, there exists $w_n\in\mathfrak{F}$, such that $D[w_n]\to d$. Let Γ'_i be an analytic Jordan curve on F, which lies in a small neighbourhood of Γ_i, such that Γ'_i, Γ_i bound a doubly connected domain Δ_i on F, where Δ_i, Δ_j $(i\neq j)$ are disjoint. Let $u^i_n(z)$ be harmonic in Δ_i, such that $u^i_n=w_n$ on Γ'_i and Γ_i. We map Δ_i conformally onto a ring domain V_i: $\rho_i<|\zeta|<1$ on the ζ-plane[3] and consider $u^i_n(z)$ and $w_n(z)$ as functions of ζ in V_i and apply Lemma 1, then since the Dirichlet integral is invariant for conformal mapping, we have $D_{\Delta_i}[u^i_n]\leq D_{\Delta_i}[w_n]$. Hence if we put $\tilde{w}_n(z)=u^i_n(z)$ in Δ_i $(i=1,2,\cdots,k)$ and $\tilde{w}_n(z)=w_n(z)$ in $F-\sum_{i=1}^{k}\Delta_i$, then $D[\tilde{w}_n]\leq D[w_n]$, so that $D[\tilde{w}_n]\to d$.

1) H. Weyl: Die Idee der Riemannschen Fläche. Leipzig-Berlin (1923) p. 86.

2) R. Courant. Dirichlet principle, conformal mapping and minimal surfaces. New York (1950).

3) This can be made simply as follows. Let $\omega_i(z)$ be the harmonic measure of Γ'_i with respect to Δ_i and $\int_{\Gamma'_i}\frac{\partial\omega_i}{\partial\nu}ds=\alpha_i>0$, where ν is the outer normal of Γ'_i with respect to Δ_i. Let $\bar{\omega}_i(z)$ be the conjugate harmonic function of $\omega_i(z)$ and $f_i(z)=\frac{2\pi}{\alpha_i}(\omega_i(z)+\sqrt{-1}\,\bar{\omega}_i(z))$. Then by $\zeta=e^{-f_i(z)}$, Δ_i is mapped conformally on V_i.

Hence we may assume that

$$w_n(z) \text{ is harmonic in } \Delta_i \quad (i=1,2,\cdots,k). \tag{1}$$

We put $w_n(z)=v_n(\zeta)$ in V_i: $\rho_i<|\zeta|<1$, then since $v_n(\zeta)-v_1(\zeta)$ is harmonic in V_i and vanishes on $|\zeta|=1$, it is harmonic in $V_i+\overline{V}_i$: $\rho_i<|\zeta|<1/\rho_i$ and

$$D_{V_i+\overline{V}_i}[v_n-v_1]=2D_{\Delta_i}[w_n-w_1]\leqq 2(D_{\Delta_i}[w_n]+2\sqrt{D_{\Delta_i}[w_n]D_{\Delta_i}[w_1]}+D_{\Delta_i}[w_1]),$$

hence $D_{V_i+\overline{V}_i}[v_n-v_1]$ is bounded for $n=1,2,\cdots$ and since $v_n-v_1=0$ on $|\zeta|=1$, we can select a partial sequence from n, which we denote by n again, such that v_n-v_1 and hence v_n converges uniformly in $V_i+\overline{V}_i$. Hence returning to the z-plane,

$$\lim_{n\to\infty} w_n(z)=w(z) \text{ uniformly in } \Delta_i \ (i=1,2,\cdots,k). \tag{2}$$

$w(z)$ is harmonic in $\sum_{i=1}^{k}\Delta_i$ and is continuous on Γ and $w(z)=w_1(z)=w_0(z)$ on Γ. Let γ_i be an analytic Jordan curve in Δ_i, which separates $\Gamma'_{i_1}\Gamma_i$ in Δ_i and let F_0 be the connected Riemann surface, bounded by $\sum_{i=1}^{k}\gamma_i$. Then by (2)

$$D_{F-F_0}[w]=\lim_{n\to\infty} D_{F-F_0}[w_n]. \tag{3}$$

Let $\varphi_n(z)$ be harmonic in F_0, such that $\varphi_n=w_n$ on γ_i $(i=1,2,\cdots,k)$. Since by the hypothesis, $w_n(z)$ is harmonic on γ_i and $\varphi_n-w_n=0$ on γ_i, φ_n-w_n and hence φ_n is harmonic in a neighbourhood U_i of γ_i, so that $\frac{\partial\varphi_n}{\partial\nu}$ is continuous on γ_i, where ν is the outer normal of γ_i. Hence

$$D_{F_0}[w_n-\varphi_n,\varphi_n]=\sum_{i=1}^{k}\int_{\gamma_i}(w_n-\varphi_n)\frac{\partial\varphi_n}{\partial\nu}ds=0,$$

so that

$$D_{F_0}[\varphi_n]\leqq D_{F_0}[w_n]. \tag{4}$$

By (4), we see easily that $D_{U_i}[\varphi_n-w_n]$ is bounded for $n=1,2,\cdots$. Since $\varphi_n-w_n=0$ on γ_i and w_n converges uniformly on γ_i, φ_n-w_n and hence φ_n converges uniformly in U_i, so that φ_n converges uniformly in

$F_0+\sum_{i=1}^{k} U_i$. Hence if we put $\lim_{n\to\infty}\varphi_n(z)=\varphi(z)$, then $\varphi(z)$ is harmonic in $F_0+\sum_{i=1}^{k} U_i$ and $\varphi(z)=w(z)$ on $\sum_{i=1}^{k}\gamma_i$ and $D_{F_0}[\varphi]=\lim_{n\to\infty} D_{F_0}[\varphi_n]$, so that by (4),

$$D_{F_0}[\varphi]\leqq \varliminf_{n\to\infty} D_{F_0}[w_n]. \tag{5}$$

Hence if we put $u(z)=\varphi(z)$ in F_0 and $u(z)=w(z)$ in $F-F_0$, then by (3), (5), $D[u]\leqq \lim_{n\to\infty} D[w_n]=d$, so that $D[u]=d$. $u(z)$ is continuous in F, $u(z)=w_0(z)$ on Γ and is harmonic in F, except on $\sum_{i=1}^{k}\gamma_i$. We shall prove that $u(z)$ is harmonic on $\sum_{i=1}^{k}\gamma_i$. Suppose that $u(z)$ is not harmonic at a point z_0 on $\sum_{i=1}^{k}\gamma_i$. Let Δ be a circular disc about z_0, which is contained in F, and $v(z)$ be harmonic in Δ, such that $v=u$ on the boundary of Δ, then by Lemma 2, $D_\Delta[v]<D_\Delta[u]$, so that if we put $\tilde{u}(z)=v(z)$ in Δ, $\tilde{u}(z)=u(z)$ in $F-\Delta$, then $D[\tilde{u}]<D[u]=d$, which is absurd. Hence $u(z)$ is harmonic on $\sum_{i=1}^{k}\gamma_i$, q. e. d.

3. Let F be a closed or an open Riemann surface spread over the z-plane and a disc $|z|\leqq R$ be contained in F. Let $0<a<R$ and

$$S(z)=\frac{x}{x^2+y^2}+\frac{x}{a^2} \quad \text{in} \quad |z|\leqq R, z=x+iy, \tag{*}$$

then $\dfrac{\partial S}{\partial \nu}=0$ on $|z|=a$, where ν is the normal of $|z|=a$.

Let $w(z)$ be a function defined on F, then we define $w^*(z)$ by

$$\begin{aligned} w^*(z)&=w(z)-S(z) \text{ in } |z|<a \\ &=w(z) \text{ outside of } |z|<a. \end{aligned} \tag{**}$$

Let $\mathfrak{F}$ be the family of $w(z)$, which is continuous on F, except at $z=0$, where $w(z)-S(z)$ is continuous and has piece-wise continuous partial derivatives of the first order on $F-\{0\}$ and the Dirichlet integral $D_F[w^*]=D[w^*]$ is finite. We see easily that $\mathfrak{F}$ is not an empty set. Let

$$d=\inf_{w\in\mathfrak{F}} D[w^*].$$

THEOREM 2.[4] *There exists a harmonic function* $u(z)$ *on* $F-\{0\}$, *such that*

$$D[u^*]=d.$$

PROOF. First we assume that F is an open Riemann surface. By the definition of d, there exists $w_n \in \mathfrak{F}$, such that $D[w_n^*]\to d$. Let $a<a_1<a_2<R$ and $u_n(z)$ be harmonic in U: $a_1<|z|<a_2$, such that $u_n=w_n$ on $|z|=a_1$ and $|z|=a_2$, then by Lemma 1, $D_U[u_n]\leqq D_U[w_n]$, so that if we put $\tilde{w}_n(z)=u_n(z)$ in U, $\tilde{w}_n(z)=w_n(z)$ in $F-U$, then $D[\tilde{w}_n^*]\leqq D[w_n^*]$, so that $D[\tilde{w}_n^*]\to d$. Hence we may assume that

$$w_n(z) \text{ is harmonic in } U: a_1<|z|<a_2. \tag{1}$$

Let K: $|z|\leqq \frac{a_1+a_2}{2}$ and $v_n(z)$ be harmonic in K, such that

$$v_n=w_n-S \quad \text{on} \quad |z|=\frac{a_1+a_2}{2} \tag{2}$$

and put

$$u_n(z)=v_n(z)+S(z) \quad \text{in} \quad K, \tag{3}$$

then

$$u_n^*(z)=v_n(z) \quad \text{in} \quad |z|<a.$$

We shall prove that

$$D_K[u_n^*]\leqq D_K[w_n^*]. \tag{4}$$

Put

$$\varphi_n(z)=w_n(z)-S(z) \quad \text{in} \quad K, \tag{5}$$

then

$$\varphi_n(z)=w_n^*(z) \quad \text{in} \quad |z|<a.$$

$\varphi_n(z)$ is continuous in K and $\varphi_n=v_n$ on $|z|=\frac{a_1+a_2}{2}$, hence by Lemma 2,

$$D_K[v_n]\leqq D_K[\varphi_n]. \tag{6}$$

Let K_1: $|z|<a$ and K_2: $a\leqq|z|\leqq\frac{a_1+a_2}{2}$, then $K=K_1+K_2$, so that

4) H. Weyl. l. c. 1). Hurwitz-Courant: Funktionentheorie. Berlin (1929).

$$D_K[u_n^*]=D_{K_1}[u_n^*]+D_{K_2}[u_n]=D_{K_1}[v_n]+D_{K_2}[v_n+S]=D_{K_1}[v_n]+D_{K_2}[v_n]$$
$$+D_{K_2}[S]+2D_{K_2}[v_n, S]=D_K[v_n]+D_{K_2}[S]+2D_{K_2}[v_n, S],$$
$$D_K[w_n^*]=D_{K_1}[w_n^*]+D_{K_2}[w_n]=D_{K_1}[\varphi_n]+D_{K_2}[\varphi_n+S]$$
$$=D_K[\varphi_n]+D_{K_2}[S]+2D_{K_2}[\varphi_n, S],$$

hence by (6)

$$D_K[w_n^*]-D_K[u_n^*]=D_K[\varphi_n]-D_K[v_n]+2D_{K_2}[\varphi_n-v_n, S]\geqq 2D_{K_2}[\varphi_n-v_n, S].$$

Since $S(z)$ is harmonic in K_2, $\frac{\partial S}{\partial \nu}=0$ on $|z|=a$ and by (2), (5), $\varphi_n-v_n=0$ on $|z|=\frac{a_1+a_2}{2}$, we have $D_{K_2}[\varphi_n-v_n, S]=0$, so that

$$D_K[w_n^*]-D_K[u_n^*]\geqq 0, \quad \text{or} \quad D_K[u_n^*]\leqq D_K[w_n^*].$$

Now by (1), $w_n(z)$ and $S(z)$ are harmonic on $|z|=\frac{a_1+a_2}{2}$ and by (2) $v_n-(w_n-S)=0$ on $|z|=\frac{a_1+a_2}{2}$, so that $v_n-(w_n-S)$ and hence

$$v_n(z) \text{ is harmonic in } |z|<\frac{a_1+a_2}{2}+\delta \tag{7}$$

for a suitable $\delta>0$.

We may assume that $v_n(0)=0$. Since by (5), (6), the Dirichlet integral of v_n in $|z|<\frac{a_1+a_2}{2}+\delta$ is bounded for $n=1, 2, \cdots$ and $v_n(0)=0$, we can select a partial sequence from n, which we denote by n again, such that

$$\lim_{n\to\infty} v_n(z)=v(z) \text{ uniformly in } |z|<\frac{a_1+a_2}{2}+\delta, \tag{8}$$

so that by (3), if we put

$$u(z)=\lim_{n\to\infty} u_n(z)=v(z)+S(z) \text{ in } K, \tag{9}$$

then by (4),

$$D_K[u^*]\leqq \varliminf_{n\to\infty} D_K[w_n^*]. \tag{10}$$

$u(z)$ is harmonic in $0<|z|<\frac{a_1+a_2}{2}+\delta$. Let $K\subset F_1\subset F_2\subset\cdots\subset F_n\to F$ be an exhaustion of F, where F_n is a compact Riemann surface, whose boundary Γ_n consists of a finite number of analytic Jordan curves.

Let $\psi_n(z)$ be harmonic in F_n-K, such that $\psi_n=w_n$ on $|z|=\frac{a_1+a_2}{2}$ and $\frac{\partial\psi_n}{\partial\nu}=0$ on Γ_n. Since $\psi_n-w_n=0$ on $|z|=\frac{a_1+a_2}{2}$ and w_n is harmonic on $|z|=\frac{a_1+a_2}{2}$, ψ_n-w_n and hence ψ_n is harmonic on $|z|=\frac{a_1+a_2}{2}$, so that $\frac{\partial\psi_n}{\partial\nu}$ is continuous on $|z|=\frac{a_1+a_2}{2}$, hence by the Green's formula $D_{F_n-K}[w_n-\psi_n,\psi_n]=0$, so that

$$D_{F_n-K}[\psi_n]\leqq D_{F_n-K}[w_n]\leqq D_{F-K}[w_n]. \tag{11}$$

Since $\psi_n-(v_n+S)=\psi_n-w_n=0$ on $|z|=\frac{a_1+a_2}{2}$ and v_n converges uniformly in $|z|<\frac{a_1+a_2}{2}+\delta$ and by (11), $D_{F_n-K}[\psi_n]$ is bounded for $n=1,2,\cdots$, we see that

$$\lim_{n\to\infty}\psi_n(z)=\psi(z) \text{ uniformly in the wider sense in } F-K, \tag{12}$$

hence from (11),

$$D_{F-K}[\psi]\leqq \varliminf_{n\to\infty} D_{F-K}[w_n]. \tag{13}$$

$\psi(z)$ is harmonic on $F-K$ and $\psi=v+S$ in $|z|=\frac{a_1+a_2}{2}$. Hence if we put $u(z)=v(z)+S(z)$ in K, $u(z)=\psi(z)$ in $F-K$, then $u(z)$ is continuous in $F-\{0\}$ and harmonic in $F-\{0\}$, except on $|z|=\frac{a_1+a_2}{2}$ and from (10), (13), $D[u^*]\leqq \lim_{n\to\infty} D[w_n^*]=d$, so that $D[u^*]=d$. We can prove as before that $u(z)$ is harmonic on $|z|=\frac{a_1+a_2}{2}$. Hence the theorem is proved, if F is an open Riemann surface. If F is a closed Riemann surface, we take off a point z_0 ($\neq 0$) from F and put $F_0=F-\{z_0\}$, then F_0 is an open Riemann surface. We construct the harmonic function $u(z)$ for F_0, then since the Dirichlet integral of $u(z)$ in a neighbourhood of z_0 is finite, $u(z)$ is harmonic at z_0. Hence the theorem is proved.

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