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On conformal Riemann spaces.

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In this paper we investigate the local conformal homeomorphism of two Riemann spaces, which we call conformal correspondence. In section **1** we define characteristic roots of the conformal correspondence and consider the case in which the characteristic roots are all equal. This case has already been investigated by A. Fialkow [5] and K. Yano [4], and section **1** will give redemonstrations of their results together with some new results. In section **2** we treat the conformally flat Riemann space of imbedding class 1 which has been already investigated by J. A. Schouten [1], M. Matsumoto [2], and L. L. Verbickiĭ [3]. This Riemann space is characterized by the property that the characteristic roots of the conformal correspondence of the space with a euclidean space are equal except one. Moreover we give new proofs of the results of [1], [2], [3] from our point of view.

Throughout the whole paper let the indices run as follows:

$$i, j, k, h=1, \cdots, n\,; \quad \alpha, \beta, \gamma, \varepsilon=2, \cdots, n\,,$$

and we shall follow the convention that the repeated indices imply summation unless otherwise mentioned.

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1. Conformally related Riemann spaces.

1.1 Let the line-elements of n-dimensional Riemann spaces S, S_1 be given by

$$ds^2=g_{ij}dx_i dx_j\,, \qquad ds_1^2=a^2 g_{ij}dx_i dx_j$$

respectively, where a and g_{ij} are functions of class C_2 of $x_1, \cdots, x_n$. We put $g_{ij}dx_i dx_j=\sum_i \omega_i^2$ with Pfaffian forms ω_i of class C_2, and so

(1) $$ds^2=\sum_i \omega_i^2, \qquad ds_1^2=a^2\sum_i \omega_i^2.$$

We can determine ω_{ij} such that

(2) $$d\omega_i=[\omega_j\omega_{ji}], \qquad \omega_{ij}=-\omega_{ji}$$

uniquely. We put

(3) $$da/a=b_i\omega_i ,$$

(4) $$\pi_i=a\omega_i , \qquad \pi_{ij}=\omega_{ij}+b_i\omega_j-b_j\omega_i .$$

Then we have by (1), (2), (3), and (4)

(5) $$ds_1^2=\sum_i \pi_i^2 ,$$

and also

$$d\pi_i=d(a\omega_i)=[da, \omega_i]+ad\omega_i=[da, \omega_i]+a[\omega_j, \omega_{ji}]$$

$$=[ab_j\omega_j, \omega_i]+[a\omega_j, \omega_{ji}]=[a\omega_j, \omega_{ji}+b_j\omega_i-b_i\omega_j] .$$

Hence we get

(6) $$d\pi_i=[\pi_j\pi_{ji}], \qquad \pi_{ij}=-\pi_{ji}$$

and so π_i, π_{ij} are the parameters of Riemannian connection attached to S_1. Let the curvature forms of S and S_1 be

$$\Omega_{ij}=d\omega_{ij}-[\omega_{ik}\omega_{kj}], \qquad \Pi_{ij}=d\pi_{ij}-[\pi_{ik}\pi_{kj}] .$$

Then we get from (4)

(7) $$\Pi_{ij}=\Omega_{ij}+[p_{ik}\omega_k, \omega_j]-[p_{jk}\omega_k, \omega_i]+b^2[\omega_i\omega_j] ,$$

where we have put

(8) $$b^2=\sum_i b_i^2,$$

(9) $$db_i+b_j\omega_{ji}-b_i da/a=p_{ik}\omega_k .$$

By taking an exterior differential of (3)

$$[db_j+b_i\omega_{ij}, \omega_j]=0 .$$

So if we put $db_j+b_i\omega_{ij}=l_{jk}\omega_k$, we get $l_{jk}=l_{kj}$. Since $b_i da/a=b_i b_k\omega_k$ we obtain $p_{ik}=p_{ki}$, too. Now, $db_i+b_k\omega_{ki}$, b_i, ω_i are the components of three vectors in the tangent spaces with respect to a rectangular frame and by a suitable rotation of a frame we can transform the symmetric matrix (p_{ij}) into a diagonal form. We take such frames at each point of the space S and use the same notation as above, thus getting

(10) $$db_i+b_k\omega_{ki}-b_i da/a=p_i\omega_i \quad \text{(not summed for } i\text{)},$$

(11) $$\Pi_{ij}=\Omega_{ij}+(b^2+p_i+p_j)[\omega_i\omega_j] \quad \text{(not summed for } i,j\text{)}.$$

We call $p_i\ (i=1, \cdots, n)$ *characteristic roots* of the conformal correspondence of S_1 with S. It should be noted that these depend not only on S_1 and S but also on the correspondence between them. We seek for the characteristic roots of the conformal correspondence of S with S_1, where the correspondence is the same as above. Putting

(12) $$c_i=b_i/a\,, \qquad q_i=p_i/a$$

we get from (10)

(13) $$dc_i+c_j\omega_{ji}=q_i\omega_i \qquad \text{(not summed for } i\text{)}.$$

We take $1/a$, π_i instead of a, ω_i and we have

$$d(1/a)/(1/a)^2=-da=-ab_i\omega_i=-ac_i\pi_i\,.$$

Hence we get $-ac_i$ instead of c_i. Putting $m_i=-ac_i$ and $c^2=\sum_i c_i^2$ we get

$$\begin{aligned} dm_i+m_k\pi_{ki} &= -d(ac_i)-ac_k(\omega_{ki}+b_k\omega_i-b_i\omega_k) \\ &= -adc_i-c_i da-ac_k\omega_{ki}-ac^2\pi_i+b_i da/a \\ &= -a(dc_i+c_k\omega_{ki})-ac^2\pi_i=-aq_i\omega_i-ac^2\pi_i \\ &= -(q_i+ac^2)\pi_i \qquad \text{(not summed for } i\text{)}. \end{aligned}$$

Hence the characteristic roots of S with respect to S_1 are given by $-(p_i+a^2c^2)$ $(i=1, \cdots, n)$.

1.2 As an application we investigate the case in which the characteristic roots of the conformal correspondence of S_1 with S are all equal. In this case the characteristic roots of S relative to S_1 are all equal by the above remark. The condition that all the characteristic roots are equal is invariant under the rotation of the rectangular frame and we can take frames at each point of S such that the components of the vector $(c_1, \cdots, c_n)$ reduce to $(c, 0, \cdots, 0)$ and yet $p_1=p_2=\cdots=p_n$. By virtue of (12) all q_i's are equal which we put q and assume that $q\neq 0$. Then (13) reduces to the following:

(14) $$dc=q\omega_1\,,\qquad c\omega_{1\alpha}=q\omega_\alpha\ (\alpha=2,\cdots,n)\,.$$

We have $da/a^2=c_i\omega_i=c\omega_1$ and so $\omega_1=da/(ca^2)$. Hence

(15) $$dc=q\,da/(ca^2)\,,\qquad d(c^2/2)=q\,da/a^2\,.$$

q ought to be a function of a and $c^2=2\int q\,da/a^2$. Hence ω_1 contains only one variable a. Let α and β run from 2 to n. Then we have

(16) $$\begin{aligned} d\omega_\alpha&=[\omega_i\omega_{i\alpha}]=[\omega_1\omega_{1\alpha}]+[\omega_\beta\omega_{\beta\alpha}]\\ &=[dc/q,\,q\omega_\alpha/c]+[\omega_\beta\omega_{\beta\alpha}]=[dc/c,\,\omega_\alpha]+[\omega_\beta\omega_{\beta\alpha}]\,,\\ d(\omega_\alpha/c)&=[\omega_\beta/c,\,\omega_{\beta\alpha}]\,,\qquad \omega_{\alpha\beta}=-\omega_{\beta\alpha}\,,\end{aligned}$$

and so by E. Cartan's lemma we can take coordinates $x_1,\cdots,x_n$ such that

$$\sum_\alpha \omega_\alpha^2/c^2=g_{\alpha\beta}\,(x_2,\cdots,x_n)\,dx_\alpha dx_\beta\,.$$

As ω_1 contains only one variable a we can take x_1 in such a way that

(17) $$\omega_1=dx_1$$

holds. Hence we get

(18) $$ds^2=\sum_i \omega_i^2=dx_1^2+c(x_1)^2g_{\alpha\beta}\,(x_2,\cdots,x_n)\,dx_\alpha dx_\beta\,.$$

On account of the relation $\omega_1=dx_1=dc/q=da/(ca^2)$ we have

(19) $$-1/a=\int c\,dx_1\,,$$

(20) $$ds_1^2=a^2ds^2=a(x_1)^2(dx_1^2+c(x_1)^2g_{\alpha\beta}\,(x_2,\cdots,x_n)\,dx_\alpha dx_\beta)\,.$$

In the case $q=0$, c is constant by (15). We assume $c\neq 0$. Then we get $\omega_{1\alpha}=0$ and $d\omega_1=0$, $d\omega_\alpha=[\omega_\beta\omega_{\beta\alpha}]$, and in this case we also have (18). In the case $c=0$ we obtain $da/a=c\omega_1=0$ and a is constant and our correspondence reduces to a similar mapping. We mean by a similar mapping the one which induces the multiplication of the Riemannian metric by a constant.

Conversely for a Riemann space with ds^2 represented by (18) we can take $\omega_1,\rho_\alpha,\ \omega_{\alpha\beta}$ in such a way that

$$\omega_1 = dx\,, \qquad \sum_\alpha \rho_\alpha^2 = g_{\alpha\beta}\,(x_2, \cdots, x_n)\, dx_\alpha dx_\beta\,,$$

$$d\rho_\alpha = [\rho_\beta \omega_{\beta\alpha}]\,, \qquad \omega_{\alpha\beta} = -\omega_{\beta\alpha}\,.$$

We put $\omega_\alpha = c\rho_\alpha$, $\omega_{1\alpha} = c'/c\omega_\alpha = -\omega_{\alpha 1}$, $q = c'$, where $c' = dc/dx_1$. Then we get

$$d\omega_1 = [\omega_\alpha \omega_{\alpha 1}]\,,$$

$$d\omega_\alpha = d(c\rho_\alpha) = [dc, \rho_\alpha] + c\, d\rho_\alpha$$

$$= [q\, dx_1, \omega_\alpha/c] + c[\rho_\beta \omega_{\beta\alpha}] = [\omega_1 \omega_{1\alpha}] + [\omega_\beta \omega_{\beta\alpha}]\,,$$

and ω_i, $\omega_{ij} = -\omega_{ji}$ are the parameters of the Riemannian connection of S. (3) and (10) hold for a determined by (19) and $(c_1, \cdots, c_n) = (c, 0, \cdots, 0)$. In summary we get

THEOREM 1. *Let S_1 and S be Riemann spaces which are conformal to each other but not similar. In order that the characteristic roots of the conformal correspondence of S_1 with S are all equal it is necessary and sufficient that by a suitable choice of coordinates the line-elements of S_1 and S are represented respectively by*

$$ds_1^2 = a(x_1)^2 ds^2\,, \qquad a(x_1) = -1/\left(\int c\, dx_1\right),$$

$$ds^2 = dx_1^2 + c(x_1)^2 g_{\alpha\beta}\,(x_2, \cdots, x_n)\, dx_\alpha dx_\beta\,.$$

This is the result of A. Fialkow [5], and the geometric characterization of the space was given by K. Yano [4]. It is the space that admits a concircular transformation. We characterize this space from another point of view in the following.

In the spaces whose line elements are given by (18) and (20) the curves determined by $x_\alpha = \text{const.}(\alpha = 2, \cdots, n)$ are geodesics corresponding to each other by our conformal correspondence. Conversely we consider two Riemann spaces S_1 and S whose line-elements are given by

$$ds_1^2 = \sum \pi_i^2\,, \qquad ds^2 = \sum \omega_i^2\,, \qquad \pi_i = a\omega_i\,.$$

A geodesic on S_1 is given by solving

$$\frac{d}{ds_1}\left(\frac{\pi_i}{ds_1}\right) + \frac{\pi_j}{ds_1}\frac{\pi_{ji}}{ds_1} = 0\,.$$

By our conformal correspondence from S_1 to S this geodesic gives rise to the curve on S which satisfies the equations

$$\frac{d}{ds_1}\left(a\frac{\omega_i}{ds_1}\right)+\frac{a\omega_j}{ds_1}\left(\frac{\omega_{ji}}{ds_1}+b_j\frac{\omega_i}{ds_1}-b_i\frac{\omega_j}{ds_1}\right)=0,$$

$$(21)\qquad \frac{d}{ds_1}\left(\frac{\omega_i}{ds_1}\right)+\frac{\omega_j}{ds_1}\frac{\omega_{ji}}{ds_1}+\frac{2}{a}\frac{da}{ds_1}\frac{\omega_i}{ds_1}-b_i\frac{\sum\omega_j^2}{ds_1^2}=0.$$

Now we assume that a geodesic on S_1 corresponds to the one on S by our correspondence. The equations of the geodesic on S is given by

$$(22)\qquad \frac{d}{ds}\left(\frac{\omega_i}{ds}\right)+\frac{\omega_j}{ds}\frac{\omega_{ji}}{ds}=0.$$

For a parameter s_1 these reduce to

$$(23)\qquad \frac{d}{ds_1}\left(\frac{\omega_i}{ds_1}\right)+\frac{\omega_j}{ds_1}\frac{\omega_{ji}}{ds_1}-\frac{\omega_i}{ds_1}\frac{d^2s}{ds_1^2}\bigg/\frac{ds}{ds_1}=0.$$

Now we assume $b_i\neq 0$ for some i. Then in order that (21) and (23) coincide it is necessary that

$$(24)\qquad \omega_i=b_i/b\,ds$$

hold along the solution, where we have referred to $\sum\omega_i^2=ds^2$, $\sum b_i^2=b^2$. So if a geodesic on S_1 corresponds to that of S by our conformal correspondence, it ought to be a solution of (24). As the relation (24) are invariant under the rotation of the frame we have the same equations for any frame. We take the frames such that (10) hold. By (3) and (24) we have

$$(25)\qquad da/a=b_i\omega_i=\sum b_i^2\,ds/b=b\,ds,$$

and by virtue of (22) and (24)

$$(26)\qquad \frac{db_i}{ds}+b_j\frac{\omega_{ji}}{ds}=\frac{b_i}{b}\frac{db}{ds},$$

and also by (10)

$$(27)\qquad \frac{db_i}{ds}+b_j\frac{\omega_{ji}}{ds}-\frac{b_i}{a}\frac{da}{ds}=p_i\frac{\omega_i}{ds}\quad \text{(not summed for } i\text{)}.$$

Putting (24), (25), (26) into these equations we get

$$b_i p_i = b_i\left(\frac{db}{ds} - b^2\right) \quad \text{(not summed for } i\text{)}.$$

In the case $b_i \neq 0\ (i=1, \cdots, n)$ we get

$$p_i = \frac{db}{ds} - b^2$$

and all $p_i\ (i=1, \cdots, n)$ are equal. Thus we get the following result.

THEOREM 2. *Let S_1 and S be conformal Riemann spaces. If the characteristic roots of the conformal correspondence of S_1 with S are all equal, S_1 and S are each generated by $(n-1)$-parametric geodesics which are mapped by our correspondence. Conversely if S_1 and S are each generated by $(n-1)$-parametric geodesics and none of $b_1, \cdots, b_n$ corresponding to certain frames satisfying* (10) *vanishes, then the characteristic roots of the conformal correspondence of S_1 with S are all equal.*

1.3 Now we consider the case in which S_1 and S are Einstein spaces. Returning to **1.1** and putting

$$\Pi_{ij} = \frac{1}{2}\bar{R}_{ijkh}[\pi_k \pi_h], \qquad \Omega_{ij} = \frac{1}{2} R_{ijkh}[\omega_k \omega_h],$$

where $\bar{R}_{ijkh} = -\bar{R}_{ijhk}$, $R_{ijkh} = -R_{ijhk}$, we get by (11)

$$\bar{R}_{ijkh} a^2 = R_{ijkh} + (b^2 + p_i + p_j)(\delta_{ik}\delta_{jh} - \delta_{ih}\delta_{jk})$$
$$\text{(not summed for } i, j\text{)}.$$

Contracting with respect to j, h we have

$$\bar{R}_{ik} a^2 = R_{ik} + \left((n-1)b^2 + (n-2)p_i + \sum_j p_j\right)\delta_{ik} \tag{28}$$
$$\text{(not summed for } i\text{)}.$$

Here $\bar{R}_{ik}$ and R_{ik} are Ricci's tensors of two spaces S_1 and S. If the two spaces S_1 and S are Einstein spaces we have

$$\bar{R}_{ik} = \bar{R}/n\,\delta_{ik}, \qquad R_{ik} = R/n\,\delta_{ik}, \tag{29}$$

and we get by (28)

$$\bar{R}/n\, a^2 = R/n + (n-1)b^2 + (n-2)p_i + \sum_j p_j \quad (i=1, \cdots, n). \tag{30}$$

Hence all p_i's are equal if $n>2$. Then we take coordinates such that the line elements of the two spaces are given by (18) and (20), and by a calculation of the curvature tensor we get the following result of H. W. Brinkmann

THEOREM 3. *If S_1 and S are n-dimensional conformal Einstein spaces ($n\geqq 3$), then the characteristic roots of the conformal correspondence of S_1 with S are all equal, and in suitably chosen coordinates $x_1, \cdots, x_n$ the line-elements of S are given by*

$$ds^2=dx_1^2+c(x_1)^2 ds_1^2, \tag{31}$$

where ds_1^2 is a line-element of an $(n-1)$-dimensional Einstein space and $c(x_1)$ is given by the followings:

$$\begin{array}{lll} \text{const}, & \exp Ax_1, & \\ \cosh Ax_1, & & \\ x, & \sinh Ax_1, & \sin Ax_1. \end{array} \qquad (A:\ \text{const.}) \tag{32}$$

1.4 Next we treat the simplest case of theorem 3 in which S_1 is a space of constant curvature and S is a euclidean space. Let the curvature of S_1 be K. Then we have $\Pi_{ij}=-K[\pi_i\pi_j]$, $\Omega_{ij}=0$ and by (10)

$$b^2+p_i+p_j=-a^2K \qquad (i\neq j). \tag{33}$$

Hence we have $p_1=p_2=\cdots=p_n$, which we put equal to p. Then for any rectangular frame characteristic roots of the conformal correspondence of S_1 with S are equal. We take a frame and coordinates $x_1, \cdots, x_n$ such that $\omega_i=dx_i$. Then we have $\omega_{ij}=0$ and by (10) $db_i-b_i da/a=p\,dx_i$, and so

$$d(b_i/a)=p/a\,dx_i. \tag{34}$$

Hence p/a contains only one variable x_i, but as i is arbitrary p/a is constant and we put $p/a=C$. In the case $C\neq 0$ we get from (34) the relation $b_i/a=Cx_i$ by a suitable choice of coordinates $x_1, \cdots, x_n$ and so

$$b^2/a^2=\sum_i (b_i/a)^2=C^2\sum x_i^2. \tag{35}$$

As $b^2+2p=-a^2K$ by (33) we get $C^2\sum x_i^2+2C/a=-K$, namely $a=-2C/(K+C^2\sum x_i^2)$ and finally

(36)
$$ds_1^2 = \frac{4C^2}{(K+C^2\sum x_i^2)^2}\sum dx_i^2, \qquad ds^2 = \sum dx_i^2.$$

In the case $K>0$ we take a similar mapping from ds_1^2 to $d\sigma_1^2 = K\,ds^2$ and from ds^2 to $d\sigma^2 = C^2/K\,ds^2$ and put $y_i = C/\sqrt{K}\,x_i$. Then we get

(37)
$$d\sigma_1^2 = \frac{4}{(1+\sum y_i^2)}\sum dy_i^2, \qquad d\sigma^2 = \sum dy_i^2$$

and in the case $K<0$ we put $d\sigma_1^2 = -K\,ds_1^2$, $d\sigma^2 = -C/K\,ds^2$, $y_i = C/\sqrt{-K}\,x_i$ and we get

(38)
$$d\sigma_1^2 = \frac{4}{(1-\sum y_i^2)^2}\sum dy_i^2, \qquad d\sigma^2 = \sum dy_i^2.$$

(37) and (38) are related to a stereographic projection in the $(n+1)$-dimensional space. In the case $K=0$ we get from (36)

(39)
$$ds_1^2 = \frac{4}{C^2(\sum x_i^2)^2}\sum dx_i^2, \qquad ds^2 = \sum dx_i^2.$$

Hence a conformal correspondence in the n-dimensional euclidean space is realized by an inversion and a similar transformation. Thus we get a new proof of Liouville's theorem.

Next we take up the case $C=0$. Then we have $p=0$ and by (34), b_i/a is constant, which we put equal to B_i, and so

$$da/a = b_i dx_i = aB_i dx_i\,.$$

By a suitable choice of rectangular coordinates $y_1, \cdots, y_n$ we get $1/a = Cy_1$ with constant C and by a similar mapping from ds_1^2 to $d\sigma_1^2 = C^2 ds_1^2$ we obtain

(40)
$$d\sigma_1^2 = \sum dy_i^2/y_1^2, \qquad ds^2 = \sum dy_i^2.$$

The former is a space of constant negative curvature and corresponds to the Poincaré's conformal representation of the non-euclidean hyperbolic space.

It is to be noted that in these cases the characteristic roots are all equal and the restriction on the dimension $n\geqq 3$ is necessary to derive the fact that all the characteristic roots are equal, and if they are all equal the above discussion holds good in the case $n=2$, too.

2. Conformally flat Riemann space of imbedding class 1.

2.1 We prove first the following theorem including the results of J. A. Schouten [1].

THEOREM 1. *Let S be an n-dimensional ($n \geqq 4$) conformally flat Riemann space of imbedding class* 1. *Then the characteristic roots of the conformal correspondence of S with an n-dimensional euclidean space are all equal except one, and when it is imbedded into an $(n+1)$-dimensional euclidean space as a hypersurface the principal curvatures are all equal except one.*

PROOF. Let Σ be a hypersurface in the $(n+1)$-dimensional euclidean space whose induced metric is given by that of S. We take a rectangular frame with origin $\boldsymbol{A}$ on Σ and one of the fundamental vectors $\boldsymbol{e}_1, \cdots, \boldsymbol{e}_{n+1}$ on the normal of Σ, which we assume to be $\boldsymbol{e}_{n+1}$, and $\boldsymbol{e}_1, \cdots, \boldsymbol{e}_n$ on the principal direction of Σ. Then the relative displacement of the frame $\boldsymbol{A}, \boldsymbol{e}_1, \cdots, \boldsymbol{e}_{n+1}$ is given by

$$(1) \qquad d\boldsymbol{A}=\pi_i \boldsymbol{e}_i\,, \quad d\boldsymbol{e}_i=\pi_{ij}\boldsymbol{e}_j+k_i\pi_i\boldsymbol{e}_{n+1} \quad \text{(not summed for } i)$$

since $\pi_{n+1}=0$ and $\pi_{i\,n+1}=-\pi_{n+1\,i}=k_i\pi_i$ (not summed for i), where k_i's are principal curvatures of Σ. The induced metric of Σ, which is that of S, is given by $ds^2=\sum_i \pi_i^2$ and we have

$$d\pi_i=[\pi_j\pi_{ji}]\,, \qquad \pi_{ij}=-\pi_{ji}\,.$$

The curvature forms of S are given by

$$(2) \qquad \Pi_{ij}=d\pi_{ij}-[\pi_{ik}\pi_{kj}]=[\pi_{i\,n+1}\pi_{n+1\,j}]=-k_ik_j[\pi_i\pi_j] \quad \text{(not summed for } i,j)\,.$$

As ds^2 is a conformally flat Riemannian metric we can put $\pi_i=a\omega_i$ where $\sum \omega_i^2$ is a metric of a euclidean space. By the relation (7) in **1.1** we get

$$(3) \qquad \Pi_{ij}=[p_{ik}\omega_k,\, \omega_j]-[p_{jk}\omega_k,\, \omega_i]+b^2[\omega_i\omega_j]\,.$$

By the comparison of (2) with (3) for h, i, j which are all different we get $p_{jh}=0\,(j \neq h)$. Hence a matrix (p_{jh}) is diagonal and we can put $p_{jj}=p_j$ (not summed for j), and we have, by virtue of (3),

$$(4) \qquad \Pi_{ij}=(b^2+p_i+p_j)[\omega_i\omega_j] \quad \text{(not summed for } i,j)\,,$$

and the frame $A, e_1, \cdots, e_n$ corresponds to the one selected on S in such a way that (10) in **1.1** holds. As we have $\pi_i = a\omega_i$ we get, by the comparison of (2) and (4),

$$b^2 + p_i + p_j = -a^2 k_i k_j \qquad (i \neq j),$$

and subtracting $b^2 + p_i + p_h = -a^2 k_i k_h \ (i \neq h)$ from this we get

$$p_j - p_h = -a^2 k_i (k_j - k_h). \tag{5}$$

Putting $j=1$, $h=2$ we obtain

$$p_1 - p_2 = -a^2 k_i (k_1 - k_2) \qquad (i = 3, 4, \cdots, n),$$

and so if $k_1 \neq k_2$, then we have $k_3 = k_4 = \cdots = k_n$ and if $k_2 \neq k_3$ we have $k_1 = k_4 = \cdots = k_n$. Thus $n-1$ of $k_1, \cdots, k_n$ are equal and by (5), $p_1, \cdots, p_n$ are all equal except one.

Thus we have proved theorem 1. Following the discussion above we derive some formulas which we use afterwards. By the equation $d\pi_{i\,n+1} = [\pi_{ij}\pi_{j\,n+1}]$ and $\pi_{i\,n+1} = k_i \pi_i$ (not summed for i) we have

$$[dk_i \pi_i] + k_i d\pi_i = [\pi_{ij}, k_j \pi_j] \qquad \text{(not summed for } i\text{)},$$

and for $i=1$ we get, by virtue of $d\pi_i = [\pi_j \pi_{ji}]$ and $k_2 = \cdots = k_n = k$,

$$[dk_1, \pi_1] = -(k_1 - k) d\pi_1 \tag{6}$$

and for $\alpha \neq 1$ we get

$$[dk\ \pi_\alpha] = (k_1 - k)[\pi_{\alpha 1} \pi_1]. \tag{7}$$

As π_i's are linearly independent we get

$$(k_1 - k)\pi_{\alpha 1} = -h\pi_\alpha + m_\alpha \pi_1, \tag{8}$$

and $$dk = h\pi_1 + n_\alpha \pi_\alpha \ \text{(not summed for } \alpha\text{)}.$$

n_α ought to be zero because of the linearly independence of π_α and the assumption $n \geqq 3$ and so

$$dk = h\pi_1. \tag{9}$$

We can characterize the hypersurface in the euclidean space whose principal curvatures are equal except one as follows.

THEOREM 2. *If the principal curvatures of a hypersurface in the* $(n+1)$*-dimensional euclidean space* $(n\geqq 3)$ *are equal except one and not equal to zero, it is an envelope of a one-parametric family of hyperspheres. Conversely the principal curvature of an envelope of a one-parametric family of hyperspheres in the* $(n+1)$*-dimensional euclidean space are all equal except one.*

PROOF. We use the same notation as in the proof of theorem 1 and assume that $k_2=\cdots=k_n\neq 0$, which we put equal to k. Hence we have $\pi_{1\,n+1}=k_1\pi_1$, $\pi_{\alpha\,n+1}=k\pi_\alpha\,(\alpha\neq 1)$ and it follows from (1) that

$$(10)\qquad \begin{aligned} d(A+1/k\,e_{n+1}) &= \pi_i e_i + 1/k\,\pi_{n+1\,i}\,e_i + d(1/k)e_{n+1} \\ &= (1-k_1/k)\pi_1 e_1 + d(1/k)e_{n+1}. \end{aligned}$$

Now by virtue of the relation (6) the equation $\pi_1=0$ is completely integrable and along the solution of the equation, which we denote by γ we have $k=$const. by (9), and (10) vanishes. Hence $A+1/k\,e_{n+1}$ is a fixed point and the $(n-1)$-dimensional subspace γ of Σ lies on the hypersphere and Σ is an envelope of these spheres.

Conversely let a hypersurface Σ be an envelope of one-parametric family of hyperspheres in an $(n+1)$-dimensional euclidean space. We take a frame $A, e_1, \cdots, e_{n+1}$ such that A lies on Σ, e_{n+1} lies on the normal of Σ at A and moreover $e_2, \cdots, e_n$ touch the $(n-1)$-dimensional subspace γ of Σ which lies on the hypersphere of radius $1/k$. Along γ we have $\pi_1=0$ and so

$$\begin{aligned} d(A+1/k\,e_{n+1}) &= \pi_i e_i + 1/k\,\pi_{n+1\,i} e_i + d(1/k)e_{n+1} \\ &= (\pi_\alpha + 1/k\,\pi_{n+1\,\alpha})e_\alpha + 1/k\,\pi_{n+1\,1} e_1 \end{aligned}$$

should vanish. It follows in general that

$$\pi_{1\,n+1}=k_1\pi_1\,,\qquad \pi_{\alpha\,n+1}=k\pi_\alpha+h_\alpha\pi_1\;(\alpha\neq 1)\,.$$

As $\pi_{n+1}=0$ we get

$$0=d\pi_{n+1}=[\pi_\alpha\pi_{\alpha\,n+1}]+[\pi_1\pi_{1\,n+1}]=h_\alpha[\pi_\alpha\pi_1]$$

and as $\pi_1, \cdots, \pi_n$ are linearly independent, we havs $h_\alpha=0$ and we get $\pi_{\alpha\,n+1}=k\pi_\alpha$. Thus $e_1, \cdots, e_n$ touch the principal direction on Σ and $n-1$ of principal curvatures are equal.

2.2 Next we shall prove a converse of the theorem 1.

THEOREM 3. *If the principal curvatures of a hypersurfaces Σ in an $(n+1)$-dimensional euclidean space $(n\geqq 3)$ are equal except one, the induced Riemannian metric of the hypersuface Σ is conformally flat.*

PROOF. Let $A, e_1, \cdots, e_n, e_{n+1}$ be a frame attached to Σ in such a way that $e_1, \cdots, e_n$ are in the principal direction of Σ. Then we have $dA=\pi_i e_i$, $de_i=\pi_{ij}e_j+k_i\pi_i e_{n+1}$ (not summed for i).

We put, by the assumption of our theorem, $k_2=\cdots=k_n=k$. For the curvature forms we have

$$\Pi_{ij}=d\pi_{ij}-[\pi_{ik}\pi_{kj}]=-k_ik_j[\pi_i\pi_j] \text{ (not summed for } i, j),$$

$$\Pi_{\alpha\beta}=-k^2[\pi_\alpha\pi_\beta] \qquad (\alpha, \beta=2, \cdots, n). \tag{11}$$

The purpose of our proof is to show the possibility of choosing a, ω_i, ω_{ij} such that

$$\pi_i=a\omega_i, \quad d\omega_i=[\omega_j\omega_{ji}], \quad \omega_{ij}=-\omega_{ji}, \quad d\omega_{ij}=[\omega_{ik}\omega_{kj}].$$

For that purpose we consider in the first the equation in l_i,

$$\rho_i=dl_i+l_k\pi_{ki}-l_il_k\pi_k-m_i\pi_i=0 \quad \text{(not summed for } i), \tag{12}$$

where m, m_i are such that

$$m_2=\cdots=m_n=m=(k^2-l^2)/2, \qquad l^2=\sum_i l_i^2, \tag{13}$$

$$-m_1+m=k^2-kk_1. \tag{14}$$

We show that (12) are completely integrable. Under the assumption $\rho_i=0$ we have

$$\begin{aligned} d(l_i\pi_i)&=[dl_i\pi_i]+l_id\pi_i \\ &=[-l_k\pi_{ki}+l_il_k\pi_k+m_i\pi_i, \pi_i]+l_i[\pi_k, \pi_{ki}]=0, \\ dl^2/2&=l_idl_i=-l_il_k\pi_{ki}+l^2l_k\pi_k+l_im_i\pi_i \\ &=(l^2+m)l_h\pi_h+(m_1-m)l_1\pi_1, \\ dm&=dk^2/2-dl^2/2=kdk-(l^2+m)l_h\pi_h-(m_1-m)l_1\pi_1. \end{aligned} \tag{15}$$

By virtue of the relation (7) and (9), which hold good in our case, and (14) we get

(16) $$[dm, \pi_1] = -(l^2+m)[l_h\pi_h, \pi_1],$$

(17) $$[dm, \pi_\alpha] = -(l^2+m)[l_h\pi_h, \pi_\alpha] - (m_1-m)l_1[\pi_1\pi_\alpha] + (m_1-m)[\pi_{\alpha 1}\pi_1].$$

So we get under the assumption $\rho_i=0$, which implies (15),

$$\begin{aligned} d\rho_i &= l_k d\pi_{ki} + [dl_k, \pi_{ki}] - [dl_i, l_k\pi_k] - [dm_i, \pi_i] - m_i d\pi_i \\ &= l_k d\pi_{ki} + [-l_k\pi_{kh} + l_h l_k \pi_k + m_h\pi_h, \pi_{hi}] \\ &\qquad - [-l_k\pi_{ki} + l_i l_k \pi_k + m_i\pi_i, l_h\pi_h] - [dm_i, \pi_i] - m_i d\pi_i \\ &= l_k \Pi_{ki} + m_h[\pi_h\pi_{hi}] - m_i[\pi_i, l_h\pi_h] - [dm_i, \pi_i] - m_i d\pi_i \\ &\qquad \text{(not summed for } i \text{ in each row)}. \end{aligned}$$

In the case $i=1$ this reduces to

$$\begin{aligned} d\rho_1 &= l_h\Pi_{h1} + m[\pi_h\pi_{h1}] - m_1[\pi_1, l_h\pi_h] - [dm_1, \pi_1] - m_1 d\pi_1 \\ &= -l_h k k_1[\pi_h\pi_1] - m_1[\pi_1, l_h\pi_h] - [dm_1, \pi_1] - (m_1-m)d\pi_1 \\ &= (m_1 - kk_1)[l_h\pi_h, \pi_1] - (m_1-m)d\pi_1 - [dm_1, \pi_1]. \end{aligned}$$

We get from (14), (16), (9), and (6),

$$\begin{aligned} [dm_1, \pi_1] &= [-2kdk + k_1 dk + kdk_1, \pi_1] + [dm, \pi_1] \\ &= [kdk_1, \pi_1] + [dm, \pi_1] = -(m_1-m)d\pi_1 - (l^2+m)[l_h\pi_h, \pi_1]. \end{aligned}$$

As $-(m_1-kk_1)=l^2+m$ by (13) and (14), we have $d\rho_1=0$. For $\alpha \neq 1$ we have, by (17) and (13),

$$\begin{aligned} d\rho_\alpha &= l_h\Pi_{h\alpha} + m[\pi_h\pi_{h\alpha}] + (m_1-m)[\pi_1\pi_{1\alpha}] - m[\pi_\alpha, l_h\pi_h] \\ &\quad - [dm, \pi_\alpha] - m d\pi_\alpha \\ &= -k^2[l_h\pi_h, \pi_\alpha] - (k_1 k - k^2)[l_1\pi_1, \pi_\alpha] + (m_1-m)[\pi_1\pi_{1\alpha}] \\ &\quad - m[\pi_\alpha, l_h\pi_h] - [dm, \pi_\alpha] \\ &= (m-k^2)[l_h\pi_h, \pi_\alpha] - (m_1-m)l_1[\pi_1\pi_\alpha] + (m_1-m)[\pi_{\alpha 1}\pi_1] - [dm\pi_\alpha] \\ &= 0. \end{aligned}$$

Thus (12) are completely integrable. As is seen from (15), $l_i\pi_i$ is a total differential and we take a such that $-da/a = l_i\pi_i$, namely

$$d\left(\frac{1}{a}\right) \Big/ \left(\frac{1}{a}\right) = l_i\pi_i.$$

Putting $\omega_{ij}=\pi_{ij}+l_i\pi_j-l_j\pi_i$, $\omega_i=\pi_i/a$,

we get by the discussion of **1.1**

$$d\omega_i=[\omega_j\omega_{ji}], \qquad \omega_{ij}=-\omega_{ji},$$

and as (12) corresponds to (10) in **1.1** we have by (11) in **1.1**,

$$\begin{aligned} d\omega_{ij}-[\omega_{ik}\omega_{kj}]&=\Pi_{ij}+(l^2+m_i+m_j)[\pi_i\pi_j] \\ &=(-k_ik_j+l^2+m_i+m_j)[\pi_i\pi_j] \text{ (not summed for } i,j), \end{aligned}$$

which vanishes by the relation

$$-k^2+l^2+2m=0, \qquad -kk_1+l^2+m+m_1=0$$

owing to (13) and (14). Hence $\sum\omega_i^2$ is flat.

In the case $n\geqq 4$ our theorem can also be proved by the vanishing of conformal curvature tensor.

2.3 Lastly we prove another converse of theorem 1.

THEOREM 4. *We assume that a Riemann space S is conformally flat and its characteristic roots relative to a euclidean space are equal to p except one and moreover that $p<-b^2/2$, where b is defined by* (8) *in* **1.1**. *Then S has an imbedding class* 1.

PROOF. We use the notations as in **1.1**. Let the metric of S be given by

(18) $$ds_1^2=\sum\pi_i^2, \qquad \pi_i=a\omega_i,$$

where $ds^2=\sum\omega_i^2$ is flat. Then we have

(19) $$d\omega_i=[\omega_j\omega_{ji}], \qquad \omega_{ij}=-\omega_{ji}, \qquad d\omega_{ij}=[\omega_{ik}\omega_{kj}].$$

Putting $p_2=\cdots=p_n=p$, $p_i/a=q_i$, $q_2=\cdots=q_n=q$, $b_i/a=c_i$ we get by (13) in **1.1**

(20) $$dc_i+c_k\omega_{ki}=q_i\omega_i \qquad \text{(not summed for } i).$$

By taking an exterior differential we obtain

$$[dc_k\omega_{ki}]+c_kd\omega_{ki}=[dq_i\omega_i]+q_id\omega_i \qquad \text{(not summed for } i).$$

Eliminating dc_k by (20) we obtain

$$-c_h[\omega_{hk}\omega_{ki}]+q_k[\omega_k\omega_{ki}]+c_kd\omega_{ki}=[dq_i\omega_i]+q_id\omega_i$$
$$\text{(not summed for } i).$$

By virtue of the relation (19) we get

$$q_h[\omega_h\omega_{hi}]=[dq_i\omega_i]+q_i d\omega_i \qquad \text{(not summed for } i).$$

For $i=1$ we have

(21) $$[dq_1, \omega_1]=(q_1-q)d\omega_1 ,$$

and for $\alpha \neq 1$ we have

(22) $$[dq, \omega_\alpha]=(q_1-q)[\omega_1\omega_{1\alpha}] ,$$

and so we get

(23) $$(q_1-q)\omega_{1\alpha}=s\omega_\alpha+r_\alpha\omega_1 , \quad dq=s\omega_1+t_\alpha\omega_\alpha \quad \text{(not summed for } \alpha).$$

As $n \geqq 3$ and ω_α $(\alpha=2, \cdots, n)$ are linearly independent we get

(24) $$dq=s\omega_1 .$$

Now take k, k_1 such that

(25) $$c^2+2q/a=-k^2 , \qquad c^2+(q+q_1)/a=-kk_1 ,$$

where $c^2=\sum c_i^2$. This is possible because by our assumption we have $c^2+2q/a=(b^2+2p)/a^2<0$. Then we have

(26) $$-(q_1-q)/a=k(k_1-k) .$$

From (20) we get

$$dc^2/2=d(\sum c_i^2)/2=c_i dc_i=c_i(-c_k\omega_{ki}+q_i\omega_i)=c_i q_i\omega_i$$
$$=qc_i\omega_i+(q_1-q)c_1\omega_1=q\, da/a^2+(q_1-q)c_1\omega_1 .$$

By taking a differential of (25) we obtain

$$dc^2-2q\, da/a^2+2\, dq/a=-2k\, dk .$$

Hence we have

(27) $$(q_1-q)c_1\omega_1+dq/a=-k\, dk .$$

Now we put $k_2=k_3=\cdots=k_n=k$, $\pi_{n+1}=0$ and

(28) $$\pi_{ij}=\omega_{ij}+ac_i\omega_j-ac_j\omega_i ,$$

$$\pi_{n+1\,n+1}=0 , \quad \pi_{i\,n+1}=-\pi_{n+1\,i}=k_i\pi_i \quad \text{(not summed for } i).$$

Then we have

$$d\pi_i=[\pi_j\pi_{ji}]+[\pi_{n+1}\pi_{n+1\,i}]\,, \qquad d\pi_{n+1}=[\pi_i\pi_{i\,n+1}]\,,$$

and by (11) in **1.1** and (25)

$$d\pi_{ij}-[\pi_{ik}\pi_{kj}]=(b^2+p_i+p_j)[\omega_i\omega_j]$$
$$=(c^2+(q_i+q_j)/a)[\pi_i\pi_j]=-k_ik_j[\pi_i\pi_j]=[\pi_{i\,n+1}\pi_{n+1\,j}]$$

(not summed for i, j in each row),

$$d\pi_{n+1\,n+1}=0=[\pi_{n+1\,i},\ \pi_{i\,n+1}]\,,$$

and so if we verify

$$d\pi_{i\,n+1}=[\pi_{ij}\pi_{j\,n+1}]$$

all the structure equations in the $(n+1)$-dimensional euclidean space are satisfied and moreover $\pi_{n+1}=0$, and S can be imbedded into an $(n+1)$-dimensional euclidean space. By virture of the relations (22), (24), (27) we get

(29) $$[dq,\ \pi_1]=[dq,\ a\omega_1]=0\,,$$

(30) $$[dk,\ \pi_1]=-1/k[(q_1-q)c_1\omega_1+dq/a,\ a\omega_1]=0\,,$$

$$[dk,\ \pi_\alpha]=-1/k[(q_1-q)c_1\omega_1+dq/a,\ a\omega_\alpha]$$
$$=-(q_1-q)/k\big(ac_1[\omega_1\omega_\alpha]+[\omega_1\omega_{1\alpha}]\big)\,.$$

Hence we get by virtue of (28) and (26)

(31) $$[dk,\ \pi_\alpha]=(k_1-k)[\pi_1\pi_{1\alpha}]\,.$$

Differentiating (26) we obtain

$$(q_1-q)da/a^2-dq_1/a+dq/a=kdk_1+k_1dk-2kdk\,.$$

Multiplying this by π_1 and considering (21), (29), (30) we get

$$(q_1-q)/a^2[da,\ \pi_1]+(q_1-q)d\omega_1=k[dk_1,\ \pi_1]\,,$$

and by virtue of (26)

$$[dk_1,\ \pi_1]=-(k_1-k)\big([da,\ \omega_1]+ad\omega_1\big)=-(k_1-k)d(a\omega_1)\,.$$

Hence we have

(32) $$[dk_1,\ \pi_1]=-(k_1-k)d\pi_1\,.$$

By virtue of (32) we get

$$d\pi_{1\,n+1}-[\pi_{1j}\pi_{j\,n+1}]=d(k_1\pi_1)-[\pi_{1\alpha}, k\pi_\alpha]$$
$$=[dk_1, \pi_1]+(k_1-k)d\pi_1=0\ ,$$

and by (30) we get

$$d\pi_{\alpha\,n+1}-[\pi_{\alpha i}\pi_{i\,n+1}]=d(k\pi_\alpha)-k[\pi_{\alpha i}\pi_i]-(k_1-k)[\pi_{\alpha 1}\pi_1]$$
$$=[dk, \pi_\alpha]-(k_1-k)[\pi_{\alpha 1}\pi_1]=0\,.$$

Thus we have proved theorem 4.

If the Riemannian metric of S is given by

$$ds^2=a^2\sum_i dx_i^2,$$

we can take frames such that $\omega_i=dx_i$ and hence $\omega_{ij}=0$. Then putting $\alpha=\log a$ we get, by virtue of (3) and (9) in **1.1**,

$$b_i=\frac{\partial\alpha}{\partial x_i}\,,\quad p_{ik}=\frac{\partial^2\alpha}{\partial x_i\partial x_k}-\frac{\partial\alpha}{\partial x_i}\frac{\partial\alpha}{\partial x_k}\,.$$

Hence we obtain the following theorem.

THEOREM 5. *In order that a Riemann space with a metric* $ds^2=a^2\sum dx_i^2$ *be of imbedding class* 1 *it is necessary and sufficient that the characteristic roots of a symmetric matrix* $\left(\frac{\partial^2\alpha}{\partial x_i\partial x_k}-\frac{\partial\alpha}{\partial x_i}\frac{\partial\alpha}{\partial x_k}\right)$ ($\alpha=\log a$) *are equal except one and the common value of these roots is smaller then* $-1/2\sum_i\left(\frac{\partial\alpha}{\partial x_i}\right)^2$.

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