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Mother-Child Combinations concerning an Inherited Character after a Panmixia

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1. Introduction.

In a previous paper[1] we have discussed from a stochastic viewpoint the distributions of genotypes in the next generation originated from an original generation with a given distribution after a panmixia. In succession, we shall now deal with an analogous problem concerning mother-child combinations.

Consider again a population of size $2N$ consisting of N females and N males, and observe a single inherited character which consists of m multiple alleles at one diploid locus denoted by

$$A_i \qquad (i=1, \cdots, m).$$

Let the given distributions of the genotypes $\{A_a A_b\}$ in females and in males be designated by

$$\mathfrak{F}=\{F_{ab}\} \quad \text{and} \quad \mathfrak{M}=\{M_{ab}\} \qquad (a, b=1, \cdots, m; a \leqq b),$$

respectively, so that

$$\sum_{a \leqq b} F_{ab} = \sum_{a \leqq b} M_{ab} = N.$$

The order of genes in a genotype being immaterial, both genotypes $A_a A_b$ and $A_b A_a$ are identified each other even when the suffices a and b are distinct. Accordingly we put $F_{ab}=F_{ba}$ and $M_{ab}=M_{ba}$.

We now observe a mother-child combination, designated by $(A_\alpha A_\beta; A_\xi A_\eta)$, which consists of a fixed pair of mother's type $A_\alpha A_\beta$ and child's type $A_\xi A_\eta$ and introduce a stochastic variable X extending

1) Y. Komatu, Distributions of genotypes after a panmixia. Journ. Math. Soc. Japan 6 (1954), 266-282.

over integers contained in an interval

$$0 \leqq X \leqq F_{\alpha\beta}.$$

We then designate by

$$\Psi(\alpha\beta;\xi\eta\,|\,X) \equiv \Psi(\alpha\beta;\xi\eta\,|\,X\,|\,\mathfrak{F};\mathfrak{M})$$

the probability that after a panmixia the mother-child combination $(A_\alpha A_\beta; A_\xi A_\eta)$ amounts to X, each mating being supposed to produce one child. The probability-generating function is then defined by

$$\Phi(\alpha\beta;\xi\eta\,|\,z) \equiv \Phi(\alpha\beta;\xi\eta\,|\,z\,|\,\mathfrak{F};\mathfrak{M}) = \sum_{X=0}^{F_{\alpha\beta}} \Psi(\alpha\beta;\xi\eta|X)z^X,$$

z designating an indeterminate variable.

The main purpose of the present paper is to establish explicit expressions for probability-generating functions of all the possible mother-child combinations.

The dependence of Ψ as well as Φ on the original data has been indicated explicitly by $\mathfrak{F};\mathfrak{M}$. However, it is a matter of course that for a given pair $(A_\alpha A_\beta; A_\xi A_\eta)$ these quantities are really independent of the F's except $F_{\alpha\beta}$ and of the M's possessing no suffices in common with the given pair. Such a reasoning will be fully availed in the following lines.

In a series of previous papers[2], we have dealt with analogous problems far extensively but with few preciseness. We have considered there a population of infinite size and studied merely the means of distributions on various definite combinations, and shown especially the following results: Suppose that the distributions of genes in female- and male-populations are both in equilibrium states and let their relative frequencies of genes be designated by

$$p_i^{(F)} \quad \text{and} \quad p_i^{(M)} \qquad (i=1,\cdots,m),$$

2) Y. Komatu, Probability-theoretic investigations on inheritance. I-XVI. Proc. Japan Acad. 27-29 (1951-1953); especially IV. Mother-child combinations. 27 (1951), 587-620 and 29 (1953), 68-77. For another sort of generalizations cf. Y. Komatu and H. Nishimiya, Probabilistic investigations on inheritance in consanguineous families. Bull. Tokyo Inst. Tech. (1954), 1-222 of which preliminary announcements are found in Y. Komatu and H. Nishimiya, Probabilities on inheritance in consanguineous families. Proc. Japan Acad. 30 (1954), 42-52, 148-155, 236-247, 636-654; cf. also Y. Komatu and H. Nishimiya, Lineal combinations on a Mendelian inherited character. Rep. Stat. Appl. Res., JUSE 3 (1953), 13-22.

respectively. Then, the values of the probability of mother-child combination $(A_\alpha A_\beta; A_\xi A_\eta)$, designated by $\pi(\alpha\beta; \xi\eta)$, are given as follows:

$$\pi(ii;\, ii) = p_i^{(F)2} p_i^{(M)}, \qquad \pi(ii;\, ik) = p_i^{(F)2} p_k^{(M)},$$

$$\pi(ij;\, ii) = p_i^{(F)} p_j^{(F)} p_i^{(M)}, \qquad \pi(ij;\, ij) = p_i^{(F)} p_j^{(F)} (p_i^{(M)} + p_j^{(M)}),$$

$$\pi(ij; ik) = p_i^{(F)} p_j^{(F)} p_k^{(M)} \qquad (i \neq j;\; k \neq i, j);$$

it is evident that $\pi(\alpha\beta; \xi\eta)$ vanishes out unless $A_\alpha A_\beta$ and $A_\xi A_\eta$ possess at least a gene in common. The results of the present paper will give a generalization of the previous results listed above.[3)]

2. Probability-generating functions.

i. $(A_iA_i; A_iA_i)$.

For mother-child combination $(A_iA_i; A_iA_i)$, the problem is reducible to one concerning two different genes. In fact, since the distinction among the genes except A_i is here a matter of indifference, these $m-1$ genes $A_b\,(b \neq i)$ may be gathered to an aggregate playing a role of an *imaginary gene*, A_ω say. We put

$$\sum_{b \neq i} M_{ib} = M_{i\omega}, \qquad \sum_{\substack{a,b \neq i \\ a \leq b}} M_{ab} = M_{\omega\omega},$$

and imagine that the male-population consists of M_{ii}, $M_{i\omega}$ and $M_{\omega\omega}$ individuals of genotypes A_iA_i, A_iA_ω and $A_\omega A_\omega$, respectively.

We consider a partition of these N males into two classes according to genotypes A_iA_i and not-A_iA_i of females to be married. Namely, let each of M_{ii}, $M_{i\omega}$, $M_{\omega\omega}$ individuals in male-population be divided into two classes, empty classes being admitted, in such a manner

$$M_{ii} = y_{ii} + (M_{ii} - y_{ii}), \qquad M_{i\omega} = y_{i\omega} + (M_{i\omega} - y_{i\omega}),$$

$$M_{\omega\omega} = y_{\omega\omega} + (M_{\omega\omega} - y_{\omega\omega}); \qquad F_{ii} = y_{ii} + y_{i\omega} + y_{\omega\omega}.$$

Let now the matings take place such that y_{ii}, $y_{i\omega}$ and $y_{\omega\omega}$ males of genotypes A_iA_i, A_iA_ω and $A_\omega A_\omega$, respectively, are combined, as a whole, with F_{ii} females of genotype A_iA_i.

3) Papers closely related to the present paper will be published in Rep. Stat. Appl. Res., JUSE 3 (1954) and Kōdai Math. Sem. Rep. (1954).

All the permutations of N males consisting of M_{ii}, $M_{i\omega}$ and $M_{\omega\omega}$ individuals of genotypes A_iA_i, A_iA_ω and $A_\omega A_\omega$, respectively, amount to

$$N!/M_{ii}!\,M_{i\omega}!\,M_{\omega\omega}!,$$

while the permutations of F_{ii} and $N-F_{ii}$ males to be married with females of genotypes A_iA_i and not-A_iA_i, these males consisting of y_{ii}, $y_{i\omega}$, $y_{\omega\omega}$ individuals and of $M_{ii}-y_{ii}$, $M_{i\omega}-y_{i\omega}$, $M_{\omega\omega}-y_{\omega\omega}$ individuals of genotypes A_iA_i, A_iA_ω, $A_\omega A_\omega$, respectively, then amount to

$$F_{ii}!/y_{ii}!\,y_{i\omega}!\,y_{\omega\omega}!, \qquad (N-F_{ii})!/(M_{ii}-y_{ii})!\,(M_{i\omega}-y_{i\omega})!\,(M_{\omega\omega}-y_{\omega\omega})!.$$

On the other hand, it is supposed that any mating produces each of four possible genotypes equally likely, that is, with probability 1/4. When some of genes involved in a mating are coincident and hence two or four of the genotypes to be produced are identical, the probability is then, of course, interpreted as the corresponding sum, namely, as 1/2 or 1, respectively.

Now, a male of genotype A_iA_i, A_iA_ω or $A_\omega A_\omega$ produces, together with a female of type A_iA_i, a child of A_iA_i with probability 1, 1/2 or 0, respectively. Consequently, the generating function for the present combination is given by

$$\Phi(ii;ii|z)\equiv\Phi(ii;ii|z|F_{ii};M_{ii},M_{i\omega},M_{\omega\omega})=\sum_{X=0}^{F_{ii}}\Psi(ii;ii|X)z^X$$

$$=\sum_{X=0}^{F_{ii}}\frac{M_{ii}!\,M_{i\omega}!\,M_{\omega\omega}!}{N!}\sum_{\mathfrak{y}}\frac{F_{ii}!}{y_{ii}!\,y_{i\omega}!\,y_{\omega\omega}!}\frac{(N-F_{ii})!}{(M_{ii}-y_{ii})!\,(M_{i\omega}-y_{i\omega})!\,(M_{\omega\omega}-y_{\omega\omega})!}$$

$$\cdot z^{y_{ii}}\binom{y_{i\omega}}{X-y_{ii}}\left(\frac{z}{2}\right)^{X-y_{ii}}\left(\frac{1}{2}\right)^{y_{i\omega}-X+y_{ii}},$$

the second summation extending over all the possible partitions represented by the sets $\mathfrak{y}=(y_{ii},y_{i\omega},y_{\omega\omega})$. Performing first, by means of binomial identity, the summation with respect to X, we get

$$\Phi(ii;ii|z)=\frac{M_{ii}!\,M_{i\omega}!\,M_{\omega\omega}!}{N!}$$

$$\times\sum_{\mathfrak{y}}\frac{F_{ii}!}{y_{ii}!\,y_{i\omega}!\,y_{\omega\omega}!}\frac{(N-F_{ii})!}{(M_{ii}-y_{ii})!\,(M_{i\omega}-y_{i\omega})!\,(M_{\omega\omega}-y_{\omega\omega})!}$$

$$\cdot z^{y_{ii}}\left(\frac{z+1}{2}\right)^{y_{i\omega}}.$$

We now introduce, with parameters t_i and t_ω, an expression defined by

$$\Phi(ii;ii|z|t_i,t_\omega)=\frac{M_{ii}!\,M_{i\omega}!\,M_{\omega\omega}!}{N!\,t_i^{M_{ii}}t_\omega^{M_{i\omega}}}$$
$$\cdot\left(t_iz+t_\omega\frac{z+1}{2}+1\right)^{F_{ii}}(t_i+t_\omega+1)^{N-F_{ii}}.$$

Our generating function $\Phi(ii;ii|z)$ is then given by the constant term, i.e. the coefficient of the term $t_i^0t_\omega^0$ in the Laurent expansion around the origin of this expression regarded as a rational function of two variables t_i and t_ω. It is convenient to introduce a further expression defined by

$$\Phi(ii;ii|z|s;t_i,t_\omega)=\frac{F_{ii}!\,(N-F_{ii})!\,M_{ii}!\,M_{i\omega}!\,M_{\omega\omega}!}{N!^2\,s^{F_{ii}}t_i^{M_{ii}}t_\omega^{M_{i\omega}}}$$
$$\cdot\left\{(sz+1)\left(t_i+\frac{t_\omega}{2}\right)+(s+1)\left(\frac{t_\omega}{2}+1\right)\right\}^N.$$

Then our generating function is given also by the constant term in the Laurent expansion around the origin of the last expression regarded as a rational function of three variables s, t_i and t_ω; here, as defined above,

$$M_{i\omega}=\sum_{b\neq i}M_{ib},\qquad M_{\omega\omega}=\sum_{\substack{a,b\neq i\\ a\leq b}}M_{ab}=N-M_{ii}-M_{i\omega}.$$

It would be noted, in passing, that, while the expression $\Phi(ii;ii|z|s;t_i,t_\omega)$ introduced above is inhomogeneous with respect to t_i and t_ω, it may be substituted by a suitable homogeneous one by appending a new parameter. In fact, if we introduce an expression defined by

$$\Phi(ii;ii|z|s;t_{ii},t_{i\omega},t_{\omega\omega})=\frac{F_{ii}!\,(N-F_{ii})!\,M_{ii}!\,M_{i\omega}!\,M_{\omega\omega}!}{N!^2\,s^{F_{ii}}t_{ii}^{M_{ii}}t_{i\omega}^{M_{i\omega}}t_{\omega\omega}^{M_{\omega\omega}}}$$
$$\cdot\left\{(sz+1)\left(t_{ii}+\frac{t_{i\omega}}{2}\right)+(s+1)\left(\frac{t_{i\omega}}{2}+t_{\omega\omega}\right)\right\}^N,$$

then our generating function can be obtained also as the constant term in the Laurent expansion around the origin of this expression regarded as a rational function of four variables s, t_{ii}, $t_{i\omega}$ and $t_{\omega\omega}$. It is evident that there holds a relation

$$\Phi(ii; ii\,|\,z\,|\,s; t_{ii}, t_{i\omega}, t_{\omega\omega}) = \Phi(ii; ii\,|\,z\,|\,s; t_{ii}/t_{\omega\omega}, t_{i\omega}/t_{\omega\omega}).$$

Quite similarly, the last expression may further be substituted by another expression defined by

$$\Phi(ii; ii\,|\,z\,|\,s_i, s_\omega; t_{ii}, t_{i\omega}, t_{\omega\omega}) = \Phi(ii; ii\,|\,z\,|\,s_i/s_\omega; t_{ii}/t_{\omega\omega}, t_{i\omega}/t_{\omega\omega}),$$

which is homogeneous with respect to five parameters s_i, s_ω, t_{ii}, $t_{i\omega}$ and $t_{\omega\omega}$.

On the other hand, the generating function $\Phi(ii; ii\,|\,z)$ itself is expressible by means of a contour integral. In fact, there holds, for instance, a relation

$$\Phi(ii; ii\,|\,z) = \frac{1}{(2\pi\sqrt{-1})^3}\iiint \Phi(ii; ii\,|\,z\,|\,s; t_i, t_\omega)\frac{ds\,dt_i\,dt_\omega}{s t_i t_\omega},$$

where the triple integration is taken along the unit circumferences

$$|s| = 1, \qquad |t_i| = 1, \qquad |t_\omega| = 1$$

in the positive sense on the respective complex planes.

Analogous remarks apply also to the combinations which will appear subsequently, though they will not be repeated explicitly.

ii. $(A_iA_i; A_iA_k)$; $k \neq i$.

For combination $(A_iA_i; A_iA_k)$ the problem is reducible to one concerning three different genes. In fact, since the distinction among the genes except A_i and A_k is here a matter of indifference, these $m-2$ genes $A_b\,(b \neq i, k)$ may be gathered to an aggregate playing a role of an imaginary gene. But the problem can be reduced really to a more simplified one. In fact, the genotypes of males which can produce, together with a female of type A_iA_i, a child of A_iA_k, are those involving at least one gene A_k. Hence, it suffices now to classify N males based on this gene A_k. We thus introduce an imaginary gene A_ω by gathering $m-1$ genes $A_b\,(b \neq k)$, and put accordingly

$$\sum_{b \neq k} M_{kb} = M_{k\omega}, \qquad \sum_{\substack{a,b \neq k \\ a \leq b}} M_{ab} = M_{\omega\omega}.$$

We consider a partition of N males under consideration into two classes according to genotypes A_iA_i and not-A_iA_i of females to be married. Namely, let each of M_{kk}, $M_{k\omega}$, $M_{\omega\omega}$ individuals in male-population be divided into two classes in such a manner

$$M_{kk}=y_{kk}+(M_{kk}-y_{kk}), \qquad M_{k\omega}=y_{k\omega}+(M_{k\omega}-y_{k\omega}),$$

$$M_{\omega\omega}=y_{\omega\omega}+(M_{\omega\omega}-y_{\omega\omega}); \qquad F_{ii}=y_{kk}+y_{k\omega}+y_{\omega\omega}.$$

Let now the matings take place such that y_{kk}, $y_{k\omega}$ and $y_{\omega\omega}$ males of genotypes A_kA_k, A_kA_ω and $A_\omega A_\omega$, respectively, are combined, as a whole, with F_{ii} females of genotype A_iA_i.

A male of genotype A_kA_k, A_kA_ω or $A_\omega A_\omega$ produces, together with a female of type A_iA_i, a child of A_iA_k with probability 1, 1/2 or 0, respectively. A reason similar to in the preceding case leads now to an expression

$$\Phi(ii;ik\,|\,z)\equiv(ii;ik\,|\,z\,|\,F_{ii};M_{kk},M_{k\omega},M_{\omega\omega})=\sum_{X=0}^{F_{ii}}\Psi(ii;ik\,|\,X)z^X$$

$$=\sum_{X=0}^{F_{ii}}\frac{M_{kk}!\,M_{k\omega}!\,M_{\omega\omega}!}{N!}\sum_{\mathfrak{y}}\frac{F_{ii}!}{y_{kk}!\,y_{k\omega}!\,y_{\omega\omega}!}\frac{(N-F_{ii})!}{(M_{kk}-y_{kk})!\,(M_{k\omega}-y_{k\omega})!\,(M_{\omega\omega}-y_{\omega\omega})!}$$

$$\cdot z^{y_{ii}}\binom{y_{k\omega}}{X-y_{kk}}\left(\frac{z}{2}\right)^{X-y_{kk}}\left(\frac{1}{2}\right)^{y_{k\omega}-X+y_{kk}}$$

$$=\frac{M_{kk}!\,M_{k\omega}!\,M_{\omega\omega}!}{N!}\sum_{\mathfrak{y}}\frac{F_{ii}!}{y_{kk}!\,y_{k\omega}!\,y_{\omega\omega}!}\frac{(N-F_{ii})!}{(M_{kk}-y_{kk})!\,(M_{k\omega}-y_{k\omega})!\,(M_{\omega\omega}-y_{\omega\omega})!}$$

$$\cdot z^{y_{kk}}\left(\frac{z+1}{2}\right)^{y_{k\omega}}.$$

Finally, if we introduce the expressions defined by

$$\Phi(ii;ik\,|\,z\,|\,t_k,t_\omega)=\frac{M_{kk}!\,M_{k\omega}!\,M_{\omega\omega}!}{N!\,t_k^{M_{kk}}\,t_\omega^{M_{k\omega}}}$$

$$\cdot\left(t_kz+t_\omega\frac{z+1}{2}+1\right)^{F_{ii}}(t_k+t_\omega+1)^{N-F_{ii}}$$

and

$$\Phi(ii;ik\,|\,z\,|\,s;t_k,t_\omega)=\frac{F_{ii}!\,(N-F_{ii})!\,M_{kk}!\,M_{k\omega}!\,M_{\omega\omega}!}{N!^2\,s^{F_{ii}}\,t_k^{M_{kk}}\,t_\omega^{M_{k\omega}}}$$

$$\cdot \left\{(sz+1)\left(t_k+\frac{t_\omega}{2}\right)+(s+1)\left(\frac{t_\omega}{2}+1\right)\right\}^N,$$

then our generating function $\Phi(ii;ik|z)$ is given by the constant term in the Laurent expansion around the origin of either of these expressions regarded as a rational function of variables t_k and t_ω or s, t_k and t_ω, respectively; here, as defined above,

$$M_{k\omega}=\sum_{b\neq k} M_{kb}, \qquad M_{\omega\omega}=\sum_{\substack{a,b\neq k\\ a\leq b}} M_{ab}=N-M_{kk}-M_{k\omega}.$$

iii. $(A_iA_j;A_iA_i)$; $i\neq j$.

For combination $(A_iA_j;A_iA_i)$, the problem can be reduced in a fairly similar manner as in the preceding case. The genotypes of males which can produce, together with a female of type A_iA_j, a child of A_iA_i, are those involving at least one gene A_i. Hence, it suffices now to classify males based on the gene A_i. We thus introduce an imaginary gene A_ω by gathering $m-1$ genes $A_b\,(b\neq i)$, and put

$$\sum_{b\neq i} M_{ib}=M_{i\omega}, \qquad \sum_{\substack{a,b\neq i\\ a\leq b}} M_{ab}=M_{\omega\omega}.$$

We consider a partition of N males into two classes according to genotypes A_iA_j and not-A_iA_j of females to be married. Put

$$M_{ii}=y_{ii}+(M_{ii}-y_{ii}), \qquad M_{i\omega}=y_{i\omega}+(M_{i\omega}-y_{i\omega}),$$

$$M_{\omega\omega}=y_{\omega\omega}+(M_{\omega\omega}-y_{\omega\omega}); \qquad F_{ij}=y_{ii}+y_{i\omega}+y_{\omega\omega},$$

and let the matings take place such that y_{ii}, $y_{i\omega}$ and $y_{\omega\omega}$ males of genotypes A_iA_i, A_iA_ω and $A_\omega A_\omega$, respectively, are combined, as a whole, with F_{ij} females of genotype A_iA_j.

A male of genotype A_iA_i, A_iA_ω or $A_\omega A_\omega$ produces, together with a female of type A_iA_j, a child of A_iA_i with probability 1/2, 1/4 or 0, respectively. A similar reason as above leads now to an expression

$$\Phi(ij;ii|z)\equiv\Phi(ij;ii|z|F_{ij};M_{ii},M_{i\omega},M_{\omega\omega})=\sum_{X=0}^{F_{ij}}\Psi(ij;ii|X)z^X$$

$$=\sum_{X=0}^{F_{ij}}\frac{M_{ii}!\,M_{i\omega}!\,M_{\omega\omega}!}{N!}\sum_{\mathfrak{y}}\frac{F_{ij}!}{y_{ii}!\,y_{i\omega}!\,y_{\omega\omega}!}\,\frac{(N-F_{ij})!}{(M_{ii}-y_{ii})!\,(M_{i\omega}-y_{i\omega})!\,(M_{\omega\omega}-y_{\omega\omega})!}$$

$$\cdot \sum_{x_i+x_\omega=X} \binom{y_{ii}}{x_i}\left(\frac{z}{2}\right)^{x_i}\left(\frac{1}{2}\right)^{y_{ii}-x_i}\binom{y_{i\omega}}{x_\omega}\left(\frac{z}{4}\right)^{x_\omega}\left(\frac{3}{4}\right)^{y_{i\omega}-x_\omega}$$

$$= \frac{M_{ii}!\,M_{i\omega}!\,M_{\omega\omega}!}{N!} \sum_{\mathfrak{y}} \frac{F_{ij}!}{y_{ii}!\,y_{i\omega}!\,y_{\omega\omega}!}\,\frac{(N-F_{ij})!}{(M_{ii}-y_{ii})!\,(M_{i\omega}-y_{i\omega})!\,(M_{\omega\omega}-y_{\omega\omega})!}$$

$$\cdot\left(\frac{z+1}{2}\right)^{y_{ii}}\left(\frac{z+3}{4}\right)^{y_{i\omega}}.$$

Finally, if we introduce the expressions defined by

$$\Phi(ij;ii|z|t_i,t_\omega)=\frac{M_{ii}!\,M_{i\omega}!\,M_{\omega\omega}!}{N!\,t_i^{M_{ii}}t_\omega^{M_{i\omega}}}$$

$$\cdot\left(t_i\frac{z+1}{2}+t_\omega\frac{z+3}{4}+1\right)^{F_{ij}}(t_i+t_\omega+1)^{N-F_{ij}}$$

and

$$\Phi(ij;ii|z|s;t_i,t_\omega)=\frac{F_{ij}!\,(N-F_{ij})!\,M_{ii}!\,M_{i\omega}!\,M_{\omega\omega}!}{N!^2\,s^{F_{ij}}\,t_i^{M_{ii}}\,t_\omega^{M_{i\omega}}}$$

$$\cdot\left\{\left(s\frac{z+1}{2}+1\right)\left(t_i+\frac{t_\omega}{2}\right)+(s+1)\left(\frac{t_\omega}{2}+1\right)\right\}^N,$$

then our generating function $\Phi(ij;ii|z)$ is given by the constant term in the Laurent expansion around the origin of either of these expressions regarded as a rational function of variables t_i and t_ω or s, t_i and t_ω, respectively; here, as defined above,

$$M_{i\omega}=\sum_{b\neq i}M_{ib},\qquad M_{\omega\omega}=\sum_{\substack{a,b\neq i\\ a\leq b}}M_{ab}=N-M_{ii}-M_{i\omega}.$$

iv. $(A_iA_j;\,A_iA_j)$; $i\neq j$.

For combination $(A_iA_j;A_iA_j)$ the problem is reducible to one concerning three different genes. In fact, since the distinction among the genes except A_i and A_j is here a matter of indifference, these $m-2$ genes $A_b\,(b\neq i,j)$ may be gathered to an aggregate playing a role of an imaginary gene, A_ω say. We put accordingly

$$\sum_{b\neq i,j}M_{ib}=M_{i\omega},\qquad \sum_{b\neq i,j}M_{jb}=M_{j\omega},\qquad \sum_{\substack{a,b\neq i,j\\ a\leq b}}M_{ab}=M_{\omega\omega},$$

and imagine that the male-population consists of M_{ii}, M_{ij}, M_{jj}, $M_{i\omega}$,

$M_{j\omega}$ and $M_{\omega\omega}$ individuals of genotypes A_iA_i, A_iA_j, A_jA_j, A_iA_ω, A_jA_ω and $A_\omega A_\omega$, respectively. But it is supposed that a male of genotype A_iA_i, A_iA_j, A_jA_j, A_iA_ω, A_jA_ω or $A_\omega A_\omega$ produces, together with a female of type A_iA_j, a child of A_iA_j with probability 1/2, 1/2, 1/2, 1/4, 1/4 or 0, respectively.

A glance at the circumstance stated just above suggests a possibility of further reduction of the problem. Namely, so far as there concerns mother-child combination $(A_iA_j; A_iA_j)$ three genotypes A_iA_i, A_iA_j and A_jA_j as well as two imaginary genotypes A_iA_ω and A_jA_ω of males behave themselves, respectively, in quite the same manner. In fact, anyone in the former group or in the latter group produces, together with a female of type A_iA_j, a child of A_iA_j with the same probability 1/2 or 1/4, respectively. Consequently, two genes A_i and A_j in the male population may also be identified and hence gathered to an aggregate A_α. We put accordingly

$$M_{ii}+M_{ij}+M_{jj}=M_{\alpha\alpha}, \qquad M_{i\omega}+M_{j\omega}=M_{\alpha\omega}.$$

We can then proceed quite similarly as in the preceding case of combination $(A_iA_j; A_iA_i)$ by substituting $M_{\alpha\alpha}$, $M_{\alpha\omega}$ and $M_{\omega\omega}$ instead of M_{ii}, $M_{i\omega}$ and $M_{\omega\omega}$ in the preceding case, respectively. We thus get for the generating function an expression

$$\Phi(ij; ij\,|\,z)\equiv\Phi(ij; ij\,|\,z\,|\,F_{ij}; M_{\alpha\alpha}, M_{\alpha\omega}, M_{\omega\omega})=\sum_{X=0}^{F_{ij}}\Psi(ij; ij\,|\,X)z^X$$

$$=\frac{M_{\alpha\alpha}!\,M_{\alpha\omega}!\,M_{\omega\omega}}{N!}\sum_{\mathfrak{y}}\frac{F_{ij}!}{y_{\alpha\alpha}!\,y_{\alpha\omega}!\,y_{\omega\omega}!}\,\frac{(N-F_{ij})!}{(M_{\alpha\alpha}-y_{\alpha\alpha})!\,(M_{\alpha\omega}-y_{\alpha\omega})!\,(M_{\omega\omega}-y_{\omega\omega})!}$$

$$\cdot\left(\frac{z+1}{2}\right)^{y_{\alpha\alpha}}\left(\frac{z+3}{4}\right)^{y_{\alpha\omega}},$$

where the last summation extends over all the sets of non-negative integers $\mathfrak{y}=(y_{\alpha\alpha}, y_{\alpha\omega}, y_{\omega\omega})$ satisfying the relations

$$y_{\alpha\alpha}\leqq M_{\alpha\alpha}, \quad y_{\alpha\omega}\leqq M_{\alpha\omega}, \quad y_{\omega\omega}\leqq M_{\omega\omega};$$

$$y_{\alpha\alpha}+y_{\alpha\omega}+y_{\omega\omega}=F_{ij}.$$

The generating function is finally given by the constant term in the Laurent expansion around the origin of either of the expressions

$$\Phi(ij;ij|z|t_\alpha,t_\omega)=\frac{M_{\alpha\alpha}!\,M_{\alpha\omega}!\,M_{\omega\omega}!}{N!\,t_\alpha^{M_{\alpha\alpha}}\,t_\omega^{M_{\alpha\omega}}}$$

$$\cdot\left(t_\alpha\frac{z+1}{2}+t_\omega\frac{z+3}{4}+1\right)^{F_{ij}}(t_\alpha+t_\omega+1)^{N-F_{ij}}$$

and

$$\Phi(ij;ij|z|s;t_\alpha,t_\omega)=\frac{F_{ij}!\,(N-F_{ij})!\,M_{\alpha\alpha}!\,M_{\alpha\omega}!\,M_{\omega\omega}!}{N!^2\,s^{F_{ij}}\,t_\alpha^{M_{\alpha\alpha}}\,t_\omega^{M_{\alpha\omega}}}$$

$$\cdot\left\{\left(s\frac{z+1}{2}+1\right)\left(t_\alpha+\frac{t_\omega}{2}\right)+(s+1)\left(\frac{t_\omega}{2}+1\right)\right\}^N$$

regarded as a rational function of variables t_α and t_ω or s, t_α and t_ω, respectively; here, as defined above,

$$M_{\alpha\alpha}=M_{ii}+M_{ij}+M_{jj},\qquad M_{\alpha\omega}=\sum_{b\neq i,j}(M_{ib}+M_{jb}),$$

$$M_{\omega\omega}=\sum_{\substack{a,b\neq i,j\\ a\leq b}}M_{ab}=N-M_{\alpha\alpha}-M_{\alpha\omega}.$$

By the way, if we had introduced an imaginary gene A_ω alone, we would obtain an expression

$$\Phi(ij;ij|z|s;t_{ii},t_{ij},t_{jj},t_{i\omega},t_{j\omega})$$

$$=\frac{F_{ij}!\,(N-F_{ij})!\,M_{ii}!\,M_{ij}!\,M_{jj}!\,M_{i\omega}!\,M_{j\omega}!\,M_{\omega\omega}!}{N!^2 s^{F_{ij}}\,t_{ii}^{M_{ii}}\,t_{ij}^{M_{ij}}\,t_{jj}^{M_{jj}}\,t_{i\omega}^{M_{i\omega}}\,t_{j\omega}^{M_{j\omega}}}$$

$$\cdot\left\{\left(s\frac{z+1}{2}+1\right)\left(t_{ii}+t_{ij}+t_{jj}+\frac{t_{i\omega}+t_{j\omega}}{2}\right)+(s+1)\left(\frac{t_{i\omega}+t_{j\omega}}{2}+1\right)\right\}^N$$

which implies the generating function

$$\Phi(ij;ij|z)\equiv\Phi(ij;ij|z|F_{ij};M_{ii},M_{ij},M_{jj},M_{i\omega},M_{j\omega},M_{\omega\omega})$$

$$=\frac{M_{ii}!\,M_{ij}!\,M_{jj}!\,M_{i\omega}!\,M_{j\omega}!\,M_{\omega\omega}!}{N!}\sum_{\mathfrak{y}}\frac{F_{ij}!}{y_{ii}!\,y_{ij}!\,y_{jj}!\,y_{i\omega}!\,y_{j\omega}!\,y_{\omega\omega}!}$$

$$\cdot\frac{(N-F_{ij})!}{(M_{ii}-y_{ii})!\,(M_{ij}-y_{ij})!\,(M_{jj}-y_{jj})!\,(M_{i\omega}-y_{i\omega})!\,(M_{j\omega}-y_{j\omega})!\,(M_{\omega\omega}-y_{\omega\omega})!}$$

$$\cdot\left(\frac{z+1}{2}\right)^{y_{ii}+y_{ij}+y_{jj}}\left(\frac{z+3}{4}\right)^{y_{i\omega}+y_{j\omega}}$$

with a range of summation given by

$$y_{ii} \leqq M_{ii},\ y_{ij} \leqq M_{ij},\ y_{jj} \leqq M_{jj},\ y_{i\omega} \leqq M_{i\omega},\ y_{j\omega} \leqq M_{j\omega},\ y_{\omega\omega} \leqq M_{\omega\omega};$$

$$y_{ii}+y_{ij}+y_{jj}+y_{i\omega}+y_{j\omega}+y_{\omega\omega}=F_{ij},$$

as the constant term in its Laurent expansion around the origin, the relevant variables being s, t_{ii}, t_{ij}, t_{jj}, $t_{i\omega}$ and $t_{j\omega}$.

v. $(A_iA_j;\ A_iA_k)$; $i \neq j$; $k \neq i, j$.

For combination $(A_iA_j; A_iA_k)$, we introduce an imaginary gene A_ω by gathering $m-1$ genes $A_b\,(b \neq k)$, and put accordingly

$$\sum_{b\neq k} M_{kb}=M_{k\omega}, \qquad \sum_{\substack{a,b\neq k\\ a\leqq b}} M_{ab}=M_{\omega\omega}.$$

A male of genotype A_kA_k, A_kA_ω or $A_\omega A_\omega$ then produces, together with a female of type A_iA_j, a child of A_iA_k with probability 1/2, 1/4 or 0, respectively. Consequently, the circumstance is quite similar as in case of combination $(A_iA_j; A_iA_i)$ or $(A_iA_j; A_iA_j)$. We thus obtain for the generating function an expression

$$\Phi(ij;ik\,|\,z)\equiv\Phi(ij;ik\,|\,z\,|\,F_{ij};M_{kk},M_{k\omega},M_{\omega\omega})=\sum_{X=0}^{F_{ij}}\Psi(ij;ik\,|X)z^X$$

$$=\frac{M_{kk}!\,M_{k\omega}!\,M_{\omega\omega}!}{N!}\sum_{\mathfrak{y}}\frac{F_{ij}!}{y_{kk}!\,y_{k\omega}!\,y_{\omega\omega}!}\,\frac{(N-F_{ij})!}{(M_{kk}-y_{kk})!\,(M_{k\omega}-y_{k\omega})!\,(M_{\omega\omega}-y_{\omega\omega})!}$$

$$\cdot\left(\frac{z+1}{2}\right)^{y_{kk}}\left(\frac{z+3}{4}\right)^{y_{k\omega}},$$

the summation extending over all the sets of non-negative integers $\mathfrak{y}=(y_{kk}, y_{k\omega}, y_{\omega\omega})$ which satisfy the relations

$$y_{kk} \leqq M_{kk},\quad y_{k\omega} \leqq M_{k\omega},\quad y_{\omega\omega} \leqq M_{\omega\omega};$$

$$y_{kk}+y_{k\omega}+y_{\omega\omega}=F_{ij}.$$

The generating function is finally given by the constant term in the Laurent expansion around the origin of either of the expressions

$$\Phi(ij;ik\,|\,z\,|\,t_k,t_\omega)=\frac{M_{kk}!\,M_{k\omega}!\,M_{\omega\omega}!}{N!\,t_k^{M_{kk}}\,t_\omega^{M_{k\omega}}}$$

$$\cdot\left(t_k\frac{z+1}{2}+t_\omega\frac{z+3}{4}+1\right)^{F_{ij}}(t_k+t_\omega+1)^{N-F_{ij}}$$

and

$$\Phi(ij;ik\,|\,z\,|\,s;t_k,t_\omega)=\frac{F_{ij}!\,(N-F_{ij})!\,M_{kk}!\,M_{k\omega}!\,M_{\omega\omega}!}{N!^2\,s^{F_{ij}}\,t_k^{M_{kk}}\,t_\omega^{M_{k\omega}}}$$

$$\cdot\left\{\left(s\frac{z+1}{2}+1\right)\left(t_k+\frac{t_\omega}{2}\right)+(s+1)\left(\frac{t_\omega}{2}+1\right)\right\}^N$$

regarded as a rational function of variables t_k and t_ω or s, t_k and t_ω, respectively; here, as defined above,

$$M_{k\omega}=\sum_{b\neq k}M_{kb},\qquad M_{\omega\omega}=\sum_{\substack{a,b\neq k\\ a\leq b}}M_{ab}=N-M_{kk}-M_{k\omega}.$$

3. Means of stochastic variables.

We have established in the preceding section analytical expressions for the generating functions in all the possible mother-child combinations. By making use of them, the means of respective stochastic variables X's can be readily calculated. In fact, for a combination $(A_\alpha A_\beta;A_\xi A_\eta)$, *the mean of* $X=X(\alpha\beta;\xi\eta)$ is defined by

$$\tilde{X}(\alpha\beta;\xi\eta)=\sum_{X=1}^{F_{\alpha\beta}}X\Psi(\alpha\beta;\xi\eta\,|\,X)=\frac{d\Phi}{dz}(\alpha\beta;\xi\eta\,|\,1).$$

Actual calculation will be performed in the following lines.

i. $(A_iA_i;A_iA_i)$.

Differentiation of $\Phi(ii;ii|z|s;t_i,t_\omega)$ with respect to z leads to

$$\frac{d\Phi}{dz}(ii;ii\,|\,z\,|\,s;t_i,t_\omega)=\frac{F_{ii}!\,(N-F_{ii})!\,M_{ii}!\,M_{i\omega}!\,M_{\omega\omega}!}{N!^2\,s^{F_{ii}}\,t_i^{M_{ii}}\,t_\omega^{M_{i\omega}}}$$

$$\cdot N\left\{(sz+1)\left(t_i+\frac{t_\omega}{2}\right)+(s+1)\left(\frac{t_\omega}{2}+1\right)\right\}^{N-1}s\left(t_i+\frac{t_\omega}{2}\right),$$

whence follows, after putting $z=1$,

$$\frac{d\Phi}{dz}(ii;ii\,|\,1\,|\,s;t_i,t_\omega)=\frac{F_{ii}!\,(N-F_{ii})!\,M_{ii}!\,M_{i\omega}!\,M_{\omega\omega}!}{N!^2\,s^{F_{ii}}\,t_i^{M_{ii}}\,t_\omega^{M_{i\omega}}}$$

$$\cdot N(s+1)^{N-1}(t_i+t_\omega+1)^{N-1}s\left(t_i+\frac{t_\omega}{2}\right).$$

Consequently, by separating the constant term in the Laurent expansion around the origin of the last quantity regarded as a function of variables s, t_i and t_ω, we obtain

$$\tilde{X}(ii;ii)=\frac{d\Phi}{dz}(ii;ii\,|\,1)$$
$$=\frac{1}{N}F_{ii}\left(M_{ii}+\frac{M_{i\omega}}{2}\right)=\frac{1}{N}F_{ii}\left(M_{ii}+\sum_{b\neq i}\frac{M_{ib}}{2}\right).$$

ii. $(A_iA_i;\,A_iA_k)$; $k\neq i$.

Similarly as above, it follows

$$\frac{d\Phi}{dz}(ii;ik\,|\,1\,|\,s;t_k,t_\omega)=\frac{F_{ii}!\,(N-F_{ii})!\,M_{kk}!\,M_{k\omega}!\,M_{\omega\omega}!}{N!^2\,s^{F_{ii}}\,t_k^{M_{kk}}\,t_\omega^{M_{k\omega}}}$$
$$\cdot N(s+1)^{N-1}(t_k+t_\omega+1)^{N-1}s\left(t_k+\frac{t_\omega}{2}\right).$$

Consequently, we obtain

$$\tilde{X}(ii;ik)=\frac{d\Phi}{dz}(ii;ik\,|\,1)$$
$$=\frac{1}{N}F_{ii}\left(M_{kk}+\frac{M_{k\omega}}{2}\right)=\frac{1}{N}F_{ii}\left(M_{kk}+\sum_{b\neq k}\frac{M_{kb}}{2}\right).$$

iii. $(A_iA_j;\,A_iA_i)$; $i\neq j$.

We get in turn

$$\frac{d\Phi}{dz}(ij;ii\,|\,1\,|\,s;t_i,t_\omega)=\frac{F_{ij}!\,(N-F_{ij})!\,M_{ii}!\,M_{i\omega}!\,M_{\omega\omega}!}{N!^2\,s^{F_{ij}}\,t_i^{M_{ii}}\,t_\omega^{M_{i\omega}}}$$
$$\cdot N(s+1)^{N-1}(t_i+t_\omega+1)^{N-1}\frac{s}{2}\left(t_i+\frac{t_\omega}{2}\right),$$

$$\tilde{X}(ij;ii)=\frac{d\Phi}{dz}(ij;ii\,|\,1)$$
$$=\frac{1}{N}\frac{F_{ij}}{2}\left(M_{ii}+\frac{M_{i\omega}}{2}\right)=\frac{1}{N}\frac{F_{ij}}{2}\left(M_{ii}+\sum_{b\neq i}\frac{M_{ib}}{2}\right).$$

iv. $(A_iA_j;\ A_iA_j);\ i \neq j.$

We get

$$\frac{d\Phi}{dz}(ij;\, ij\,|\,1\,|\,s;\, t_\alpha, t_\beta) = \frac{F_{ij}!\,(N-F_{ij})!\,M_{\alpha\alpha}!\,M_{\alpha\omega}!\,M_{\omega\omega}!}{N!^2\, s^{F_{ij}}\, t_\alpha^{M_{\alpha\alpha}}\, t_\omega^{M_{\alpha\omega}}}$$

$$\cdot N(s+1)^{N-1}(t_\alpha+t_\omega+1)^{N-1}\frac{s}{2}\left(t_\alpha+\frac{t_\omega}{2}\right),$$

$$\widetilde{X}(ij;\, ij) = \frac{d\Phi}{dz}(ij;\, ij\,|\,1)$$

$$= \frac{1}{N}\frac{F_{ij}}{2}\left(M_{\alpha\alpha}+\frac{M_{\alpha\omega}}{2}\right)$$

$$= \frac{1}{N}\frac{F_{ij}}{2}\left(M_{ii}+\sum_{b\neq i}\frac{M_{ib}}{2}+M_{jj}+\sum_{b\neq j}\frac{M_{jb}}{2}\right).$$

v. $(A_iA_j;\ A_iA_k);\ i \neq j;\ k \neq i, j.$

We get

$$\frac{d\Phi}{dz}(ij;\, ik\,|\,1\,|\,s;\, t_k, t_\omega) = \frac{F_{ij}!\,(N-F_{ij})!\,M_{kk}!\,M_{k\omega}!\,M_{\omega\omega}!}{N!^2\, s^{F_{ij}}\, t_k^{M_{kk}}\, t_\omega^{M_{k\omega}}}$$

$$\cdot N(s+1)^{N-1}(t_k+t_\omega+1)^{N-1}\frac{s}{2}\left(t_k+\frac{t_\omega}{2}\right),$$

$$\widetilde{X}(ij;\, ik) = \frac{d\Phi}{dz}(ij;\, ik\,|\,1)$$

$$= \frac{1}{N}\frac{F_{ij}}{2}\left(M_{kk}+\frac{M_{k\omega}}{2}\right) = \frac{1}{N}\frac{F_{ij}}{2}\left(M_{kk}+\sum\frac{M_{kb}}{2}\right).$$

These results are quite plausible. In fact, the relative frequencies of the genes $A_i\ (i=1, \cdots, m)$ in male-population are given by

$$p_i^{(M)} = \frac{1}{N}\left(M_{ii}+\sum_{b\neq i}\frac{M_{ib}}{2}\right) \qquad (i=1, \cdots, m),$$

respectively. By making use of the quantities thus defined, the results derived above can be expressed in the form

$$\widetilde{X}(ii;\ ii) = F_{ii}p_i^{(M)}, \qquad \widetilde{X}(ii;\, ik) = F_{ii}p_k^{(M)},$$

$$\tilde{X}(ij;\ ii)=F_{ij}\frac{1}{2}p_i^{(M)},\qquad \tilde{X}(ij;ij)=F_{ij}\frac{1}{2}(p_i^{(M)}+p_j^{(M)}),$$

$$\tilde{X}(ij;ik)=F_{ij}\frac{1}{2}p_k^{(M)}\qquad (i\neq j;\ k\neq i,j).$$

If the male-population is, in particular, in an equilibrium state, the quantities p's now defined coincide, of course, with those used in § 1. Further, if the female-population is also in an equilibrium state, the relative frequencies of genotypes are expressed by

$$F_{bb}/N=p_b^{(F)2},\qquad F_{ab}/N=2p_a^{(F)}p_b^{(F)}\qquad (a,b=1,\cdots,m;\ a<b).$$

Accordingly, the means X's are then expressible in the form

$$\tilde{X}(ii;\ ii)=Np_i^{(F)2}p_i^{(M)},\qquad \tilde{X}(ii;ik)=Np_i^{(F)2}p_k^{(M)},$$

$$\tilde{X}(ij;\ ii)=Np_i^{(F)}p_j^{(F)}p_i^{(M)},\qquad \tilde{X}(ij;\ ij)=Np_i^{(F)}p_j^{(F)}(p_i^{(M)}+p_j^{(M)}),$$

$$\tilde{X}(ij;ik)=Np_i^{(F)}p_j^{(F)}p_k^{(M)},\qquad (i\neq j;\ k\neq i,j),$$

and consequently coincide identically with the π's multiplied by N:

$$\tilde{X}(\alpha\beta;\xi\eta)=N\pi(\alpha\beta;\xi\eta)\qquad (\alpha,\beta,\xi,\eta=1,\cdots,m).$$

4. Variances.

In order to express the quantities concerning the moments of higher orders in clearer forms, we again avail an abbreviated notation defined by

$$[A^n]^!=\frac{A!}{(A-n)!}=A(A-1)\cdots(A-n+1)\qquad (n=0,1,\cdots,A),$$

or more generally

$$[\Omega(A_1,\cdots,A_\omega)]^!=\sum_n a_{n_1\cdots n_\omega}\prod_{\kappa=1}^{\omega}[A_\kappa^{n_\kappa}]^!$$

for a polynomial of any number of arguments

$$\Omega(A_1,\cdots,A_\omega)=\sum_n a_{n_1\cdots n_\omega}\prod_{\kappa=1}^{\omega}A_\kappa^{n_\kappa}.$$

Now, it is readily shown that the differentiation of $\Phi(ii;ii|z|s;t_i,t_\omega)$ n times with respect to z leads to

$$\frac{d^n\Phi}{dz^n}(ii;\, ii\,|\,z\,|\,s;\, t_i,\, t_\omega)= \frac{F_{ii}!\,(N-F_{ii})!\,M_{ii}!\,M_{i\omega}!M_{\omega\omega}!}{N!^2\, s^{F_{ii}}\, t_i^{M_{ii}}\, t_\omega^{M_{i\omega}}}$$

$$\cdot\,[N^n]^!\left\{(sz+1)\left(t_i+\frac{t_\omega}{2}\right)+(s+1)\left(\frac{t_\omega}{2}+1\right)\right\}^{N-n}s^n\left(t_i+\frac{t_\omega}{2}\right)^n,$$

whence follows, after putting $z=1$,

$$\frac{d^n\Phi}{dz^n}(ii;\, ii\,|\,1\,|\,s;\, t_i,\, t_\omega)= \frac{F_{ii}!\,(N-F_{ii})!\,M_{ii}!\,M_{i\omega}!\,M_{\omega\omega}!}{N!^2\, s^{F_{ii}}\, t_i^{M_{ii}}\, t_\omega^{M_{i\omega}}}$$

$$\cdot\,[N^n]^!\,(s+1)^{N-n}(t_i+t_\omega+1)^{N-n}s^n\left(t_i+\frac{t_\omega}{2}\right)^n.$$

Consequently, by separating the constant term in the Laurent expansion, we get

$$\frac{d^n\Phi}{dz^n}(ii;\, ii\,|\,1) = \frac{1}{[N^n]^!}\,[(N\tilde{X}(ii;\, ii))^n]^!,$$

where the relevant variables in the bracket-notation $[\ \ ]^!$ for the second factor are F_{ii} and $M_{ib}\,(b=1, \cdots, m)$; as shown above, we have

$$N\tilde{X}(ii;\, ii)=F_{ii}\left(M_{ii}+\sum_{b\neq i}\frac{M_{ib}}{2}\right).$$

In quite a similar manner, we can conclude that there holds a general formula

$$\frac{d^n\Phi}{dz^n}(\alpha\beta;\, \xi\eta\,|\,1) = \frac{1}{[N^n]^!}\,[(N\tilde{X}(\alpha\beta;\, \xi\eta))^n]^!\,;$$

the relevant variables in the bracket-notation for the second factor are $F_{\alpha\beta}$ and the M_{ab} appearing essentially in the expression for $N\tilde{X}(\alpha\beta;\, \xi\eta)$.

By means of the last formula, any moment of an arbitrary order can be readily computed. As an illustration, we shall here deal with the variances.

Now, *the variance of* $X(\alpha\beta;\, \xi\eta)$ is given by

$$\operatorname{var} X(\alpha\beta;\, \xi\eta)= \sum_{X=0}^{F_{\alpha\beta}}(X-\tilde{X}(\alpha\beta;\, \xi\eta))^2\Psi(\alpha\beta;\, \xi\eta\,|\,X)$$

$$= \sum_{X=0}^{F_{\alpha\beta}}([X^2]^!+X)\Psi(\alpha\beta;\, \xi\eta\,|\,X)-\tilde{X}(\alpha\beta;\, \xi\eta)^2$$

$$= \frac{d^2\Phi}{d^2z}(\alpha\beta;\xi\eta\,|\,1) + \tilde{X}(\alpha\beta;\xi\eta) - \tilde{X}(\alpha\beta;\xi\eta)^2.$$

Since the value of $\tilde{X}(\alpha\beta;\xi\eta)$ has been already determined, it is only necessary to substitute the value of the derivative of the second order which follows readily from a general formula established above. For the sake of completeness, we shall set out below the values of variance, after classified according to several distinguished combinations, in more concrete forms.

In general, if the relevant variables in the bracket-notation are $Y_{ab}\,(a, b=1, \cdots, m;\, a \leqq b;\, Y_{ab}=Y_{ba})$, then we get

$$\left[\left(Y_{ii}+\sum_{b\neq i}\frac{Y_{ib}}{2}\right)^2\right]^! = \left(Y_{ii}+\sum_{b\neq i}\frac{Y_{ib}}{2}\right)^2 - \left(Y_{ii}+\sum_{b\neq i}\frac{Y_{ib}}{4}\right),$$

$$\left[\left(Y_{ii}+\sum_{b\neq i}\frac{Y_{ib}}{2}\right)\left(Y_{jj}+\sum_{b\neq j}\frac{Y_{jb}}{2}\right)\right]^!$$

$$= \left(Y_{ii}+\sum_{b\neq i}\frac{Y_{ib}}{2}\right)\left(Y_{jj}+\sum_{b\neq j}\frac{Y_{jb}}{2}\right) - \frac{Y_{ij}}{4} \qquad (i \neq j).$$

Hence, introducing, as before, notations defined by

$$p_i^{(M)} = \frac{1}{N}\left(M_{ii}+\sum_{b\neq i}\frac{M_{ib}}{2}\right) \qquad (i=1, \cdots, m),$$

we obtain the following results:

$$\operatorname{var} X(ii;ii) = \frac{F_{ii}(F_{ii}-1)}{N(N-1)}\left(N^2 p_i^{(M)2} - \frac{Np_i^{(M)}+M_{ii}}{2}\right) + F_{ii}p_i^{(M)} - F_{ii}^2 p_i^{(M)2}$$

$$= \frac{NF_{ii}}{N-1}\left\{p_i^{(M)}(1-p_i^{(M)})\left(1-\frac{F_{ii}}{N}\right) + \frac{1}{2}\left(p_i^{(M)} - \frac{M_{ii}}{N}\right)\frac{F_{ii}-1}{N}\right\},$$

$$\operatorname{var} X(ii;ik) = \frac{F_{ii}(F_{ii}-1)}{N(N-1)}\left(N^2 p_k^{(M)2} - \frac{Np_k^{(M)}+M_{kk}}{2}\right) + F_{ii}p_k^{(M)} - F_{ii}^2 p_k^{(M)2}$$

$$= \frac{NF_{ii}}{N-1}\left\{p_k^{(M)}(1-p_k^{(M)})\left(1-\frac{F_{ii}}{N}\right) + \frac{1}{2}\left(p_k^{(M)} - \frac{M_{kk}}{N}\right)\frac{F_{ii}-1}{N}\right\}$$

$$(k \neq i),$$

$$\operatorname{var} X(ij;ii) = \frac{F_{ij}(F_{ij}-1)}{N(N-1)}\left(N^2\frac{p_i^{(M)2}}{4} - \frac{Np_i^{(M)}+M_{ii}}{8}\right) + F_{ij}\frac{p_i^{(M)}}{2} - F_{ij}^2\frac{p_i^{(M)2}}{4}$$

$$= \frac{NF_{ij}}{N-1}\left\{\frac{1}{4}p_i^{(M)}(2-p_i^{(M)})\left(1-\frac{F_{ij}}{N}\right) + \frac{1}{8}\left(3p_i^{(M)}-\frac{M_{ii}}{N}\right)\frac{F_{ij}-1}{N}\right\} \qquad (i\neq j),$$

$$\operatorname{var} X(ij;ij) = \frac{F_{ij}(F_{ij}-1)}{N(N-1)}\left(N^2\frac{p_i^{(M)2}}{4} - \frac{Np_i^{(M)}+M_{ii}}{8} + N^2\frac{p_j^{(M)2}}{4} - \frac{Np_j^{(M)}+M_{jj}}{8} + N^2\frac{p_i^{(M)}p_j^{(M)}}{2} - \frac{M_{ij}}{8}\right) + F_{ij}\frac{p_i^{(M)}+p_j^{(M)}}{2} - F_{ij}^2\frac{(p_i^{(M)}+p_j^{(M)})^2}{4}$$

$$= \frac{NF_{ij}}{N-1}\left\{\frac{1}{4}(p_i^{(M)}+p_j^{(M)})(2-p_i^{(M)}-p_j^{(M)})\left(1-\frac{F_{ij}}{N}\right) + \frac{1}{8}\left(3(p_i^{(M)}+p_j^{(M)})-\frac{M_{ii}+M_{ij}+M_{jj}}{N}\right)\frac{F_{ij}-1}{N}\right\} \qquad (i\neq j),$$

$$\operatorname{var} X(ij;ik) = \frac{F_{ij}(F_{ij}-1)}{N(N-1)}\left(N^2\frac{p_k^{(M)2}}{4} - \frac{Np_k^{(M)}+M_{kk}}{8}\right) + F_{ij}\frac{p_k^{(M)}}{2} - F_{ij}^2\frac{p_k^{(M)2}}{4}$$

$$= \frac{NF_{ij}}{N-1}\left\{\frac{1}{4}p_k^{(M)}(2-p_k^{(M)})\left(1-\frac{F_{ij}}{N}\right) + \frac{1}{8}\left(3p_k^{(M)}-\frac{M_{kk}}{N}\right)\frac{F_{ij}-1}{N}\right\} \qquad (i\neq j;\ k\neq i,j).$$

If, in particular, the original distributions $\mathfrak{F}$ and $\mathfrak{M}$ show the same *equilibrium state*, i. e. when there hold

$$F_{bb}=M_{bb}=Np_b^2, \quad F_{ab}=M_{ab}=N2p_ap_b \qquad (a, b=1, \cdots, m;\ a<b),$$

the expressions can then be reduced to fairly simple forms:

$$\operatorname{var} X(ii;ii) = \frac{N^2}{N-1} p_i^3(1-p_i)\left(1-\frac{p_i^2}{2}-\frac{1}{2N}\right),$$

$$\operatorname{var} X(ii;ik) = \frac{N^2}{N-1} p_i^2 p_k(1-p_k)\left(1-\frac{p_i^2}{2}-\frac{1}{2N}\right) \qquad (k\neq i),$$

$$\operatorname{var} X(ij;ii) = \frac{N^2}{N-1} p_i^2 p_j\left(1-\frac{p_i}{2}-p_ip_j\frac{1-p_i}{2}-\frac{3-p_i}{2N}\right) \qquad (i\neq j),$$

$$\operatorname{var} X(ij;ij) = \frac{N^2}{N-1} p_ip_j(p_i+p_j)\left(1-\frac{p_i+p_j}{2}-p_ip_j\frac{1-p_i-p_j}{2}-\frac{3-p_i-p_j}{2N}\right) \qquad (i\neq j),$$

$$\operatorname{var} X(ij;ik) = \frac{N^2}{N-1} p_ip_jp_k\left(1-\frac{p_k}{2}-p_ip_j\frac{1-p_k}{2}-\frac{3-p_k}{2N}\right) \qquad (i\neq j;\ k\neq i,j).$$

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