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On the convergence-region of interpolation polynomials.

By Tetsujiro KAKEHASHI

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The convergence of interpolation polynomials to a given function in the points which satisfy a certain condition has been studied by Walsh and others. (Cf. Walsh: Interpolations and Approximations; American Mathematical Society Colloquium Publications, 1935.)

If the sequence of polynomials which interpolate to a given function in the points is a series, an exact region of the convergence can be studied in a manner similar to that of the power series, but if the sequence is not a series, the exact region of the convergence, except in some particular cases, has not yet been established, as far as I know.

In the particular case, where the points of interpolation are defined by the zeros of polynomials $z^n-1=0$; $n=1, 2, 3, \cdots$, the exact region of the convergence of interpolation polynomials has been determined by Walsh. (The divergence of sequences of polynomials interpolating in roots of unity; Bulletin of the American Mathematical Society, 1936, Vol. 42, page 715.)

The purpose of this paper is to generalize the results given in the above-mentioned paper by Walsh, and to determine the exact region of the convergence of interpolation polynomials in more generalized point sets.

1. Let the function $f(z)$ be analytic throughout the interior of the circle Γ_ρ: $|z|=\rho>1$ but not analytic on Γ_ρ. Let $\lambda(z)$ be an analytic function with positive modulus exterior to the unit circle Γ: $|z|=1$. For t on Γ_R $(\rho>R>1)$ and for a fixed point z which lies between Γ and Γ_R, we consider the series

$$\frac{\lambda(t)-\lambda(z)}{\lambda(t)}\frac{1}{t-z}+\frac{\lambda(z)}{\lambda(t)t}\left(1+\frac{z}{t}+\frac{z^2}{t^2}+\cdots+\frac{z^n}{t^n}\right)$$

$$= \frac{1}{t-z}\left(1 - \frac{\lambda(z)z^{n+1}}{\lambda(t)t^{n+1}}\right); \quad n=0, 1, 2, \cdots.$$

$\lambda(t)$ being an analytic function with positive modulus on Γ_R, the series converges to the function $\dfrac{1}{t-z}$ uniformly for t on Γ_R as n tends to infinity.

Accordingly, we can define approximating functions of $f(z)$, which is analytic throughout the interior of Γ_ρ $(\rho > R > 1)$, by

$$(1.1) \qquad S_n(z; f) = \frac{1}{2\pi i}\int_{\Gamma_R} \frac{\lambda(t)t^{n+1} - \lambda(z)z^{n+1}}{\lambda(t)t^{n+1}} \frac{f(t)}{t-z}\,dt$$

$$= \frac{1}{2\pi i}\int_{\Gamma_R} \frac{\lambda(t) - \lambda(z)}{\lambda(t)} \frac{f(t)}{t-z}\,dt + \textstyle\sum_{k=0}^{n} \alpha_k \lambda(z) z^k,$$

where

$$(1.2) \qquad \alpha_k = \frac{1}{2\pi i}\int_{\Gamma_R} \frac{f(t)}{\lambda(t)t^{k+1}}\,dt; \qquad k=0, 1, 2, \cdots.$$

Then we have

$$(1.3) \qquad f(z) - S_n(z; f) = \frac{1}{2\pi i}\int_{\Gamma_R} \frac{\lambda(z)z^{n+1}}{\lambda(t)t^{n+1}} \frac{f(t)}{t-z}\,dt.$$

The series $S_n(z; f)$ is defined for z exterior to Γ and even for z exterior to Γ_ρ, but if $\lambda(z)$ is suitably defined for z on and interior to Γ, we can define $S_n(z; f)$ for z on and interior to Γ. $S_n(z; f)$ is a certain interpolation formula of $f(z)$ and has properties similar to those of the power series of $f(z)$.

THEOREM 1. *Let the function $f(z)$ be analytic throughout the interior of the circle Γ_ρ: $|z| = \rho > 1$ but not analytic on Γ_ρ. Let $\lambda(z)$ be an analytic function with positive modulus exterior to the unit circle Γ: $|z| = 1$.*

Then the series $S_n(z; f)$ defined by (1.1) *converges to $f(z)$ throughout the interior of the region between Γ and Γ_ρ, uniformly on any closed region between Γ and Γ_ρ, and diverges at every point exterior to Γ_ρ. Moreover, we have*

$$(1.4) \qquad \overline{\lim_{n\to\infty}}\, |f(z) - S_n(z; f)|^{1/n} \leq R'/\rho$$

for z on $\Gamma_{R'}$ $(1 < R' < \rho)$, and for z exterior to Γ_ρ

(1.5) $$\overline{\lim_{n\to\infty}}\ |S_n(z;f)|^{1/n}=|z|/\rho\,.$$

The first part of the theorem can be proved from the last part of the theorem, that is, from the relations (1.4) and (1.5).

If we choose a circle Γ_R between Γ_ρ and $\Gamma_{R'}$, the equation (1.3) is valid for z on $\Gamma_{R'}$. Thus, for z on $\Gamma_{R'}$,

$$\overline{\lim_{n\to\infty}}\ |f(z)-S_n(z;f)|^{1/n}\leqq R'/R<1$$

follows immediately from (1.3). Allowing R to approach ρ, then yields the relation (1.4).

Let α_k; $k=0,1,2,\cdots$ be the coefficients defined by (1.2). If we expand the function $f(z)/\lambda(z)$ into *Laurent's series*, we have

$$f(z)/\lambda(z)=\sum_{k=0}^{\infty}\alpha_k z^k+\sum_{k=1}^{\infty}\beta_k z^{-k}\,.$$

Then the equality

(1.6) $$\overline{\lim_{n\to\infty}}\ |\alpha_n|^{1/n}=1/\rho$$

can be verified easily.

For n sufficiently large we have

$$|\alpha_n|<\left|\frac{1}{\rho}+\varepsilon\right|^n,$$

so the sequence $|\alpha_k|\Big/\left(\frac{1}{\rho}+\varepsilon\right)^k$ has a finite upper bound K, thus

$$|\alpha_k|\leqq K\left(\frac{1}{\rho}+\varepsilon\right)^k,$$

$$|S_n(z;f)|\leqq K'+K|\lambda(z)|\sum_{k=0}^{n}\left(\frac{1}{\rho}+\varepsilon\right)^k|z|^k$$

$$=K'+K|\lambda(z)|\frac{\left(\frac{1}{\rho}+\varepsilon\right)^{n+1}|z|^{n+1}-1}{\left(\frac{1}{\rho}+\varepsilon\right)|z|-1},$$

$$\overline{\lim_{n\to\infty}}\ |S_n(z;f|^{1/n}\leqq\left(\frac{1}{\rho}+\varepsilon\right)|z|$$

for z exterior to Γ_ρ, where K' is the absolute value of $\frac{1}{2\pi i}\int_{\Gamma_R}\times$ $\times\frac{\lambda(t)-\lambda(z)}{\lambda(t)}\frac{f(t)}{t-z}dt$, which depends on z but not on $R\,(1<R<\rho)$. Allowing ϵ to approach zero, then yields the relation

$$\overline{\lim_{n\to\infty}}\ |S_n(z;f)|^{1/n}\leqq |z|/\rho\,.$$

If we now assume the inequality

$$\overline{\lim_{n\to\infty}}\ |S_n(z;f)|^{1/n}<A<|z|/\rho\,,$$

for any fixed z exterior to Γ_ρ, we shall reach a contradiction. For n sufficiently large we have

$$|S_{n-1}(z;f)|<A^{n-1}, \qquad |S_n(z;f)|<A^n,$$

$$|S_n(z;f)-S_{n-1}(z;f)|=|\alpha_n\lambda(z)z^n|<A^{n-1}(A+1),$$

$$|\alpha_n|<A^{n-1}(A+1)/|\lambda(z)z^n|,$$

$$\overline{\lim_{n\to\infty}}\ |\alpha_n|^{1/n}\leqq A/|z|<1/\rho\,,$$

which contradicts (1.6). Equation (1.5) has been proved. Thus the theorem is established.

2. Let a function $f(z)$ be analytic within the circle Γ_ρ: $|z|=\rho>1$ but not analytic on Γ_ρ, and be given a set of points

$$(2.1)\qquad \begin{cases} z_1^{(0)} \\ z_1^{(1)},\ z_2^{(1)} \\ z_1^{(2)},\ z_2^{(2)},\ z_3^{(2)} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ z_1^{(n)},\ z_2^{(n)},\ z_3^{(n)},\ \cdots,\ z_{n+1}^{(n)} \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{cases}$$

which does not lie exterior to the unit circle Γ. The sequence of polynomials $P_n(z;f)$ of respective degrees n found by interpolation to $f(z)$ in the points $z_1^{(n)}, z_2^{(n)}, \cdots, z_{n+1}^{(n)}$ is defined for $R\,(1<R<\rho)$ by

$$(2.2)\qquad P_n(z;f)=\frac{1}{2\pi i}\int_{\Gamma_R}\frac{\varphi_{n+1}(t)-\varphi_{n+1}(z)}{\varphi_{n+1}(t)}\frac{f(t)}{t-z}dt\,,$$

and the relation

$$f(z)-P_n(z;f)=\frac{1}{2\pi i}\int_{\Gamma_R}\frac{\varphi_{n+1}(z)}{\varphi_{n+1}(t)}\frac{f(t)}{t-z}dt \tag{2.3}$$

is valid for z interior to $\Gamma_R\,(\rho>R>1)$, where

$$\varphi_{n+1}(z)=(z-z_1^{(n)})(z-z_2^{(n)})\cdots(z-z_{n+1}^{(n)})\,.$$

Let the points (2.1) satisfy the condition that the sequence $\varphi_n(z)/z^n$ converges to a function $\lambda(z)$ with positive modulus like a geometric series for z exterior to the unit circle Γ: $|z|=1$. That is, we have, for any positive number $R_1(>1)$ and for a certain positive number independent of n and z, the relation

$$\left|\frac{\varphi_n(z)}{z^n}-\lambda(z)\right|<M\alpha^n \tag{2.4}$$

uniformly for z on and exterior to Γ_{R_1}, where M is a positive number independent of n and z. This condition can be replaced by the existence of the function $\lambda(z)$ which satisfies

$$\overline{\lim_{n\to\infty}}\,|\varphi_n(z)-\lambda(z)z^n|<|z| \quad \text{for} \quad |z|>1\,. \tag{2.5}$$

It is clear that the condition (2.4) or (2.5) yields the relation

$$\lim_{n\to\infty}|\varphi_n(z)|^{1/n}=|z| \tag{2.6}$$

for z exterior to the unit circle Γ, and uniformly for $|z|\geqq R>1$, and the relation

$$\overline{\lim_{n\to\infty}}\,|\varphi_n(z)|^{1/n}\leqq 1 \qquad \text{for} \qquad |z|\leqq 1 \tag{2.7}$$

can be verified by the principle of maximum from (2.6).

THEOREM 2. *Let $f(z)$ be the function which satisfies the condition in the theorem* 1, *and $\varphi_n(z)$ be the sequence of polynomials of respective degrees n such as the sequence $\varphi_n(z)/z^n$ converges to a function $\lambda(z)$ with positive modulus in such a way that* (2.5) *or* (2.6) *holds for z exterior to the unit circle Γ. Let $P_n(z;f)$ be the unique polynomial of degree n which interpolates to $f(z)$ in all the zeros of $\varphi_{n+1}(z)$. Then the sequence of polynomials $P_n(z;f)$ converges to $f(z)$ throughout the* *region* $|z|<\rho$, *and uniformly on any closed set interior to* Γ_ρ. *The sequence* $P_n(z;f)$ *diverges at every point exterior to* Γ_ρ.

Moreover, we have

$$\overline{\lim_{n\to\infty}}\ |f(z)-P_n(z;f)|^{1/n}\leqq R_1/\rho \tag{2.8}$$

for z *on* Γ_{R_1} $(1<R_1<\rho)$,

$$\overline{\lim_{n\to\infty}}\ |f(z)-P_n(z;f)|^{1/n}\leqq 1/\rho \quad \textit{for} \quad |z|\leqq 1, \tag{2.9}$$

and

$$\overline{\lim_{n\to\infty}}\ |P_n(z;f)|^{1/n}=|z|/\rho \tag{2.10}$$

for z *exterior to* Γ_ρ.

The first part of the theorem follows immediately from the last part of the theorem, that is, from the relations (2.8), (2.9) and (2.10). The inequalities (2.8) and (2.9) can be verified respectively from the relations

$$\overline{\lim_{n\to\infty}}\ |f(z)-P_n(z;f)|^{1/n}\leqq R_1/R \quad (1<R_1<R<\rho)$$

and

$$\overline{\lim_{n\to\infty}}\ |f(z)-P_n(z;f)|^{1/n}\leqq 1/R \quad \text{for} \quad |z|\leqq 1$$

which can be estimated directly from the equation (2.3) by (2.6) and (2.7).

In our proof of the equation (2.10), it is convenient to have the following lemma by Walsh. (Cf. The divergence of sequences of polynomials interpolating in roots of unity; Bulletin of the American Mathematical Society, 1936, Vol. 42, page 715.)

LEMMA. *The relations*

$$\overline{\lim_{n\to\infty}}\ |A_n|^{1/n}=a, \qquad \overline{\lim_{n\to\infty}}\ |A_n+B_n|=b<a \tag{2.11}$$

imply

$$\overline{\lim_{n\to\infty}}\ |B_n|^{1/n}=\overline{\lim_{n\to\infty}}\ |A_n|^{1/n}=a. \tag{2.12}$$

We are now to prove the theorem. Subtraction of (2.2) from (1.1) side by side yields the relation

$$S_n(z;f)-P_n(z;f)=\frac{1}{2\pi i}\int_{\Gamma_R}\left[\frac{\varphi_{n+1}(z)}{\varphi_{n+1}(t)}-\frac{\lambda(z)z^{n+1}}{\lambda(t)t^{n+1}}\right]\frac{f(t)}{t-z}\,dt. \tag{2.13}$$

It is seen that the left-hand side of (2.13) represents a function of z which is analytic for all finite values of z exterior to the unit cirle Γ, so the equation (2.13) is valid for all finite values of z, even for z interior to Γ, if $\lambda(z)$ is suitably defined there.

Substitution of $\varepsilon_{n+1}(z)=\varphi_{n+1}(z)-\lambda(z)z^{n+1}$ for (2.13) yields

$$S_n(z;f)-P_n(z;f)=\frac{1}{2\pi i}\int_{\Gamma_R}\frac{\varepsilon_{n+1}(z)\lambda(t)t^{n+1}-\varepsilon_{n+1}(t)\lambda(z)z^{n+1}}{\lambda(t)t^{n+1}[\lambda(t)t^{n+1}+\varepsilon_{n+1}(t)]}\frac{f(t)}{t-z}dt,$$

and applying (2.4) to this equation, we have for any finite value of z exterior to Γ_ρ

$$\overline{\lim_{n\to\infty}}|S_n(z;f)-P_n(z;f)|^{1/n}\leqq\alpha|z|/R\quad(\alpha<1).$$

R can be allowed to approach ρ, whence we have the relation

$$\overline{\lim_{n\to\infty}}|S_n(z;f)-P_n(z;f)|^{1/n}\leqq\alpha|z|/\rho<|z|/\rho \tag{2.14}$$

for z exterior to Γ_ρ.

Accordingly, from (1.5), (2.14) and the lemma we can verify the relation

$$\overline{\lim_{n\to\infty}}|P_n(z;f)|^{1/n}=|z|/\rho$$

for z exterior to the circle Γ_ρ. Thus the sequence can not be bounded when $|z|>\rho$, hence can not converge. The theorem is thus established.

3. In this paragraph, we consider some examples of polynomials which satisfy the condition in the previous paragraph.

It is clear that the sequence of polynomials

$$\varphi_n(z)=z^n-1;\qquad n=1,2,\cdots \tag{3.1}$$

satisfies the condition in theorem 2 for z exterior to the unit circle Γ, and $\lambda(z)$ can be determined so that we have $\lambda(z)\equiv1$. The sequence of polynomials

$$\varphi_n(z)=1+z+z^2+\cdots+z^n;\quad n=1,2,\cdots \tag{3.2}$$

also satisfies the same condition and $\lambda(z)$ is given by $z/z-1$.

Next we consider an important example. Let $D(z)$ be a function analytic and non-zero for z interior to the unit circle Γ: $|z|=1$. Let

$D_n(z)$ be the partial sums of power series of $D(z)$ about $z=0$, that is,

(3.3) $$D_n(z)=a_0+a_1z+a_2z^2+\cdots+a_nz^n\ ;\quad n=1, 2, \cdots.$$

Then we can verify the relation

(3.4) $$\lim_{n\to\infty} |D(z)-D_n(z)|^{1/n}\leqq|z|<1$$

for z interior to Γ.

Let $D_n^*(z)$ be the *reciprocal polynomial* of $D_n(z)$, that is

(3.5) $$D_n^*(z)=z^n\bar{D}_n(z^{-1})=\bar{a}_0z^n+\bar{a}_1z^{n-1}+\cdots+\bar{a}_n\,.$$

Then we have from (3.4) the relation

$$\overline{\lim_{n\to\infty}} |D_n^*(z)-\bar{D}(z^{-1})z^n|\leqq 1<|z|$$

for z exterior to Γ. Thus we can verify that the sequence of polynomials $D_n^*(z)$ satisfies the condition of theorem 2.

The following example is also important. Let $W(\theta)$ be the positive weight function which setisfies the relation

(3.6) $$W(\theta)=\{T_m(\theta)\}^{-1}>0\ ;\quad 0\leqq\theta\leqq 2\pi\,,$$

where $T_m(\theta)$ is a positive trigonometric polynomial of degree m. Then we know that there exists a unique polynomial $h_m(z)$ of degree m which satisfies the conditions

(3.7) $$T_m(\theta)=|h_m(e^{i\theta})|^2>0$$

and

(3.8) $$h_m(0)>0\,,\quad |h_m(z)|>0\quad \text{for}\quad |z|\leqq 1\,.$$

(This result has been obtained by Fejér. Cf. Szegö: Orthogonal Polynomials, American Mathematical Society Colloquium Publications.)

Then, it is known that the polynomials

(3.9) $$\psi_n(z; m)=\bar{h}_m(z^{-1})z^n\ ;\qquad n=m, m+1, \cdots\cdots$$

form the ortho-normal set of polynomials associated with the weight function $W(\theta)$. Indeed, for $\rho(z)$ an arbitrary polynomial of degree less than n, we have

$$\frac{1}{2\pi}\int_0^{2\pi} W(\theta)\psi_n(z;m)\bar{\rho}(z)d\theta = \frac{1}{2\pi}\int_{|z|=1}\{h_m(z)\bar{h}_m(z^{-1})\}^{-1}z^n\bar{h}_m(z^{-1})\bar{\rho}(z^{-1})\frac{dz}{iz}$$

$$= \frac{1}{2\pi i}\int_{|z|=1}\frac{z^{n+1}\bar{\rho}(z^{-1})}{h_m(z)}dz = 0: \quad z=e^{i\theta}$$

according to *Cauchy's theorem*, and

$$\frac{1}{2\pi}\int_0^{2\pi} W(\theta)|\psi_n(z;m)|^2 d\theta = \frac{1}{2\pi}\int_0^{2\pi}|h_m(z)|^{-2}|z^n\bar{h}_m(z^{-1})|^2 d\theta = 1\,.$$

Accordingly, it is clear that the sequence of polynomials $\psi_n(z;m)$ defined by (3.9) satisfies the condition in theorem 2.

More generally, let $F(z)$ be a function analytic and positive on the unit circle Γ and $W(\theta)$ be the weight function defined by

$$W(\theta) = F(e^{i\theta}) > 0\,.$$

Let $\phi_n(z)$ be the set of ortho-normal polynomials associated with the weight function $W(\theta)$. Then we can prove that the sequence of polynomials $\phi_n(z)$ satisfies the same condition. This problem we shall consider in paragraph 5.

4. In this paragraph, we consider a generalization of the results obtained in paragraph 2.

Let D be a closed limited point set whose complement K with respect to the extended plane is connected and regular in the sense that K possesses a *Green's function* with pole at infinity. Let $w=\phi(z)$ map K onto the region $|w|>1$ so that the points at infinity correspond to each other. Let $C_R\,(R>1)$ be the level curve determined by $|w|=R>1$.

Given a function $f(z)$ analytic throughout the interior of the level curve $C_\rho\,(\rho>1)$, but not analytic on C_ρ, and given a set of points (2.1) which lie on D, the sequence of polynomials $P_n(z;f)$ of respective degrees n found by interpolation to $f(z)$ in the points $z_1^{(n)}, z_2^{(n)}, \cdots, z_{n+1}^{(n)}$ is defined, for any positive number R less than ρ but greater than unity, by

$$P_n(z;f) = \frac{1}{2\pi i}\int_{C_R}\frac{\varphi_{n+1}(t)-\varphi_{n+1}(z)}{\varphi_{n+1}(t)}\frac{f(t)}{t-z}dt\,. \tag{4.1}$$

And the relation

$$(4.2) \qquad f(z)-P_n(z;f)=\frac{1}{2\pi i}\int_{C_R}\frac{\varphi_{n+1}(z)}{\varphi_{n+1}(t)}\frac{f(t)}{t-z}dt$$

is valid for z interior to C_R $(\rho>R>1)$, where

$$\varphi_{n+1}(z)=(z-z_1^{(n)})(z-z_2^{(n)})\cdots(z-z_{n+1}^{(n)}).$$

Let the set of points (2.1) satisfy the condition that the sequence of function $\varphi_n(z)/\varDelta^n w^n$ converges to an analytic function $\lambda(w)=\lambda[\phi(z)]$ with positive modulus like a geometric series for z exterior to D, where $\varDelta$ is the *capacity* of D. That is, we have, for a certain positive number $\alpha\,(<1)$ independent of n and z, the relation

$$(4.3) \qquad |\varphi_n(z)[\varDelta w]^{-n}-\lambda(w)|<M\alpha^n: \quad w=\phi(z)$$

uniformly for z on any closed region interior to K, where M is a positive number independent of n and z. This condition can be replaced by the existence of the function $\lambda(w)$ which satisfies

$$(4.4) \qquad \overline{\lim_{n\to\infty}}\,|\varphi_n(z)-\lambda(w)[\varDelta w]^n|^{1/n}<\varDelta|w| \quad \text{for} \quad |w|>1.$$

It is clear that the relation (4.3) or (4.4) yields

$$(4.5) \qquad \lim_{n\to\infty}|\varphi_n(z)|^{1/n}=\varDelta|\phi(z)|$$

uniformly on any closed limited points set interior to K.

Now we can define the sequence of approximating functions $S_n(z;f)$ similarly to those in paragraph 1, that is, for any positive number R between 1 and ρ,

$$(4.6) \qquad S_n(z;f)=\frac{1}{2\pi i}\int_{C_R}\left\{1-\frac{\lambda(z)[\phi(z)]^{n+1}}{\lambda(t)[\phi(t)]^{n+1}}\right\}\frac{f(t)}{t-z}dt.$$

And for z between C: $|w|=1$ and C_R, we have

$$(4.7) \qquad f(z)-S_n(z;f)=\frac{1}{2\pi i}\int_{C_R}\frac{\lambda(z)[\phi(z)]^{n+1}}{\lambda(t)[\phi(t)]^{n+1}}\frac{f(t)}{t-z}dt.$$

The properties of $S_n(z;f)$ analogous to those of theorem 1 can be verified in a manner similar to that of theorem 1. Thus the following theorem can be verified as the generalization of theorem 2.

THEOREM 3. *Let D be a closed limited point set whose complement K with respect to the extended plane is connected and regular in* *the sense that K possesses a Green's function with pole at infinity. Let $w=\phi(z)$ map K onto the region $|w|>1$ so that the points at infinity correspond to each other.*

Let the function $f(z)$ be analytic throughout the interior of the level curve C_ρ: $|\phi(z)|=\rho>1$ but not analytic on C_ρ, and let a set of polynomials $\varphi_n(z)$ of respective degrees n satisfy the condition (4.3) *or* (4.4).

Then the sequence of polynomials $P_n(z;f)$ of respective degress n which interpolates to $f(z)$ in all the zeros of $\varphi_{n+1}(z)$ converges to $f(z)$ throughout the region $|w|<\rho$, uniformly on any closed set interior to C_ρ, and diverges at every point exterior to C_ρ. Moreover, we have

$$\varlimsup_{n\to\infty} |f(z)-P_n(z;f)|^{1/n} \leqq R_1/\rho \tag{4.8}$$

for z on C_{R_1}; $|w|=R_1$ $(1<R_1<\rho)$,

$$\varlimsup_{n\to\infty} |f(z)-P_n(z;f)|^{1/n} \leqq 1/\rho \quad \textit{for} \quad |\phi(z)|\leqq 1 \tag{4.9}$$

and

$$\varlimsup_{n\to\infty} |P_n(z;f)|^{1/n} = |w|/\rho \tag{4.10}$$

for z exterior to C_ρ.

Examples considered in paragraph 3 can be applied to this generalized case. The following example is to be noticed.

The polynomials $\varphi_n(z)$ of respective degrees n found by the orthogonalization of the set $1, z, z^2, \cdots$ on the line segment $-1\leqq z\leqq 1$ with respect to the weight function $(1-z^2)^{-1/2}$ are known as *Tchebycheff's polynomials.* In this case, by the transformation

$$z=\frac{1}{2}(w+w^{-1}),$$

the exterior of the real interval $[-1, 1]$ is transformed onto the exterior of the unit circle Γ: $|w|=1$ so that the points at infinity correspond to each other. And the polynomials are given by

$$\varphi_n(z)=2^{-n}(w^n+w^{-n}),$$

which satisfy the condition of theorem 3.

Accordingly, the sequence of polynomials $P_n(z;f)$ of respective degrees n found by interpolation to $f(z)$, which is analytic throughout the interior of C_ρ but not analytic on C_ρ, in all the zeros of Tchebycheff's polynomials converge to $f(z)$ for z interior to C_ρ, uniformly on any closed point set interior to C_ρ and diverges at every point exterior to C_ρ. In this case, the exact region of the convergence of $P_n(z;f)$ is determined by the interior of the ellipse with foci at ± 1 and with semi-axes $\frac{1}{2}(\rho+\rho^{-1})$ and $\frac{1}{2}(\rho-\rho^{-1})$. That is equivalent to the region on which the Fourier-expansion of $f(z)$ by Tchebycheff's polynomials converges.

More generally, we can prove that the set of orthogonal polynomials $\varphi_n(z)$ associated with the positive weight function

$$P(z)=F(z)(1-z^2)^{-1/2}: \quad -1<z<1,$$

where the function $F(z)$ is analytic and positive on $[-1, 1]$, satisfies the condition (4.4) or (4.5). In this case, the mapping function $w=\phi(z)$ is also given by $z=\frac{1}{2}(w+w^{-1})$ and the capacity Δ of $[-1, 1]$ is 1/2.

Accordingly, the exact region of the ccnvergence of $P_n(z;f)$ found by interpolation to $f(z)$ in all the zeros of $\varphi_{n+1}(z)$ is identical to that of the convergence of Fourier-expansion by the set of $\varphi_n(z)$.

We shall study such a problem in paragraph 6.

5. In this paragraph, we consider asymptotic properties of the set of polynomials which are ortho-normal with respect to a certain positive weight function defined on the unit circle $\Gamma: z=e^{i\theta}$.

Let $\phi_n(z)$ be the set of ortho-normal polynomials associated with a weight function, on the unit circle $z=e^{i\theta}$, which satisfies a certain condition. Then the asymptotic behavior of $\phi_n(z)$ has been studied by Szegö and been found to be as follows;

$$(5.1) \qquad \lim_{n\to\infty} \phi_n(z)/z^n=\lambda(z) \quad \text{uniformly for} \quad |z|\geqq R>1,$$

where $\lambda(z)$ is an analytic function with positive modulus exterior to the unit circle Γ. (Cf. Szegö: Orthogonal polynomials, American Mathematical Society Colloquium Publications.)

The corresponding result for polynomials ortho-normal with respect to a weight function on the real segment $[-1, 1]$ is

$$(5.2) \qquad \lim_{n\to\infty} \phi_n(z)/w^n=\lambda(w) \quad \text{uniformly for} \quad |w|\geqq R>1,$$

where z is in complex plane cut along the segment $[-1, 1]$ and

$$z=\frac{1}{2}(w+w^{-1}).$$

If we add a certain condition to such a weight function, for the corresponding set of ortho-normal polynomials $\phi_n(z)$, the sequence of functions $\phi_n(z)/z^n$ or $\phi_n(z)/w^n$ will converge to $\lambda(z)$ or $\lambda(w)$ like a geometric series for z exterior to the unit circle $|z|=1$ or $|w|=1$; that is, we shall have the following asymptotic relation

$$\varlimsup_{n\to\infty} |\phi_n(z)-\lambda(z)z^n|^{1/n}<|z| \quad \text{for} \quad |z|>1,$$

or

$$\varlimsup_{n\to\infty} |\phi_n(z)-\lambda(w)w^n|^{1/n}<|w| \quad \text{for} \quad |w|>1,$$

corresponding to (5.1) and (5.2), respectively.

Such a property is applied to the study of the divergence problem of polynomials which interpolate to an analytic function in all the zeros of $\phi_n(z)$. (Cf. paragraphs 3 and 4.)

THEOREM 4. *Let $F(z)$ be a function analytic and non-zero throughout the interior of the region between the circles Γ_R: $|z|=R>1$ and $\Gamma_{R^{-1}}$: $|z|=R^{-1}<1$; but which has singularities or zeros on Γ_R or $\Gamma_{R^{-1}}$, and which is real and positive on the unit circle Γ: $|z|=1$. Let $\phi_n(z)$ be the set of ortho-normal polynomials associated with the weight function*

$$W(\theta)=F(e^{i\theta})>0: \quad 0\leqq\theta\leqq 2\pi$$

Then we have the asymptotic relations

$$\varlimsup_{n\to\infty} |\phi_n(z)-\bar{h}(z^{-1})z^n|^{1/n}\leqq|z|/R \quad \textit{for} \quad |z|\geqq 1, \tag{5.3}$$

and

$$\lim_{n\to\infty} |\phi_n(z)|^{1/n}= \begin{cases} |z| & \textit{for} \quad R^{-1}<|z|<1, \\ R^{-1} & \textit{for} \quad |z|\leqq R^{-1}, \end{cases} \tag{5.4}$$

where $h(z)$ is the function analytic and non-zero throughout the interior of the circle Γ_R, and uniquely determined under the conditions

$$\begin{cases} |h(e^{i\theta})|^2=\{F(e^{i\theta})\}^{-1}=\{W(\theta)\}^{-1}>0, \\ h(0)>0. \end{cases} \tag{5.5}$$

In our proof of this theorem, it is convenient to have several lemmas.

LEMMA 1. *Let $F(z)$ be a function analytic and non-zero throughout the interior of the region between the circles Γ_R and $\Gamma_{R^{-1}}$, and which is positive on the unit circle Γ. Then the function $h(z)$, analytic and non-zero throughout the interior of the circle Γ_R $(R>1)$ is uniquely determined under the conditions* (5.5).

Let $S_n(z)$ be the partial sums $S_n(z)=\sum_{k=-n}^{n} a_k z^k$ of Laurent's series $-\log\{F(z)\}=\sum_{-\infty}^{\infty} a_k z^k$, where $\log\{F(z)\}$ is analytic for $R^{-1}<|z|<R$ and real on the unit circle $z=e^{i\theta}$. Let $R_n(e^{i\theta})$ be the real parts of $S_n(e^{i\theta})$, that is,

(5.6) $$R_n(e^{i\theta})=\alpha_0+\sum_{k=1}^{n}(\alpha_k\cos k\theta+\beta_k\sin k\theta),$$

where

(5.7) $$\begin{cases} \alpha_0=\dfrac{1}{2\pi i}\displaystyle\int_{|z|=1}\dfrac{-\log\{F(t)\}}{t}\,dt=\dfrac{-1}{2\pi}\int_0^{2\pi}\log\{F(e^{i\theta})\}\,d\theta, \\ \alpha_k=\dfrac{1}{2}(a_k+a_{-k}), \\ \beta_k=\dfrac{1}{2i}(a_k-a_{-k}), \end{cases}$$

because of the relation $a_{-k}=\bar{a}_k$ which can be verified by the reality of $\log\{F(e^{i\theta})\}$.

Then we can verify the relation

$$\overline{\lim_{n\to\infty}}\{|\alpha_k|+|\beta_k|\}^{1/n}=R^{-1}<1$$

from the following property of Laurent's series of $-\log\{F(z)\}$

$$\max\{\overline{\lim_{n\to\infty}}|a_n|^{1/n},\ \overline{\lim_{n\to\infty}}|a_{-n}|^{1/n}\}=R^{-1}.$$

Accordingly, for any positive number r less than R, we can define the harmonic function $R(re^{i\theta})$ by

$$R(re^{i\theta})=\alpha_0+\sum_{k=1}^{\infty}r^k(\alpha_k\cos k\theta+\beta_k\sin k\theta);\qquad 0\leq r<R,$$

which converges for $r<R$ and uniformly for $r\leqq R_1<R$, and satisfies

$$R(e^{i\theta})=-\log\{F(e^{i\theta})\}, \qquad R(0)=\alpha_0.$$

Now $R(z)=R(re^{i\theta})$ is completed to an analytic function

$$\varphi(z)=\alpha_0+\sum_{k=1}^{\infty}(\alpha_k-i\beta_k)z^k,$$

and

$$h(z)=\exp\left\{\frac{1}{2}\varphi(z)\right\}$$

has been determined under the conditions

$$|h(e^{i\theta})|^2=\{F(e^{i\theta})\}^{-1}>0,$$

$$h(0)=\exp\alpha_0>0.$$

Thus the lemma is established.

Let $h_m(z)$ be partial sums of the power series of $h(z)$, that is

(5.8) $$h_m(z)=h(0)+h'(0)z+h''(0)\,z^2/2!+\cdots+h^{(m)}(0)\,z^m/m!;$$
$$m=0,1,2,\cdots,$$

which are non-zero on and interior to the unit circle Γ for m sufficiently large. Let $W_m(\theta)$ be the weight functions defined by the trigonometric polynomials

(5.9) $$\{W_m(\theta)\}^{-1}=|h_m(e^{i\theta})|^2>0$$

of respective degrees m sufficiently large. Then it is clear that the polynomials

(5.10) $$\psi_n(z;m)=\bar{h}_m(z^{-1})\,z^n; \qquad n=m,\ m+1,\cdots$$

form the ortho-normal set of polynomials associated with the weight function $W_m(\theta)$. (Cf. paragraph 3.)

Now we shall have the following lemma.

LEMMA 2. *Let $h(z)$ be a function analytic and non-zero within the circle Γ_R $(R>1)$, which has a singularity or zero on Γ_R. Let $W(\theta)$ and $W_n(\theta)$ be the positive weight functions defined respectively by*

$$W(\theta)=|h(e^{i\theta})|^{-2}>0,$$

and

$$W_n(\theta)=|h_n(e^{i\theta})|^{-2}>0 \qquad \textit{for } n \textit{ sufficiently large,}$$

where $h_n(z)$ are partial sums of the power series of $h(z)$ for n sufficiently large. Then we have

$$\overline{\lim_{n\to\infty}} |W(\theta)-W_n(\theta)|^{1/n}=\overline{\lim_{n\to\infty}} ||h(e^{i\theta})|^{-2}-|h_n(e^{i\theta})|^{-2}|^{1/n}\leqq R^{-1}. \tag{5.11}$$

This lemma can be proved easily by the following property of the power series of $h(z)$

$$\overline{\lim_{n\to\infty}} |h(e^{i\theta})-h_n(e^{i\theta})|^{1/n}\leqq R^{-1}, \tag{5.12}$$

which can be verified in a manner similar to that in paragraph 1, and the boundedness of $h(z)$ and $h_n(z)$ on Γ.

LEMMA 3. *Let $\psi_\nu(z;n)$; $\nu=0,1,2,\cdots$ be the set of ortho-normal polynomials associated with the weight function*

$$W_n(\theta)=|h_n(e^{i\theta})|^{-2}$$

defined in lemma 2. Let $K_n(\zeta,z)$ and $L_n(z)$ represent respectively

$$K_n(\zeta, z)=\sum_{\nu=0}^{h-1}\overline{\psi_\nu(\zeta, n)}\,\psi_\nu(z, n) \qquad \text{for } \zeta \text{ on } \Gamma, \tag{5.13}$$

and

$$L_n(z)=\int_0^{2\pi}|K_n(\zeta, z)|\,dt \qquad : \zeta=e^{it}. \tag{5.14}$$

Then we have

$$\overline{\lim_{n\to\infty}}\{L_n(z)\}^{1/n}=\begin{cases} |z| & \text{for} \quad |z|>1, \\ 1 & \text{for} \quad |z|\leqq 1. \end{cases} \tag{5.15}$$

The *kernel polynomials* $K_n(\zeta,z)$ can be calculated by a method similar to the proof of the *Christoffel-Darboux formula* as follows

$$\begin{aligned} K_n(\zeta, z)&=\frac{\overline{\psi_n^*(\zeta;n)}\,\psi_n^*(z;n)-\overline{\psi_n(\zeta;n)}\,\psi_n(z;n)}{1-\bar{\zeta}z} \\ &=\frac{\overline{h_n(\zeta)}\,h_n(z)-\overline{h_n^*(\zeta)}\,h_n(z)}{1-\bar{\zeta}z}, \end{aligned} \tag{5.16}$$

where * represents the reciprocal polynomial, that is,

$$p^*(z)=z^n p(z^{-1})=\bar{a}_n+\bar{a}_{n-1}z+\cdots+\bar{a}_0 z^n.$$

The last identity of (5.16) follows by the relation (5.10).

$h_n(z)$ being partial sums of the power series of $h(z)$, we can prove that $h_n(z)$ and $h_n^*(z)$ satisfy respectively the asymptotic relations

$$\overline{\lim_{n\to\infty}}\,|h_n(z)|^{1/n}=\begin{cases}|z|/R & \text{for} \quad |z|\geqq R>1, \\ 1 & \text{for} \quad |z|<R,\end{cases} \tag{5.17}$$

and

$$\overline{\lim_{n\to\infty}}\,|h_n^*(z)|^{1/n}=\begin{cases}|z| & \text{for} \quad |z|>R^{-1}, \\ R^{-1} & \text{for} \quad |z|\leqq R^{-1}.\end{cases} \tag{5.18}$$

These equations can be verified by a method similar to the proof of theorem 1.

For z which does not lie on the unit circle Γ, the modulus of denominator $|1-\zeta z|$ of (5.16) being positive, the validity of (5.15) can be verified by (5.17) and (5.18).

For z on Γ, we can prove that

$$\int_0^{2\pi}\left|\frac{\overline{h_n(\zeta)}\,h_n(z)-\overline{h_n^*(\zeta)}\,h_n^*(z)}{1-\zeta z}\right|dt=O(\log n);\quad \zeta=e^{it}. \tag{5.19}$$

Indeed, the numerater is a polynomial of degree n in z, which vanishes for $z=\zeta$. Therefore, we see that the integrand is $O(n)$ by the *theorem of Bernstein.* Thus the contribution of the arc $|\zeta-z|\leqq n^{-1}$ is $O(1)$, while the complementary arc $|\zeta-z|>n^{-1}$ supplies

$$O(1)\int_{|\zeta-z|>n^{-1}}\frac{dt}{|1-\zeta z|}=O(\log n).$$

Accordingly, the relation (5.15) is valid for this case. The lemma 3 is thus established.

LEMMA 4. *Let κ_n and κ_n' be the highest coefficients respectively of $\phi_n(z)$ and $\psi_n(z\,;n)$ which are defined respectively in theorem 4 and lemma 3. Then we have*

$$\kappa_n'=h_n(0)=h(0)>0 \tag{5.20}$$

and

$$\overline{\lim_{n\to\infty}}\,|\kappa_n-\kappa_n'|^{1/n}=\overline{\lim_{n\to\infty}}\,|\kappa_n-h(0)|^{1/n}\leqq R^{-1}. \tag{5.21}$$

The identity (5.20) can be verified by (5.10) and (5.8).

Let $\rho(z)=z^n+\cdots$ be an arbitrary polynomial of degree n with the highest term z^n. We know that the minimum value of

$$\frac{1}{2\pi}\int_0^{2\pi} W(\theta)\,|\rho(z)|^2 d\theta \quad : \quad z=e^{i\theta}$$

is κ_n^{-2}, attained for $\rho(z)=\kappa_n^{-1}\,\phi_n(z)$.

If $\rho(z)$ is any one of such polynomials, $z\rho(z)$ is a polynomial with the highest term z^{n+1}, and $|z\rho(z)|=|\rho(z)|$ for $z=e^{i\theta}$. Thus we can verify that

$$\kappa_n^{-2}\leqq\kappa_{n+1}^{-2} \qquad \text{or} \qquad \kappa_n\leqq\kappa_{n+1}$$

Consequently, $\lim_{n\leftarrow\infty}\kappa_n^{-2}=\mu\geqq 0$ exists. Szegö has proved

$$\lim_{n\to\infty}\kappa_n^{-2}=\{h(0)\}^{-2}>0$$

or

$$\lim_{n\to\infty}\kappa_n=h(0)>0$$

under the weaker condition of $W(\theta)$. (Cf. Orthogonal polynomials, page 293.)

Moreover, we have

$$\{h(0)\}^{-2}\leqq\kappa_n^{-2}\leqq\frac{1}{2\pi}\int_0^{2\pi} W(\theta)\,|\{h(0)\}^{-1}h_n^*(z)|^2 d\theta$$

$$=\{h(0)\}^{-2}\frac{1}{2\pi}\int_0^{2\pi}|h_n(z)\{h(z)\}^{-1}|^2 d\theta$$

or

$$h(0)\geqq\kappa_n\geqq h(0)\left[\frac{1}{2\pi}\int_0^{2\pi}|h_n(z)\{h(z)\}^{-1}|^2 d\theta\right]^{-1/2} : \ z=e^{i\theta}.$$

Now the relation (5.21) follows by

$$\overline{\lim_{n\to\infty}}\left[\frac{1}{2\pi}\int_0^{2\pi}|h_n(z)\{h(z)\}^{-1}|^2 d\theta-1\right]^{1/n}\leqq R^{-1}: \quad z=e^{i\theta},$$

which can be verified by (5.12). Thus lamma 4 is established.

We are now to prove the theorem. We shall express the polynomial $\phi_n(z)$, associated with $W(\theta)$, in terms of polynomials $\psi_\nu(z;n)$

corresponding to $W_n(\theta)$:

$$\phi_n(z)=\sum^{n}\alpha_\nu\psi_\nu(z;n)+\frac{1}{2\pi}\int_0^{2\pi}W_n(\theta)\,\phi_n(\zeta)\,K_n(\zeta,z)\,d\theta$$

$$=\kappa_n\{h(0)\}^{-1}\bar{h}_n(z^{-1})z^n+\frac{1}{2\pi}\int_0^{2\pi}\{W_n(\theta)-W(\theta)\}\phi_n(\zeta)\,K_n(\zeta,z)\,d\theta$$

$$+\frac{1}{2\pi}\int_0^{2\pi}W(\theta)\,\phi_n(\zeta)\,K_n(\zeta,z)\,d\theta \qquad : \zeta=e^{it}.$$

The last term vanishes because of the orthogonality of $\phi_n(z)$ with $\psi_\nu(\zeta;n)$ $(\nu<n)$. Thus we have

$$\phi_n(z)=\kappa_n\{h(0)\}^{-1}h_n^*(z)+\frac{1}{2\pi}\int_0^{2\pi}\{W_n(\theta)-W(\theta)\}\phi_n(\zeta)\,K_n(\zeta,z)d\theta: \quad \zeta=e^{it}. \tag{5.22}$$

Next we shall try to estimate $M_n=\max_{|z|=1}|\phi_n(z)|$. Using the lemmas 2, 4 and the relation (5.19), we find from (5.22) for $r\,(R>r>1)$, $M_n\leq O(1)+M_nO(r^{-n}\log n)$ so that $M_n=O(1)$.

Now the relation (5.3) follows from

$$\phi_n(z)-\bar{h}(z^{-1})z^n=[\kappa_n\{h(0)\}^{-1}-1]\,\bar{h}_n(z^{-1})z^n+\{\bar{h}_n(z^{-1})-\bar{h}(z^{-1})\}\,z^n$$

$$+\frac{1}{2\pi}\int_0^{2\pi}\{W_n(\theta)-W(\theta)\}\,\phi_n(\zeta)K_n(\zeta,z)d\theta$$

by the use of (5.11), (5.12), (5.18) and (5.21). The relation (5.4) also follows from (5.22). Thus the theorem can be established by lemma 1.

The following theorem can be verified easily by theorems 2 and 3.

THEOREM 5. *Let $f(z)$ be a function analytic throughout the interior of the circle Γ_ρ: $|z|=\rho>1$, but not analytic on Γ_ρ. Let $F(z)$ be a function analytic and positive on the unit circle Γ, and $\phi_n(z)$: $n=0,1,2,\cdots$ be the set of ortho-normal polynomials associated with the positive weight function $W(\theta)=F(e^{i\theta})$ on Γ. Then the sequence of polynomials $P_n(z;f)$ which interpolate to $f(z)$ in all the zeros of $\phi_n(z)$ converges to $f(z)$ throughout the region $|z|<\rho$ and uniformly on any closed set interior to that region. The sequence $P_n(z;f)$ diverges at every point exterior to Γ_ρ.*

Moreover, we have

(5.23) $$\overline{\lim_{n\to\infty}} |f(z)-P_n(z;f)|^{1/n} \leqq R/\rho$$

for z on $\Gamma_R\,(1<R<\rho)$,

(5.24) $$\overline{\lim_{n\to\infty}} |f(z)-P_n(z;f)|^{1/n} \leqq 1/\rho \quad for \quad |z|\leqq 1,$$

and

(5.25) $$\lim_{n\to\infty} |P_n(z;f)|^{1/n} = |z|/\rho$$

for z exterior to Γ_ρ.

6. In this paragraph, we consider the set of ortho-normal polynomials on the real segment $[-1, 1]$ which have an asymptotic relation corresponding to that of paragraph 5.

Let $\phi_n(w)$ be the set of ortho-normal polynomials associated with a weight function $W(\theta)=F(e^{i\theta})$, on the unit circle $w=e^{i\theta}$, which satisfies the condition of theorem 4. Let $P_n(z)$ be the set of ortho-normal polynomials associated with the weight function

(6.1) $$P(x)=W(\theta)|\sin\theta|^{-1}=F(e^{i\theta})\sqrt{1-x^2}; \quad -1<x<1,$$

where

$$z=\frac{1}{2}(w+w^{-1}), \qquad x=\frac{1}{2}(e^{i\theta}+e^{-i\theta})=\cos\theta .$$

Then the following relation is known

(6.2) $$P_n(z)=(2\pi)^{-1/2}\left\{1+\frac{\phi_{2n}(0)}{\kappa_{2n}}\right\}^{-1/2}\{w^{-n}\phi_{2n}(w)+w^n\phi_{2n}(w^{-1})\},$$

where κ_n represents the highest coefficient of $\phi_n(w)$ respectively for each n. (Cf. Orthogonal Polynomials, page 287.)

Conversely, we considered a function $G(z)$ which is analytic and non-zero throughout the interior of the ellipse $C_R\,(R>1)$ with foci at ± 1 and with semi-axes $\frac{1}{2}(R+R^{-1})$, $\frac{1}{2}(R-R^{-1})$, but not analytic nor non-zero on C_R, and positive on the real segment $[-1, 1]$. Such a function can be expanded by Tchebycheff polynomials as follows:

(6.3) $$G(z)=\sum_{k=0}^{\infty}a_k(w^k+w^{-k})\equiv F(w),$$

and the coefficients a_k satisfy

$$\overline{\lim_{n\to\infty}}\ |a_n|^{1/n}=R^{-1}.$$

It is clear that the function $F(w)$ defined by (6.3) is analytic and non-zero throughout the interior of the region between the circles $|w|=R$ and $|w|=R^{-1}$ but not analytic nor non-zero on Γ_R or $\Gamma_{R^{-1}}$.

Now the following theorem is ready to be proved.

THEOREM 6. *Let $G(z)$ be the function which is analytic and non-zero within the ellipse C_R $(R>1)$ with foci at ± 1 and with semi-axes $\frac{1}{2}(R+R^{-1})$, $\frac{1}{2}(R-R^{-1})$, but not analytic nor non-zero on C_R, and positive on the real segment $[-1, 1]$. Let $P_n(z)$ be the set of orthonormal polynomials associated with positive weight function*

$$P(x)=G(x)/\sqrt{1-x^2}>0;\quad -1<x<1.$$

Then we have the asymptotic relation

(6.4) $$\overline{\lim_{n\to\infty}}\ |P_n(z)-(2\pi)^{-1/2}h(w^{-1})w^n|^{1/n}\leqq|w|^{-1}\quad\text{for}\quad 1\leqq|w|<R,$$

$$\leqq|w|/R^2\quad\text{for}\quad|w|\leqq R,$$

where $h(w)$ is the function analytic and non-zero throughout the interior of the circle Γ_R: $|w|=R$, and uniquely determined under the conditions

(6.5) $$\begin{cases}|h(e^{i\theta})|^2=\{F(e^{i\theta})\}^{-1}=\{G\cos\theta)\}^{-1}>0,\\ h(0)>0,\end{cases}$$

where the relation between $F(w)$ and $G(z)$ is given by (6.3).

Such a function $h(w)$ can be determined from $F(w)$ by a method similar to the case of $h(z)$ in theorem 4.

Next we can verify the identity

(6.6) $$\overline{\lim_{n\to\infty}}\left|\left\{1+\frac{\phi_{2n}(0)}{\kappa_{2n}}\right\}^{-1/2}-1\right|^{1/n}=R^{-2}$$

by the relation (5.4) in theorem 4 and (5.20) in lemma 4. The relations

$$\varlimsup_{n\to\infty} |w^{-n}\phi_{2n}(w)-w^n h(w^{-1})|^{1/n} \leqq |w|/R^2 \quad \text{for} \quad |w|>1, \tag{6.7}$$

and

$$\varlimsup_{n\to\infty} |\phi_{2n}(w^{-1})|^{1/n} = \begin{cases} |w|^{-1} & \text{for} \quad R>|w|>1, \\ |w|/R^2 & \text{for} \quad |w| \geqq R \end{cases} \tag{6.8}$$

can be verified by the result of theorem 4.

Consequently, the asymptotic relation (6.4) of $P_n(z)$ can be obtained from (6.2), (6.6), (6.7) and (6.8). Thus the theorem has been established.

Now the following theorem which corresponds to theorem 5 can be proved by theorem 6 and one of the examples of theorem 3.

THEOREM 7. *Let $G(z)$ be a function analytic and positive on the real segment $[-1,1]$, and $P_n(z)$ be the set of ortho-normal polynomials associated with the positive weight function*

$$P(x)=G(x)/\sqrt{1-x^2}; \qquad -1<x<1.$$

Let $f(z)$ be a function analytic throughout the interior of the ellipse C_ρ $(\rho>1)$ with foci at ± 1 and with semi-axes $\frac{1}{2}(\rho+\rho^{-1})$ and $\frac{1}{2}(\rho-\rho^{-1})$, but not analytic on C_ρ.

Then the sequence of polynomials $P_n(z;f)$ which interpolate to $f(z)$ respectively in all the zeros of $P_{n+1}(z)$ converges to $f(z)$ throughout the interior of C_ρ, uniformly on any closed set interior to C_ρ, but diverges at every point exterior to C_ρ.

Moreover, we have

$$\varlimsup_{n\to\infty} |f(z)-P_n(z;f)|^{1/n} \leqq R/\rho \tag{6.9}$$

for z on $C_R (1<R<\rho)$,

$$\varlimsup_{n\to\infty} |f(z)-P_n(z;f)|^{1/n} \leqq 1/\rho \tag{6.10}$$

for z on the real segment $[-1,1]$, and

$$\varlimsup_{n\to\infty} |P_n(z;f)|^{1/n} = R'/\rho \tag{6.11}$$

for z *on* $C_{R'}$ $(R'>\rho)$.

By this theorem we can understand that the exact region of convergence of the sequence $P_n(z;f)$ which interpolates to $f(z)$ in all the zeros of $P_{n+1}(z)$ defined in the theorem is equivalent to that of the Fourier expansion of $f(z)$ by $P_n(z)$. But if the set of orthogonal polynomials $P_n(z)$ is given by the association with a weight function which satisfies a certain condition (cf. the equation (5.1) or (5.2)) weaker than that of theorem 5 or 7, the exact region of uniform convergence of interpolation polynomials is known, but the divergence at all points exterior to that region can not be determined.

This problem is quite similar to the sequence of interpolation polynomials in all the zeros of polynomials $\varphi_n(z)$ which satisfy only the condition

$$\lim_{n\to\infty} \varphi_n(z)/z^n=\lambda(z) \qquad \text{for} \quad |z|>1,$$

or

$$\lim_{n\to\infty} \varphi_n(z)/\Delta^n w^n=\lambda(w) \quad \text{for} \quad |w|>1.$$

But if the singularities of a function on Γ_ρ or C_ρ are not complicated, as when the singularities are all poles, the divergence of interpolation polynomials at all points exterior to Γ_ρ or C_ρ can be verified. Accordingly, the research of singularities on Γ_ρ or C_ρ may probably bring a finer result.

7. In this paragraph, we consider the divergence of polynomials found by interpolation to $f(z)$, which is analytic throughout the interior of the circle Γ_ρ: $|z|=\rho>1$ and on Γ_ρ has only a finite number of poles, in all the zeros of polynomials $\varphi_{n+1}(z)$ which satisfy a condition more general than that in previous paragraphs.

THEOREM 8. *Let* $\varphi_n(z)$ *be the sequence of polynomials of respective degrees* n *with highest terms* z^n, *which satisty the condition*

$$\lim_{n\to\infty} \varphi_n(z)/z^n=\lambda(z) \tag{7.1}$$

for z *exterior to the unit circle* Γ: $|z|=1$ *and uniformly for* $|z|\geqq R>1$, *where* $\lambda(z)$ *is a function analytic and non-zero exterior to* Γ. *Let* $f(z)$ *be a function analytic throughout the interior of the circle* Γ_ρ: $|z|=\rho>1$ *but on* Γ_ρ *having a finite number of poles.*

Then the sequence of polynomials $P_n(z;f)$ *of respective degrees* n

found by interpolation to $f(z)$ in all the zeros of $\varphi_{n+1}(z)$ diverges at every point exterior to Γ_ρ. Moreover, we have

$$\overline{\lim_{n\to\infty}} \, |\rho^n P_n(z;f)/n^{p-1}z^n|>0; \quad for \quad |z|>\rho, \tag{7.2}$$

where p is the maximum order of poles of $f(z)$ on Γ_ρ.

In the proof of this theorem, we shall prove the following lemma.

LEMMA. *Let $A_n^{(k)}$: $k=1,2,\cdots,m$; $n=1,2,\cdots$ be a given set of complex numbers which satisfy*

$$\lim_{n\to\infty} A_n^{(k)}=A^{(k)}; \qquad k=1,2,\cdots,m, \tag{7.3}$$

where $A^{(k)}$ are complex numbers not all equal to zeros. Let θ_k : $k=1, 2,\cdots,m$ be mutually distinct angles between 0 and 2π ($0\leqq\theta_k<2\pi$). Then we have

$$\overline{\lim_{n\to\infty}} \, \left|\sum_{k=1}^{m} A_n^{(k)}e^{-in\theta_k}\right|>0. \tag{7.4}$$

If we assume the equation

$$\lim_{n\to\infty} \left\{\sum_{k=1}^{m} A_n^{(k)}e^{-in\theta_k}\right\}=0,$$

we have, for $A^{(1)}$ which can be assumed to be not zero,

$$\lim_{n\to\infty} \left\{A^{(1)}+\sum_{k=2}^{m} A^{(k)}e^{-in(\theta_k-\theta_1)}\right\}=0 \tag{7.5}$$

by the relation (7.3). While the *arithmetic means*

$$\frac{1}{n}\sum_{\nu=0}^{n-1}\left\{A^{(1)}+\sum_{k=2}^{m} A^{(k)}e^{-i\nu(\theta_k-\theta_1)}\right\}=A^{(1)}+\frac{1}{n}\sum_{k=2}^{m}\frac{1-e^{-in(\theta_k-\theta_1)}}{1-e^{-i(\theta_k-\theta_1)}}$$

converge clearly to $A_1\neq0$ for the reason that all denominators of the last terms are non-vanishing. This contradicts (8.5). Thus the lemma has been proved.

Let R be an arbitrary positive number less than ρ but greater than unity. The sequence of polynomials $P_n(z;f)$ which interpolate to $f(z)$ in all the zeros of $\varphi_{n+1}(z)$ (we can assume that $\varphi_n(z)$ are non-vanishing exterior to Γ for n sufficiently large) can be represented by

$$P_n(z;f)=\frac{1}{2\pi i}\int_{\Gamma_R}\frac{\varphi_{n+1}(t)-\varphi_{n+1}(z)}{\varphi_{n+1}(t)}\frac{f(t)}{t-z}dt. \tag{7.6}$$

Let $S_n(z;f)$ be the sequence of functions defined by

$$S_n(z;f)=\frac{1}{2\pi i}\int_{\Gamma_R}\frac{\lambda(t)t^{n+1}-\lambda(z)z^{n+1}}{\lambda(t)t^{n+1}}\frac{f(t)}{t-z}dt \tag{7.7}$$

for $|z|>1$. Then we have

$$S_n(z;f)-P_n(z;f)=\frac{1}{2\pi i}\int_{\Gamma_R}\left\{\frac{\varphi_{n+1}(z)}{\varphi_{n+1}(t)}-\frac{\lambda(z)z^{n+1}}{\lambda(t)t^{n+1}}\right\}\frac{f(t)}{t-z}dt \tag{7.8}$$

for z exterior to the unit circle Γ.

Let $f(z)$ be the function which has on Γ_ρ m poles of respective orders p_k at $z_k=\rho e^{i\theta_k}$; $k=1,2,\cdots,m$. For any z exterior to Γ_ρ, we can choose a positive number R greater than ρ but less than $|z|$, such that the function $f(z)$ is analytic on and within $\Gamma_{R'}$ except on Γ_ρ, by the condition of the theorem. For such a point z exterior to Γ_ρ, let $F_k(t,z)$ be the function defined by

$$F_k(t,z)=\frac{f(t)(t-z_k)^{p_k}}{t-z};\qquad k=1,2,\cdots,m, \tag{7.9}$$

which is analytic at $t=z^k$.

Then the equation (7.8) yields

$$\begin{aligned}S_n(z;f)-P_n(z;f)=&\frac{1}{2\pi i}\int_{\Gamma_{R'}}\left\{\frac{\varphi_{n+1}(z)}{\varphi_{n+1}(t)}-\frac{\lambda(z)z^{n+1}}{\lambda(t)t^{n+1}}\right\}\frac{f(t)}{t-z}dt\\&-\sum_{k=1}^{m}\left[\frac{d^{p_k-1}}{dt^{p_k-1}}\left\{\left(\frac{\varphi_{n+1}(z)}{\varphi_{n+1}(t)}-\frac{\lambda(z)z^{n+1}}{\lambda(t)t^{n+1}}\right)F_k(t,z)\right\}\right]_{t=z_k}\end{aligned} \tag{7.10}$$

for $|z|\geqq R'>\rho$: $|z_k|=\rho$. Furthermore, for z exterior to Γ_ρ, we have

$$\lim_{n\to\infty}\left[\frac{d^{p_k-1}}{dt^{p_k-1}}\left\{\left(\frac{\varphi_{n+1}(z)}{\varphi_{n+1}(t)}-\frac{\lambda(z)z^{n+1}}{\lambda(t)t^{n+1}}\right)F_k(t,z)\right\}\right]_{t=z_k}n^{p_k-1}\rho^n|z|^{-n}=0 \tag{7.11}$$

by the relation

$$\left[\frac{d^{p_k-1}}{dt^{p_k-1}}\left\{\left(\frac{\varphi_{n+1}(z)}{\varphi_{n+1}(t)}-\frac{\lambda(z)z^{n+1}}{\lambda(t)t^{n+1}}\right)F_k(t,z)\right\}\right]_{t=z_k}$$

$$=(-1)^{p_k-1}(n+1)(n+2)\cdots(n+p_k-1)$$
$$\left[\frac{z^{n+1}}{t^{n+p_k-1}}\left\{\left(\frac{t^{n+1}\varphi_{n+1}(z)}{z^{n+1}\varphi_{n+1}(t)}-\frac{\lambda(z)}{\lambda(t)}\right)F_k(t,z)\right\}\right]_{t=z_k}+O\left(n^{p_k-2}\left|\frac{z}{\rho}\right|^n\right)$$

for t on Γ_ρ, and by the condition (7.1). Accordingly, for any positive numbers ε_1 and ε_2, we have for n sufficiently large, from (7.10) and (7.11)

$$|S_n(z;f)-P_n(z;f)|<\varepsilon_1\frac{|z|^{n+1}}{R'^{n+1}}+\varepsilon_2 n^{p-1}\frac{|z|^{n+1}}{\rho^{n+1}}$$

for $|z|\geqq R'>\rho$, where p is the maximum of p_k. Thus we have

$$\lim_{n\to\infty}\frac{\rho^n}{n^{p-1}z^n}\left\{S_n(z;f)-P_n(z;f)\right\}=0 \quad \text{for} \quad |z|>R'>\rho\,, \tag{7.12}$$

while

$$S_n(z;f)=\frac{1}{2\pi i}\int_{\Gamma_{R'}}\frac{f(t)}{t-z}dt-\sum_{k=1}^{m}\frac{1}{(p_k-1)!}\left[\frac{d^{p_k-1}}{dt^{p_k-1}}F_k(t,z)\right]_{t=z_k} \tag{7.13}$$
$$-\frac{1}{2\pi i}\int_{\Gamma_{R'}}\frac{\lambda(z)z^{n+1}}{\lambda(t)t^{n+1}}\frac{f(t)}{t-z}dt+\lambda(z)z^{n+1}\sum_{k=1}^{m}n^{p_k-1}\rho^{-n}B_n^{(k)}e^{-in\theta_k}\,,$$

where

$$B_n^{(k)}=\frac{1}{(p_k-1)!}\left[\frac{d^{p_k-1}}{dt^{p_k-1}}\left\{\frac{F_k(t,z)}{\lambda(t)t^{n+1}}\right\}\right]_{t=z_k}\Big/ n^{p_k-1}z_k^n$$
$$=\frac{(-1)^{p_k-1}}{(p_k-1)!}\frac{(n+1)(n+2)\cdots(n+p_k-1)}{n^{p_k-1}}\frac{F(z_k,z)\rho^{p_k}}{\lambda(z_k)z_k^{p_k}}+O\left(\frac{1}{n}\right)$$

which converge respectively to

$$B^{(k)}=\frac{(-1)^{p_k-1}}{(p_k-1)!}\frac{F(z_k,z)}{\lambda(z_k)}e^{-ip_k\theta_k}\neq 0$$

as n tends to infinity.

Let p be the maximum value of p_k. Now the relation (7.13) yields by the lemma

$$\overline{\lim_{n\to\infty}}\left|\frac{\rho^n S_n(z;f)}{n^{p-1}z^n}\right|>0 \quad \text{for} \quad |z|>R'>\rho\,. \tag{7.14}$$

Thus we can verify by (7.12) and (7.14) the following relation:

(7.15) $$\overline{\lim_{n\to\infty}}\left|\frac{\rho^n P_n(z;f)}{n^{p-1}z^n}\right|>0$$

for z exterior to Γ_ρ. Hence the sequence $P_n(z;f)$ can not converge for z exterior to Γ_ρ. Thus the theorem has been established.

The generalization of this theorem to a more generalized point set can be verified by the method of paragraph 3.

THEOREM 9. *Let D be a closed limited point set with the capacity Δ whose complement K with respect to the extended plane is connected and regular. Let $w=\phi(z)$ map K onto the region $|w|>1$ so that the points at infinity correspond to each other. Let $\varphi_n(z)$ be the polynomials of respective degrees n such that the sequence of functions $\varphi_n(z)/\Delta^n w^n$ converges to a function $\lambda(w)$ analytic and non-zero on K and uniformly on any closed set interior to K as n tends to infinity. Let $f(z)$ be a function analytic throughout the interior of the level curve C_ρ: $|\phi(z)|=\rho>1$ and having a finite number of poles on C_ρ as the function of w.*

Then the sequence of polynomials $P_n(z;f)$ of respective degrees n found by interpolation to $f(z)$ in all the zeros of $\varphi_{n+1}(z)$ diverges at every point exterior to C_ρ. Moreover, we have

(7.16) $$\lim_{n\to\infty}|\rho^n P_n(z;f)/n^{p-1}[\phi(z)]^n|>0$$

for $|w|=|\phi(z)|>\rho$, where p is the maximum order of poles of $f(z)$ on C_ρ.