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The boundary distortion on conformal mapping.

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CONTENTS

1. Main Theorems.

1. Let D de a domain on the $w=\xi+i\eta$-plane, which is bounded by a Jordan curve C, which passes through $w=0$ and touches the real axis at $w=0$ and its inner normal at $w=0$ coincides with the positive η-axis. We map D conformally on the upper half $\Im z>0$ of the $z=x+iy$-plane by $w=w(z)$, $w(0)=0$. There are many researches concerning the existence of $\lim_{z\to 0}\frac{w(z)}{z}$. Among others, we state the following theorems.

THEOREM 1. (*Carathéodory*)[1]. *If there are two circles K_1, K_2, which touch the real axis at $w=0$, such that K_1 lies in D and K_2 lies outside of D, then*

$$\lim_{z\to 0}\frac{w(z)}{z}=\lim_{z\to 0} w'(z)=\gamma\,, \qquad 0<\gamma<\infty\,,$$

exists uniformly, when $z\to 0$ in any Stolz domain, whose vertex is at $z=0$.

THEOREM 2. (*Besonoff-Lavrentieff*)[2]. *If in a neighbourhood of $w=0$,* (i) *C lies between two curves:*

$$H:\ \eta=|\xi|^{1+\alpha} \quad \textit{and} \quad \overline{H}:\ \eta=-|\xi|^{1+\alpha} \quad (0<\alpha<1)$$

1) C. Carathéodory: Über die Winkelderivierte von beschränkten analytischen Funktionen. Sitzber. der Berl. Akad. 1929.

2) P. Besonoff et M. Lavrentieff: Sur l'existence de la derivée limite. Bull. Soc. Math. 58 (1930).

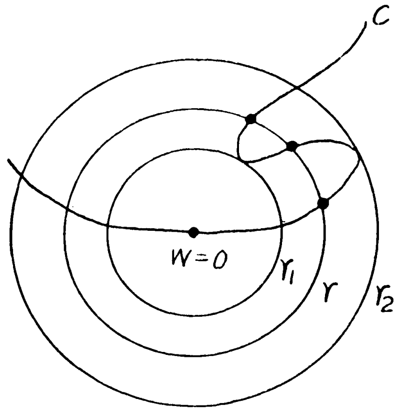
and (ii) *C is rectifiable and is represented by* $w=w(s)=\xi(s)+i\eta(s)$, *where* s *is the arc length, measured from* $w=0$, *such that*

$$\lim_{s\to 0}\frac{\xi(s)}{s}=1,$$

then $\lim_{z\to 0}\frac{w(z)}{z}=\gamma\ (0<\gamma<\infty)$ *exists, when* $z\to 0$ *from the inside of* $\Im z\geqq 0$ *and* $\lim_{z\to 0} w'(z)=\gamma$ *uniformly, when* $z\to 0$ *in any Stolz domain, whose vertex is at* $z=0$.

2. In this paper, we shall prove a theorem, which contains the above two theorems as special cases.

Let C be represented by a parameter t: $w=w(t)=\xi(t)+i\eta(t)$, $w(0)=0$ $(|t|\leqq 1)$, such that for a small $\delta>0$, $0<t\leqq\delta$ corresponds to the part of C, which lies on the right of the imaginary axis. Let C meet the circle $|w|=r$ $(r>0)$ and $M(r)$ be the set of t $(0<t\leqq\delta)$, such that $|w(t)|=r$ and put



$$\underline{t}=\underline{t}(r)=\inf_{t\in M(r)} t,\qquad \bar{t}=\bar{t}(r)=\sup_{t\in M(r)} t,$$

$$r_1=r_1(r)=\operatorname*{Min}_{\underline{t}\leqq t\leqq\bar{t}}|w(t)|,\qquad r_2=r_2(r)=\operatorname*{Max}_{\underline{t}\leqq t\leqq\bar{t}}|w(t)|,$$

$$r_1(r)\leqq r\leqq r_2(r).$$

If C satisfies the condition:

$$\lim_{r\to 0}\frac{r_1(r)}{r}=1,\qquad \lim_{r\to 0}\frac{r_2(r)}{r}=1 \tag{2}$$

and the similar relation on the left of the imaginary axis, then we say that C satisfies *the condition* (W) *at* $w=0$, since it is first introduced by Warschawski[3].

Similarly we define the condition (W^*) as follows.

Let L: $\Re w=\text{const.}=\xi$ (>0) be a line parallel to the imaginary axis and $M(\xi)$ be the set of t $(0<t\leqq\delta)$, such that $\Re w(t)=\xi$ and put

3) S. Warschawski: Über die Randverhalten der Ableitung der Abbildungsfunktion bei konformer Abbildung. Math. Zeits. 35 (1932).

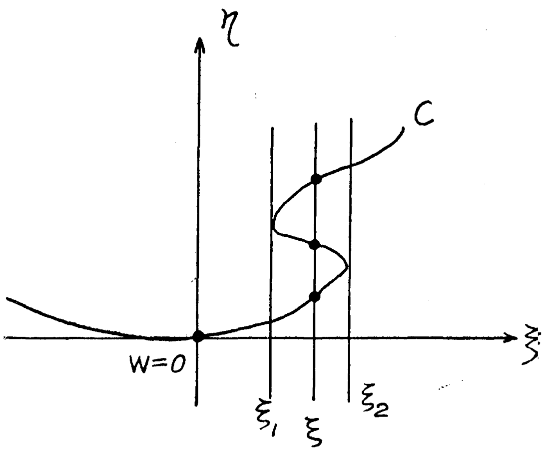
$$\underline{t}=\underline{t}(\xi)=\inf_{t\in M(\xi)} t,$$

$$\bar{t}=\bar{t}(\xi)=\sup_{t\in M(\xi)}. t,$$

$$\xi_1=\xi_1(\xi)=\operatorname*{Min}_{\underline{t}\leq t\leq \bar{t}} \Re w(t),$$

$$\xi_2=\xi_2(\xi)=\operatorname*{Max}_{\underline{t}\leq t\leq \bar{t}} \Re w(t),$$

$$\xi_1(\xi)\leqq\xi\leqq\xi_2(\xi). \qquad (1^*)$$



If C satisfies the condition:

$$\lim_{\xi\to 0}\frac{\xi_1(\xi)}{\xi}=1, \qquad \lim_{\xi\to 0}\frac{\xi_2(\xi)}{\xi}=1 \qquad (2^*)$$

and the similar relation on the left of the imaginary axis, then we say that C satisfies *the condition* (W^*) *at* $w=0$.

Now we shall state our main theorems.

THEOREM 3. (i) *If in a neighbourhood of* $w=0$, C *lies between two curves* H *and* $\overline{H}$, *each of which is symmetric to the imaginary axis and whose parts on the right of the imaginary axis are*

$$H:\ \eta=h(\xi) \quad and \quad \overline{H}:\ \eta=-h(\xi) \quad (0\leqq\xi\leqq\delta), \quad h(0)=0,$$

where $h(t)>0$ *is a continuous increasing function of* $t>0$, *such that*

$$\int_0^\delta \frac{h(t)}{t^2}\,dt<\infty,$$

then

$$\lim_{z\to 0}\frac{w(z)}{z}=\lim_{z\to 0} w'(z)=\gamma, \qquad 0<\gamma<\infty,$$

exists uniformly, when $z\to 0$ *in any Stolz domain, whose vertex is at* $z=0$. (ii) *If* C *lies between the above two curves and further satisfies the condition* (W) *or the condition* (W^*) *at* $w=0$, *then*

$$\lim_{z\to 0}\frac{w(z)}{z}=\gamma, \qquad 0<\gamma<\infty,$$

exists, when $z \to 0$ *from the inside of* $\Im z \geqq 0$.

Theorem 1 is a special case of (i) and Theorem 2 is that of (ii). (ii) is due to Warschawski[3], though under a different enunciation. Warschawski's proof is very complicated. His fundamental lemma is proved simply by Wolf[4], under the hypothesis that D lies on the upper half-plane. By modifying his method, and by means of Green's functions, we shall prove our theorem simply.

The condition $\int_0^{\delta} \frac{h(t)}{t^2}\, dt < \infty$ is essential, since the following theorem holds.

THEOREM 4. *Let D be a domain on the $w=\xi+i\eta$-plane, which is bounded by a Jordan curve C, which passes through $w=0$ and touches the real axis at $w=0$ and its inner normal at $w=0$ coincides with the positive η-axis. We suppose that in a neighbourhood of $w=0$, C is represented by one of two forms:*

$$\text{(i)}\quad \eta = h(\xi), \qquad \textit{or} \qquad \text{(ii)}\quad \eta = -h(\xi) \qquad (|\xi| \leqq \delta), \quad h(0)=0,$$

where $h(t)>0$ is a continuous function of t ($|t| \leqq \delta$), which is decreasing for $t<0$ and is increasing for $t>0$. If we map D conformally on $\Im z>0$ by $w=w(z)$, $w(0)=0$, then

$$\lim_{z\to 0} \frac{w(z)}{z} = \gamma$$

exists, when $z \to 0$ from the inside of $\Im z \geqq 0$, where

$$0<\gamma \leqq \infty \quad \textit{in case (i) and} \quad 0 \leqq \gamma < \infty \quad \textit{in case (ii)}.$$

In each case, the necessary and sufficient condition that $0<\gamma<\infty$ is

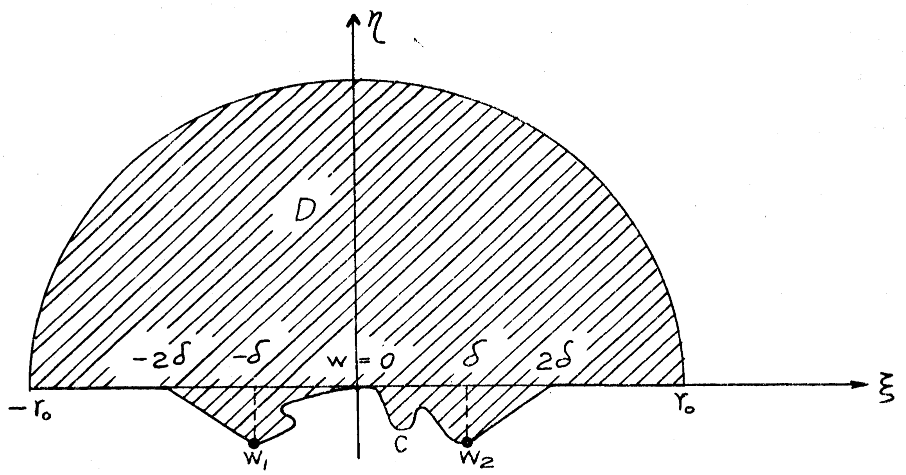
$$\int_{-\delta}^{\delta} \frac{h(t)}{t^2}\, dt < \infty .$$

We remark the following.

Let D_1, D_2 be two domains, which have a common boundary in a neighbourhood of $w=0$. If Theorem 3 holds for D_1, then it holds for D_2. Hence we may assume that D is the following special domain. Let C be represented by a parameter t as before: $w=w(t)$ ($|t| \leqq 1$). If

4) J. Wolf: Sur la représentation conforme des bandes. Compositio Math. 1 (1935).

we make t increase from $t=0$, then C meets the line $\xi=\delta$ $(\delta>0)$ at first at w_2, so that the arc $\widehat{0w_2}$ lies between two lines $\xi=0$, $\xi=\delta$. Similarly we define w_1 on the left of the imaginary axis, such that the arc $\widehat{w_1 0}$ lies between two lines $\xi=-\delta$, $\xi=0$. By the above remark, we assume that the boundary of D consists of the following lines.



(i) the arc $\widehat{w_1 0 w_2}$, (ii) a rectilinear segment, which connects w_2 to $w=2\delta$, (iii) a segment on the real axis $2\delta\leqq\xi\leqq r_0$, (iv) a semi-circle $w=r_0e^{i\theta}$ $(0\leqq\theta\leqq\pi)$, (v) a segment on the real axis $-r_0\leqq\xi\leqq-2\delta$, (vi) a rectilinear segment, which connects $w=-2\delta$ to w_1.

2. Some lemmas.

First we shall prove some lemmas. In this paper, $K_{\rho_0}(\varphi_0)$ denotes a sector, which is bounded by a circle of radius ρ_0 about the origin 0 and two lines through 0, each of which makes an angle φ_0 $\left(<\frac{\pi}{2}\right)$ with the positive imaginary axis.

LEMMA 1. *Under the condition* (i) *of Theorem 3,*

$$0<A|z|\leqq|w(z)|\leqq B|z|, \qquad z\in K_{\rho_0}(\varphi_0),$$

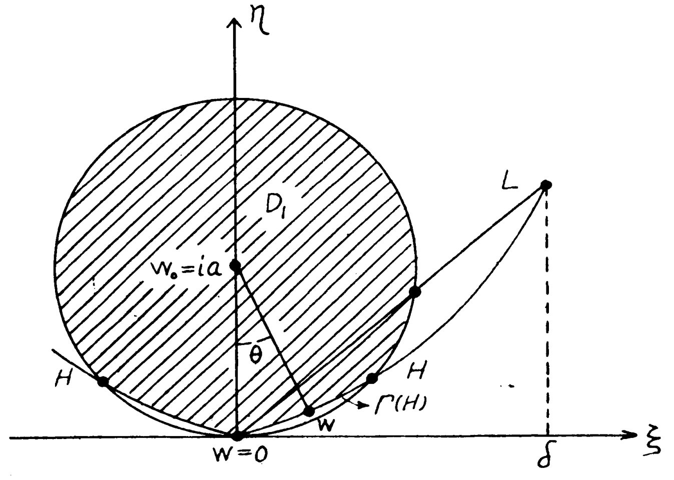
where $A>0$, $B>0$ *are constants.*

PROOF. (i) *Proof of* $|w(z)|\leqq B|z|$, $z\in K_{\rho_0}(\varphi_0)$, *if the part of* C, *which lies in a neighbourhood of* $w=0$, *lies below the curve* H.

We take a $(0<a<\delta)$ so small that $w_0=ia\in D$ and K: $|z-ia|=a$ be a circle. Let D_1 be the common part of the inside of K and the domain defined by $\eta\geqq h(\xi)$, which lies above the curve H and let Γ be its boundary. Then $D_1\subset D$. If H has no points in K, then we may take K instead of H, hence we assume that H has points in K and let $\Gamma(H)$ be the part of Γ, which belongs to H. Let L: $\arg w=\varepsilon$, $\varepsilon=\tan^{-1}(h(\delta)/\delta)$, be a line, which connects $w=0$ to $w=\delta+ih(\delta)$ and let (ξ_0, η_0) be its intersection with K, then $\xi_0=a\sin 2\varepsilon$. We take a so small that $\Gamma(H)$ lies below L. Let $G_{D_1}(w, ia)$ be the Green's function of D_1, with ia as its pole, then

$$G_{D_1}(w, ia)=\log\frac{a}{r}-v(w), \qquad r=|w-ia|, \tag{1}$$

where $v(w)$ is harmonic in D_1, such that $v=\log\frac{a}{r}$ on Γ.



Since $\Gamma(H)$ lies below L, if $w\in\Gamma(H)$, then $r\geqq a\cos\varepsilon$. Since $\int_0^\delta\frac{h(t)dt}{t^2}<\infty$ and $h(t)$ is increasing, $\lim_{t\to 0}\frac{h(t)}{t}=0$, so that we take δ so small that $\cos\varepsilon\geqq 1/2$, then

$$v(w)=\log\frac{a}{r}=\log\left(1+\frac{a-r}{r}\right)\leqq\frac{a-r}{r}\leqq\frac{a-r}{a\cos\varepsilon}\leqq 2\frac{a-r}{a}$$

$$\text{on } \Gamma(H). \tag{2}$$

Let $w=\xi+i\eta\in\Gamma(H)$ and θ be the angle between the vector $\overrightarrow{w_0w}$ $(w_0=ia)$ and the negative η-axis, then $\xi=r\sin\theta$, $\eta=a-r\cos\theta$, so that $a-r\cos\theta=h(r\sin\theta)$, $a-r\leqq h(\xi)$, hence by (2),

$$v(w)\leqq\frac{2h(\xi)}{a}\qquad\text{on}\quad\Gamma(H)\,. \tag{3}$$

We consider

$$u(w)=\frac{1}{\pi}\int_{-a}^{a}h(t)\frac{\eta dt}{\eta^2+(\xi-t)^2}\,,\quad w=\xi+i\eta\,. \tag{4}$$

Then $u(w)$ is harmonic in $\Im w>0$. Let $w_1=\xi_1+i\eta_1\in\Gamma(H)$, $\xi_1>0$, then since $\xi_1\leqq\xi_0=a\sin 2\epsilon$, $\eta_1=h(\xi_1)$, we have $\xi_1+\eta_1<a$, if δ is small, so that

$$u(w_1)\geqq\frac{1}{\pi}\int_{\xi_1}^{\xi_1+\eta_1}\frac{h(t)\eta_1 dt}{\eta_1^2+(\xi_1-t)^2}\geqq\frac{1}{\pi}\int_{\xi_1}^{\xi_1+\eta_1}\frac{h(t)\eta_1 dt}{\eta_1^2+\eta_1^2}=\frac{1}{2\pi\eta_1}\int_{\xi_1}^{\xi_1+\eta_1}h(t)dt$$

$$\geqq\frac{h(\xi_1)}{2\pi}\,,$$

so that by (3),

$$u(w)\geqq\frac{a}{4\pi}v(w)\qquad\text{on}\quad\Gamma(H)\,.$$

A similar relation holds on the left of the imaginary axis. Since $v=0$ on $\Gamma-\Gamma(H)$ and $u>0$, we have $u\geqq\frac{a}{4\pi}v$ on Γ, so that by the maximum principle,

$$u(w)\geqq\frac{a}{4\pi}v(w)\qquad\text{in}\quad D_1\,. \tag{5}$$

Let $w=\xi+i\eta=\rho e^{i(\frac{\pi}{2}-\varphi)}\in K_{\rho_0}(\varphi_0)$, $\eta=\rho\cos\varphi$, then by (4),

$$u(w)=\frac{\rho\cos\varphi}{\pi}\int_{-a}^{a}\frac{h(t)dt}{t^2-2t\rho\sin\varphi+\rho^2}\,.$$

Since $t^2\cos^2\varphi\leqq t^2-2t\rho\sin\varphi+\rho^2$, we have

$$u(w)\leqq\frac{\rho}{\pi\cos\varphi}\int_{-a}^{a}\frac{h(t)dt}{t^2}\leqq\frac{2\rho}{\pi\cos\varphi_0}\int_{0}^{a}\frac{h(t)dt}{t^2}\,,$$

so that by (5),

$$v(w) \leqq \frac{8\rho}{a\cos\varphi_0}\int_0^a \frac{h(t)dt}{t^2}, \qquad w \in K_{\rho_0}(\varphi_0). \tag{6}$$

Let $w \in K_{\rho_0}(\varphi_0)$ and $|w|=\rho$ is small and $r=|w-ia|$, then

$$\log\frac{a}{r} = \log\left(1+\frac{a-r}{r}\right) \geqq \text{const.}\,\frac{a-r}{a} \geqq \text{const.}\,\frac{\rho}{a},$$

so that if a is small,

$$G_{D_1}(w, ia) = \log\frac{a}{r} - v(w) \geqq \frac{\rho}{a}\left(\text{const.} - \frac{8}{\cos\varphi_0}\int_0^a \frac{h(t)dt}{t^2}\right)$$

$$\geqq \text{const.}\,\frac{\rho}{a}.$$

Hence

$$G_{D_1}(w, ia) \geqq \text{const.}\,|w|, \qquad w \in K_{\rho_0}(\varphi_0). \tag{7}$$

Let $G_D(w, ia)$ be the Green's function of D, then since $D_1 \subset D$,

$$G_D(w, ia) \geqq G_{D_1}(w, ia) \geqq \text{const.}\,|w|, \qquad w \in K_{\rho_0}(\varphi_0). \tag{8}$$

Let by $w=w(z)$, $w_0=ia$ correspond to z_0, then

$$G_D(w, ia) = \log\left|\frac{z-\bar{z}_0}{z-z_0}\right| \leqq \text{const.}\,|z|,$$

so that by (8),

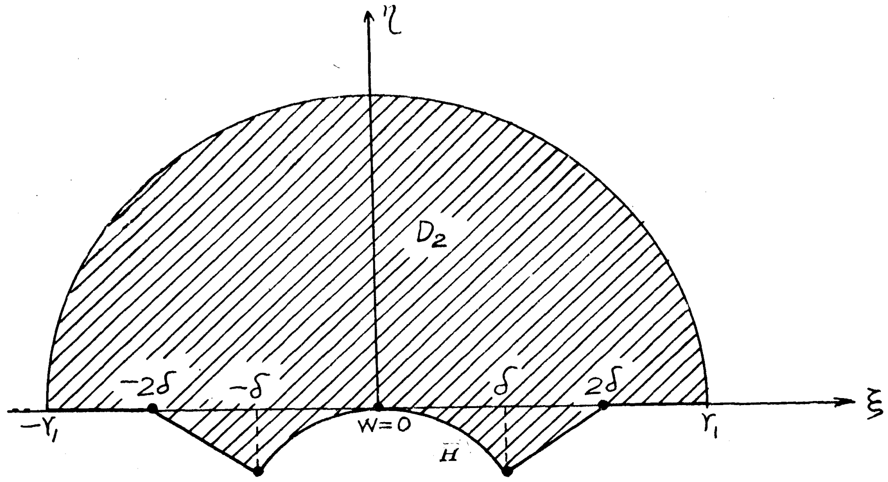
$$|w(z)| \leqq B|z|, \qquad z \in K_{\rho_0}(\varphi_0), \tag{9}$$

where $B>0$ is a constant.

(ii) *Proof of* $0<A|z| \leqq |w(z)|$, $z \in K_{\rho_0}(\varphi_0)$, *if the part of* C, *which lies in a neighbourhood of* $w=0$, *lies above the curve* $\overline{H}$.

We consider a domain $D_2 \supset D$, which is symmetric to the imaginary axis and whose boundary C_2 consists of the following lines.

(i) The curve $\overline{H}$: $\eta=-h(\xi)$ $(0 \leqq \xi \leqq \delta)$, (ii) a rectilinear segment, which connects $w=\delta-ih(\delta)$ to $w=2\delta$, (iii) a segment on the real axis $2\delta \leqq \xi \leqq r_1$ $(r_0<r_1)$, where r_0 is defined before, (iv) a part of the circle $w=r_1e^{i\theta}$ $(0 \leqq \theta \leqq \pi/2)$. By symmetry, we define the part of C_2 on the left of the imaginary axis. We map D_2 conformally on $\Im z>0$ by



$w=w_2(z)$, $w_2(0)=0$, such that three points $w=0$, $w=r_1e^{i\frac{\pi}{4}}=\frac{1+i}{\sqrt{2}}r_1$, $w=ir_1$ correspond to $z=0$, $z=1$, $z=\infty$ respectively, and let $w=\delta-ih(\delta)$, $w=2\delta$ correspond to $z=\alpha$, $z=\beta$ $(0<\alpha<\beta<1)$ respectively.

Let $\Delta\subset D_2$ be a half-disc: $|w|\leqq r_1$, $\Im w\geqq 0$ and $G_\Delta(w, ia)$ $(0<a<r_1)$ be its Green's function, then $G_\Delta(w, ia)\geqq$ const. $|w|$ in $K_{\rho_0}(\varphi_0)$.

Since $\Delta\subset D_2$, $G_{D_2}(w, ia)\geqq G_\Delta(w, ia)$ and as before, $G_{D_2}(w, ia)\leqq$ const. $|z|$, so that

$$|w_2(z)|\leqq K|z|, \qquad z\in K_{\rho_0}(\varphi_0), \tag{1}$$

where $K>0$ is a constant, which is independent of δ, as seen from the proof. This is important in the sequel.

Let $z_0=r$ $(0<r\leqq\alpha)$ correspond to $w_0=w_2(z_0)\in \overline{H}$. Now D_2 is contained in an angular domain: $\left|\arg\left(\frac{w}{i}\right)\right|\leqq\psi_0\left(\psi_0=\frac{\pi}{2}+\varepsilon\right)$, where $\varepsilon\to 0$ with $\delta\to 0$. We map D_2 on a domain contained in $|\zeta|<1$ by

$$v=i\left(\frac{w}{i}\right)^{\frac{\pi}{2\psi_0}}, \qquad v_0=i\left(\frac{w_0}{i}\right)^{\frac{\pi}{2\psi_0}},$$

$$u=\frac{v}{|v_0|}, \qquad u_0=\frac{v_0}{|v_0|}, \tag{2}$$

$$\zeta=\frac{u-i}{u+i}$$

and put

$$\zeta = \frac{u-i}{u+i} = \zeta(z). \tag{3}$$

Then $u=1,\ i,\ -1$ correspond to $\zeta=-i, 0, i$ respectively and the semi-circle $|u|=1$, $\Im u \geqq 0$ is mapped on the diameter of $|\zeta|=1$ through $i,\ -i$.

Let

$$A(t) = \int_0^t \int_0^\pi |\zeta'(z_0+te^{i\theta})|^2 t\,dt\,d\theta, \qquad z_0 = r, \tag{4}$$

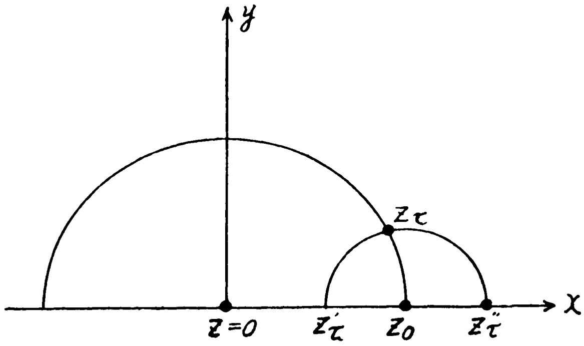
$$L(t) = \int_0^\pi |\zeta'(z_0+te^{i\theta})|\, t\,d\theta,$$

then since $A(t)$ is the area of a domain in $|\zeta|<1$, $A(t)\leqq\pi$ and by Schwarz's inequality, we have for $0<\rho<1$

$$\int_{\rho^2 r}^{\rho r} \frac{(L(t))^2}{t}\,dt \leqq \int_0^{\rho r} \frac{(L(t))^2}{t}\,dt \leqq \pi A(\rho r) \leqq \pi^2.$$

Hence there exists τ $(\rho^2 r \leqq \tau \leqq \rho r)$, such that $(L(\tau))^2 \log 1/\rho \leqq \pi^2$, so that if we take ρ small, then

$$L(\tau) < \varepsilon < \frac{1}{6}. \tag{5}$$



Let z_τ be the common point of two circles $|z|=r$ and $|z-z_0|=\tau$ and $z'_\tau = z_0-\tau$, $z''_\tau = z_0+\tau$ and $\widehat{z'_\tau z''_\tau}$ be the semi-circle: $|z-z_0|=\tau$, $\Im z \geqq 0$ and w_τ, w'_τ, w''_τ, $\widehat{w'_\tau w''_\tau}$ be the image of z_τ, z'_τ, z''_τ, $\widehat{z'_\tau z''_\tau}$ on the w-plane respectively and similary we define v_τ, v'_τ, v''_τ, $\widehat{v'_\tau v''_\tau}$ etc. Since $\widehat{u'_\tau u''_\tau}$ meets $|u|=1$ at u_0, $\widehat{\zeta'_\tau \zeta''_\tau}$ meets the imaginary axis and since by (5), its length is $<\varepsilon$, $\widehat{\zeta'_\tau \zeta''_\tau}$ lies outside of $|\zeta-1|=1-\varepsilon$. Since $u = i\,\dfrac{1+\zeta}{1-\zeta}$, we have for any z on $\widehat{z'_\tau z''_\tau}$,

$$\left|\frac{du}{dz}\right| = \frac{2}{|1-\zeta|^2}\left|\frac{d\zeta}{dz}\right| \leqq \frac{2}{(1-\varepsilon)^2}\left|\frac{d\zeta}{dz}\right|,$$

so that by (5),

$$\int_0^{\pi}|u'(z_0+\tau e^{i\theta})|\tau d\theta \leqq \frac{2}{(1-\varepsilon)^2}\int_0^{\pi}|\zeta'(z_0+\tau e^{i\theta})|\tau d\theta < \frac{2\varepsilon}{(1-\varepsilon)^2} \leqq \frac{1}{2}.$$

Since $\widehat{u'_\tau u''_\tau}$ meets $|u|=1$ and its length is $\leqq 1/2$, the image of the half-disc $\varDelta_\tau$: $|z-z_0|\leqq\tau$, $\Im z\geqq 0$ on the u-plane is contained in a ring domain: $\frac{1}{2}\leqq|u|\leqq\frac{3}{2}$, so that $\frac{|v_0|}{2}\leqq|v|\leqq\frac{3|v_0|}{2}$, hence

$$0<A\,|w_0|\leqq|w|\leqq B\,|w_0|, \qquad z\in\varDelta_\tau, \tag{6}$$

where $A=(1/2)^{\frac{2\psi_0}{\pi}}$, $B=(3/2)^{\frac{2\psi_0}{\pi}}$.

Hence especially,

$$A\,|w_0|\leqq|w_\tau|, \qquad w_\tau=w_2(z_\tau). \tag{7}$$

Since $\arg z_\tau\geqq 2\sin^{-1}(\rho^2/2)$, we have by (1), $|w_\tau|\leqq$ const. $|z_\tau|=$const. $|z_0|$, so that by (7)

$$|w_0|=|w_2(z_0)|\leqq K\,|z_0|, \tag{8}$$

where K is a constant, independent of δ.

Now for any $\varepsilon>0$, we take $\delta>0$ so small that

$$\int_0^{\delta}\frac{dh(\xi)}{\xi}=\frac{h(\delta)}{\delta}+\int_0^{\delta}\frac{h(\xi)}{\xi^2}\,d\xi<\varepsilon. \tag{9}$$

Let $0<x\leqq\alpha$ and $w_2(x)=\xi_2(x)+i\eta_2(x)=\xi+i\eta=\xi-ih(\xi)$. We put $h(\xi)=h^*(x)=-\eta_2(x)$, then since by (8), $0<\xi\leqq Kx$, we have by (9),

$$\int_0^{\alpha}\frac{dh^*(x)}{x}\leqq K\int_0^{\delta}\frac{dh(\xi)}{\xi}<K\varepsilon. \tag{10}$$

Since

$$\int_0^{\alpha}\frac{dh^*(x)}{x}=\frac{h^*(\alpha)}{\alpha}+\int_0^{\alpha}\frac{h^*(x)dx}{x^2}\geqq\int_0^{\alpha}\frac{h^*(x)dx}{x^2},$$

we have by (9), (10),

$$\int_0^{\alpha}\frac{|\eta_2(x)|dx}{x^2}=\int_0^{\alpha}\frac{h^*(x)dx}{x^2}<K\varepsilon, \tag{11}$$

$$\int_{\alpha}^{\beta}\frac{|\eta_2(x)|dx}{x^2}\leqq h(\delta)\int_{\alpha}^{\beta}\frac{dx}{x^2}\leqq\frac{h(\delta)}{\alpha}\leqq K\frac{h(\delta)}{\delta}<K\varepsilon, \tag{12}$$

so that

$$\int_0^{\beta} \frac{|\eta_2(x)|dx}{x^2} < 2K\varepsilon . \tag{13}$$

Let $w_2(z)=\xi_2(z)+i\eta_2(z)$, $z=x+iy$, then since D_2 is a bounded domain, $\eta_2(z)$ can be expressed by a Poisson integral with respect to $\eta_2(x)$, so that

$$\frac{\eta_2(iy)}{y} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\eta_2(t)dt}{y^2+t^2} = \frac{2}{\pi}\int_0^{\infty} \frac{\eta_2(t)dt}{y^2+t^2} . \tag{14}$$

Since $\frac{|\eta_2(t)|}{y^2+t^2} \leqq \frac{|\eta_2(t)|}{t^2}$ and $\frac{|\eta_2(t)|}{t^2}$ is integrable by (13), we have by Lebesgue's theorem,

$$\lim_{y\to 0} \frac{\eta_2(iy)}{y} = \frac{2}{\pi}\int_0^{\infty} \frac{\eta_2(t)dt}{t^2} . \tag{15}$$

Since $\eta_2(t)\geqq 0$ for $\beta\leqq t\leqq 1$ and $\eta_2(t)\geqq \frac{r_1}{\sqrt{2}}$ for $1\leqq t<\infty$, we have by (13),

$$\lim_{y\to 0} \frac{\eta_2(iy)}{y} \geqq \frac{2}{\pi}\left(-\int_0^{\beta} \frac{|\eta_2(t)|dt}{t^2} + \frac{r_1}{\sqrt{2}}\int_1^{\infty}\frac{dt}{t^2}\right)$$

$$\geqq \frac{2}{\pi}\left(-2K\varepsilon + \frac{r_1}{\sqrt{2}}\right) .$$

Since K is independent of δ, if we choose δ so small that $-2K\varepsilon+\frac{r_1}{\sqrt{2}}>0$, then

$$\infty > \lim_{y\to 0} \frac{\eta_2(iy)}{y} > 0 . \tag{16}$$

Since by Lindelöf's theorem, $\xi_2(iy)/\eta_2(iy)\to 0$, we have $\xi_2(iy)/y\to 0$, so that

$$\lim_{y\to 0} \frac{w_2(iy)}{iy} = \gamma_2 , \qquad 0<\gamma_2<\infty . \tag{17}$$

Since by (1), $\frac{w_2(z)}{z}$ is bounded in $K_{\rho_0}(\varphi_0)$, we have by (17) and Montel's theorem,

$$\lim_{z\to 0} \frac{w_2(z)}{z} = \gamma_2\,, \qquad 0<\gamma_2<\infty\,, \tag{18}$$

uniformly, when $z\to 0$ in $K_{\rho_0}(\varphi_0)$.

Hence

$$0<A_2|z| \leqq |w_2(z)| \leqq B_2|z|\,, \qquad z\in K_{\rho_0}(\varphi)\,, \tag{19}$$

where $A_2>0$, $B_2>0$ are constants.

Let $G_{D_2}(w, ia)$ $(a>0)$ be the Green's function of D_2. If by $w=w_2(z)$ $w_0=ia$ corresponds to z_0, then by (19)

$$G_{D_2}(w, ia) = \log\left|\frac{z-\bar{z}_0}{z-z_0}\right| \leqq \text{const.}\,|z| \leqq \text{const.}\,|w|\,.$$

Let $G_D(w, ia)$ be the Green's function of D, then since $D\subset D_2$, we have

$$G_D(w, ia) \leq G_{D_2}(w, ia) \leq \text{const.}\,|w|\,, \qquad w\in K_{\rho_0}(\varphi_0)\,. \tag{20}$$

If by $w=w(z)$, $w_0=ia$ corresponds to z_0^*, then

$$G_D(w, ia) = \log\left|\frac{z-\bar{z}_0^*}{z-z_0^*}\right| \geqq \text{const.}\,|z|\,, \qquad z\in K_{\rho_0}(\varphi_0)\,,$$

so that by (20),

$$0<A|z| \leqq |w(z)|\,, \qquad z\in K_{\rho_0}(\varphi_0)\,, \tag{21}$$

where $A>0$ is a constant. Hence the lemma is proved.

LEMMA 2. *Let $f(z)$ be regular and $\frac{f(z)}{z}$ be bounded in a sector $\varDelta$: $0<|z|\leqq R$, $|\arg z|\leqq\theta_0$, then $f'(z)$ is bounded in $0<|z|\leqq R$, $|\arg z|\leqq\theta_1<\theta_0$. If $\lim\limits_{z\to 0}\frac{f(z)}{z}=\gamma$ uniformly for $|\arg z|\leqq\theta_0$, then $\lim\limits_{z\to 0} f'(z)=\gamma$ uniformly for $|\arg z|\leqq\theta_1<\theta_0$.*

PROOF. Let $\left|\frac{f(z)}{z}\right|\leqq M$ in $\varDelta$ and $\delta=\theta_0-\theta_1$. Let $z=re^{i\theta}$ $(|\theta|\leqq\theta_1)$, then the circle C: $|\zeta-z|=r\sin\delta$ is contained in $\varDelta$, if r is small. Hence if $\zeta-z=r\sin\delta e^{i\varphi}$,

$$|f'(z)| \leqq \frac{1}{2\pi}\int_c \frac{|f(\zeta)|\,|d\zeta|}{|\zeta-z|^2} \leqq \frac{M}{2\pi}\int_c \frac{|\zeta|d\varphi}{r\sin\delta} \leqq M\,\frac{(r+r\sin\delta)}{r\sin\delta}\,,$$

so that

$$|f'(z)| \leqq M \frac{(1+\sin\delta)}{\sin\delta}. \tag{1}$$

Next suppose that $\lim\limits_{z\to 0}\frac{f(z)}{z}=\gamma$ uniformly for $|\arg z|\leqq\theta_0$. We put $F(z)=\frac{f(z)}{z}-\gamma$, then $\lim\limits_{z\to 0}F(z)=0$ uniformly in $|\arg z|\leqq\theta_0$ and $f'(z)=F(z)+zF'(z)+\gamma$. We take $r_0>0$ so small, that $|F(z)|<\varepsilon$, if $z\in\Delta$, $|z|=r\leqq r_0$. Then

$$|F'(z)| \leqq \frac{1}{2\pi}\int_c \frac{|F(\zeta)|\,|d\zeta|}{|\zeta-z|^2} < \frac{\varepsilon}{r\sin\delta},$$

so that $\lim\limits_{z\to 0} zF'(z)=0$ uniformly for $|\arg z|\leqq\theta_1<\theta_0$. Hence

$$\lim_{z\to 0} f'(z)=\gamma \tag{2}$$

uniformly for $|\arg z|\leqq\theta_1<\theta_0$.

3. Proof of Main Theorems.

1. PROOF of THEOREM 3.

(i) *Proof of the part* (i).

Let $D_2\supset D$ be the domain defined before. We map D on $\Im z>0$ by $w=w(z)$, $w(0)=0$ and D_2 on $\Im\zeta>0$ by $w=w_2(\zeta)$, $w_2(0)=0$, then since $D\subset D_2$, D is mapped on a bounded domain Δ in $\Im\zeta>0$. By Lemma 1 and (18), (19) of the proof of Lemma 1,

$$0<A|z|\leqq|w(z)|\leqq B|z|, \qquad z\in K_{\rho_0}(\varphi_0), \tag{1}$$

$$0<A_2|\zeta|\leqq|w_2(\zeta)|\leqq B_2|\zeta|, \qquad \zeta\in K_{\rho_0}(\varphi_0), \tag{2}$$

$$\lim_{\zeta\to 0}\frac{w_2(\zeta)}{\zeta}=\gamma_2, \qquad 0<\gamma_2<\infty, \qquad \zeta\in K_{\rho_0}(\varphi_0). \tag{3}$$

Now Δ is mapped on $\Im z>0$ by $\zeta=\zeta(z)=\xi(z)+i\eta(z)$, $\zeta(0)=0$, so that

$$\frac{\eta(iy)}{y}=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\eta(t)dt}{y^2+t^2}, \qquad z=x+iy.$$

Since $\eta(t)\geqq 0$,

$$\lim_{y\to 0}\frac{\eta(iy)}{y}=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\eta(t)dt}{t^2}=\gamma_1, \qquad 0<\gamma_1\leqq\infty, \tag{4}$$

exists. Since by (1), (2),

$$\left|\frac{\zeta}{z}\right|=\left|\frac{\zeta}{w}\right|\cdot\left|\frac{w}{z}\right|\leqq\frac{B}{A_2}, \qquad z\in K_{\rho_0}(\varphi_0), \tag{5}$$

we have

$$\frac{\eta(iy)}{y}\leqq\left|\frac{\zeta(iy)}{iy}\right|\leqq\frac{B}{A_2},$$

so that $0<\gamma_1<\infty$. Since by Lindelöf's theorem, $\xi(iy)/\eta(iy)\to 0$, we have $\xi(iy)/y\to 0$, so that

$$\lim_{y\to 0}\frac{\zeta(iy)}{iy}=\gamma_1, \qquad 0<\gamma_1<\infty. \tag{6}$$

Since by (5), $\left|\frac{\zeta(z)}{z}\right|$ is bounded in $K_{\rho_0}(\varphi_0)$, we have by (6) and Montel's theorem,

$$\lim_{z\to 0}\frac{\zeta(z)}{z}=\gamma_1, \qquad 0<\gamma_1<\infty, \tag{7}$$

uniformly, when $z\to 0$ in $K_{\rho_0}(\varphi_0)$.

Since $\frac{w(z)}{z}=\frac{w_2(\zeta)}{\zeta}\cdot\frac{\zeta(z)}{z}$, we have by (3), (7),

$$\lim_{z\to 0}\frac{w(z)}{z}=\gamma_2\gamma_1=\gamma, \quad 0<\gamma<\infty, \quad z\in K_{\rho_0}(\varphi_0), \tag{8}$$

hence by Lemma 2, $\lim_{z\to 0} w'(z)=\gamma$ uniformly in $K_{\rho_0}(\varphi_1)$ $(\varphi_1<\varphi_0)$. Hence the part (i) is proved.

(ii) *Proof of the part* (ii).

First we assume that C satisfies the condition (W) at $w=0$. Let $z_0=r$ $(r>0)$ be small and $w_0=w(z_0)$.

By (2) of the proof of (ii) of Lemma 1, we map D conformally on a domain contained in $|\zeta|<1$. Then by (4) of the proof of (ii) of Lemma 1, there exists τ $(\rho^2 r\leqq\tau\leqq\rho r)$, such that

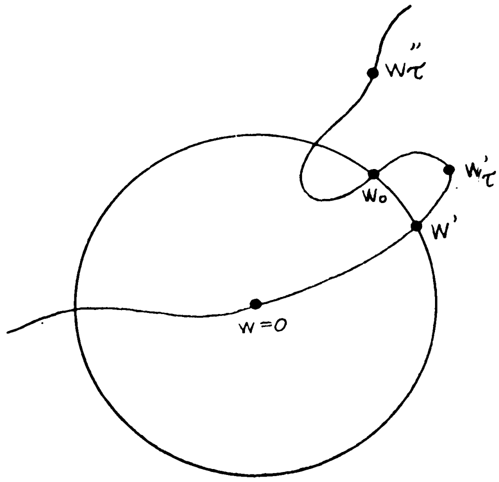
$$L(\tau)=\int_0^{\pi}|\zeta'(z_0+\tau e^{i\theta})|\tau d\theta<\varepsilon_1<\frac{1}{4}. \tag{1}$$

With the same notation as before, if the arc $\widehat{\zeta'_\tau\zeta''_\tau}$ has common points with $|\zeta-1|\leqq 1/4$, then by (1), $\widehat{\zeta'_\tau\zeta''_\tau}$ lies in $|\zeta-1|\leqq 1/2$, hence for such ζ,

$$|u|=\left|\frac{1+\zeta}{1-\zeta}\right|\geqq\frac{1}{|1-\zeta|}\geqq 2,$$ so that $|u'_\tau|\geqq 2$, hence $|v'_\tau|\geqq 2|v_0|$, so that

$$|w'_\tau|\geqq k|w_0|,\qquad k=2^{\frac{2\psi_0}{\pi}}>1, \tag{2}$$

hence the image $\widehat{0w'_\tau}$ of the segment $\overline{0z'_\tau}$ meets the circle $|w|=|w_0|$ at a point $w'(w'\neq w_0, |w'|=|w_0|)$. Since by the condition (W), $|w'_\tau|<k|w_0|$, if $|w_0|$ is small, which contradicts (2).



Hence $\widehat{\zeta'_\tau\zeta''_\tau}$ lies outside of $|\zeta-1|=1/4$, so that for any z on $\widehat{z'_\tau z''_\tau}$,

$$\left|\frac{du}{dz}\right|=\frac{2}{|1-\zeta|^2}\left|\frac{d\zeta}{dz}\right|\leqq 32\left|\frac{d\zeta}{dz}\right|,$$

hence by (1),

$$\int_0^\pi |u'(z_0+\tau e^{i\theta})|\tau d\theta<\varepsilon_2,\quad \varepsilon_2=32\varepsilon_1. \tag{3}$$

If $|u''_\tau|\leqq 1$, then the image of the half-line $\overline{z''_\tau\infty}$ meets $|u|=1$ at a point $u''(|u''|=1, u''\neq u_0)$, hence by the condition (W), $|u''_\tau|>1-\varepsilon_2$, if $|w_0|$ is small, so that by (3), $\widehat{u'_\tau u''_\tau}$ lies in a ring domain:

$$1-2\varepsilon_2\leqq|u|\leqq 1+\varepsilon_2.$$

If $|u''_\tau|>1$ and $|u'_\tau|\leqq 1$, then $\widehat{u'_\tau u''_\tau}$ has a common point with $|u|=1$, so that $\widehat{u'_\tau u''_\tau}$ lies in a ring domain: $1-\varepsilon_2\leqq|u|\leqq 1+\varepsilon_2$. If $|u''_\tau|>1$ and $|u'_\tau|>1$, then the image $\widehat{ou'_\tau}$ of the segment $\overline{0z'_\tau}$ meets $|u|=1$ at a point $u'(|u'|=1, u'\neq u_0)$, so that by the condition (W), $|u'_\tau|<1+\varepsilon_2$, if $|w_0|$ is small, hence $\widehat{u'_\tau u''_\tau}$ lies in a ring domain:

$$1-\varepsilon_2\leqq|u|\leqq 1+2\varepsilon_2.$$

Hence in any case, $\widehat{u'_\tau u''_\tau}$ lies in a ring domain: $1-2\varepsilon_2\leqq|u|\leqq 1+2\varepsilon_2$. By this and the condition (W), we can prove easily that if z_0 is small, then the image of the half-disc $\Delta_\tau: |z-z_0|\leqq\tau$, $\Im z\geqq 0$ lies in a ring domain: $1-3\varepsilon_2 \leqq |u| \leqq 1+3\varepsilon_2$, hence for any $z\in\Delta_\tau$,

$$(1-\varepsilon_3)|w_0| \leqq |w(z)| \leqq (1+\varepsilon_3)|w_0|, \tag{4}$$

where $\varepsilon_3 \to 0$ with $\varepsilon_2 \to 0$. Hence especially,

$$(1-\varepsilon_3)|w_0| \leqq |w_\tau| \leqq (1+\varepsilon_3)|w_0|, \qquad w_\tau = w(z_\tau),$$

so that by (4),

$$(1-\varepsilon)|w_\tau| \leqq |w(z)| \leqq (1+\varepsilon)|w_\tau|, \qquad z\in\Delta_\tau, \tag{5}$$

where $\varepsilon \to 0$ with $\varepsilon_3 \to 0$.

Since $\arg z_\tau \geqq 2\sin^{-1}(\rho^2/2)$, we have by the part (i),

$$\lim_{z_\tau \to 0} \left|\frac{w_\tau}{z_\tau}\right| = \gamma, \qquad 0<\gamma<\infty. \tag{6}$$

Since $\varepsilon>0$ is arbitrary, we have by (5), (6) and the part (i), we have for any $z\in\Delta_\tau$, $|z|=|z_\tau|$,

$$\lim_{z\to 0} \left|\frac{w(z)}{z}\right| = \gamma, \qquad 0<\gamma<\infty, \tag{7}$$

when $z\to 0$ in $\Im z \geqq 0$. Since by Lindelöf's theorem, $\lim\limits_{z\to 0} \arg \dfrac{w(z)}{z} = 0$, we have

$$\lim_{z\to 0} \frac{w(z)}{z} = \gamma, \qquad 0<\gamma<\infty, \tag{8}$$

when $z\to 0$ in $\Im z \geqq 0$.

Similarly we can prove (8), if C satisfies the condition (W^*) at $w=0$. Hence Theorem 3 is proved.

2. PROOF of THEOREM 4.

Let $w=w(z)=\xi(z)+i\eta(z)$, then

$$\frac{\eta(z)}{y} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\eta(t)dt}{|z-t|^2}, \qquad z=x+iy. \tag{1}$$

First we consider the case (i), then we may assume that D lies on the upper half-plane, so that $\eta(t) \geqq 0$, hence

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\eta(t)}{t^2}\, dt = \gamma, \qquad 0<\gamma\leqq\infty, \tag{2}$$

exists. If $z\in K_{\rho_0}(\varphi_0)$, then as we have proved before, $\frac{1}{|z-t|^2}\leqq\frac{1}{t^2\cos^2\varphi_0}$, so that if $0<\gamma<\infty$, then by Lebesgue's theorem,

$$\lim_{y\to 0}\frac{\eta(z)}{y}=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\eta(t)dt}{t^2}=\gamma. \tag{3}$$

If $\gamma=\infty$, then by Fatou's lemma,

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\eta(t)}{t^2}dt\leqq\varliminf_{y\to 0}\frac{\eta(z)}{y},\quad\text{so that}\quad\lim_{y\to 0}\frac{\eta(z)}{y}=\infty,$$

hence (3) holds in any case.

Let $w=\rho e^{i\varphi}$, $z=re^{i\theta}\in K_{\rho_0}(\varphi_0)$, then

$$\frac{w(z)}{z}=\frac{\eta(z)(i+\cot\varphi)}{y(i+\cot\theta)}.$$

Since by Lindelöf's theorem, $\lim_{z\to 0}(\varphi-\theta)=0$, we have by (3),

$$\lim_{z\to 0}\frac{w(z)}{z}=\gamma,\quad 0<\gamma\leqq\infty,\quad z\in K_{\rho_0}(\varphi_0). \tag{4}$$

Now by (5) of the proof of the part (ii) of Theorem 3,

$$(1-\varepsilon)|w_\tau|\leqq|w(z)|\leqq(1+\varepsilon)|w_\tau|,\qquad z\in\Delta_\tau, \tag{5}$$

where $\varepsilon\to 0$ with $z\to 0$. Since $z_\tau\in K_{\rho_0}(\varphi_0)$, $\lim_{z\to 0}\left|\frac{w_\tau}{z_\tau}\right|=\gamma$ and since $\varepsilon>0$ is arbitrary, we have $\lim_{z\to 0}\left|\frac{w(z)}{z}\right|=\gamma$, $0<\gamma\leqq\infty$, $\Im z\geqq 0$. Since by Lindelöf's theorem, $\lim_{z\to 0}\arg\frac{w(z)}{z}=0$, we have

$$\lim_{z\to 0}\frac{w(z)}{z}=\gamma,\quad 0<\gamma\leqq\infty,\quad \Im z\geqq 0. \tag{6}$$

Suppose that $0<\gamma<\infty$, then

$$0<A|z|\leqq|w(z)|\leqq B|z|,\qquad \Im z\geqq 0, \tag{7}$$

where $A>0$, $B>0$ are constants.

Let $w=-\delta$, $w=\delta$ correspond to $z=-\alpha$, $z=\beta$ $(\alpha>0,\ \beta>0)$ respectively. Let $z=x$ $(0<x\leqq\beta)$ and $w(x)=\xi(x)+i\eta(x)=\xi+ih(\xi)$, $(h(\xi)=\eta(x))$, then since $0<\gamma<\infty$,

$$\int_0^\beta \frac{\eta(x)dx}{x^2}<\infty\,,\quad \text{hence}\quad \int_0^\beta \frac{d\eta(x)}{x}<\infty\,.$$

Since by (7), $0<x\leqq K\xi$ ($K=$ const.), we have

$$\int_0^\delta \frac{dh(\xi)}{\xi}<\infty\,,\quad \text{hence}\quad \int_0^\delta \frac{h(\xi)d\xi}{\xi^2}<\infty\,.$$

Similarly we can prove that $\int_{-\delta}^0 \frac{h(\xi)d\xi}{\xi^2}<\infty$.

Hence if $0<\gamma<\infty$,

$$\int_{-\delta}^\delta \frac{h(t)dt}{t^2}<\infty\,. \tag{8}$$

If (8) holds, then by Theorem 3, $0<\gamma<\infty$. Hence in the case (i), (8) is the necessary and sufficient condition that $0<\gamma<\infty$.

Next we consider the case (ii). By taking account of $\frac{\eta(iy)}{y}>0$, we can prove as the case (i), that

$$\lim_{z\to 0}\frac{w(z)}{z}=\frac{1}{\pi}\int_{-\infty}^\infty \frac{\eta(t)}{t^2}\,dt=\gamma\,,\quad 0\leqq\gamma<\infty\,,\quad \Im z\geqq 0\,. \tag{9}$$

Since $\eta(t)\leqq 0$ in a neighbourhood of $t=0$, we have from (9)

$$\int_{-\infty}^\beta \frac{|\eta(t)|dt}{t^2}<\infty\,. \tag{10}$$

By means of (10), we can prove similarly as the case (i), (8) is the necessary and sufficient condition that $0<\gamma<\infty$.

4. Extension of Valiron's theorem.

1. The following theorem, which is analogous to (i) of Theorem 3 is proved by Valiron[6], under the hypothesis that D lies on the upper half-plane.

6) G. Valiron: Sur la derivée angulaire dans la représentaion conforme. Bull. Sci. Math. (1932).

THEOREM 5. *Let D be a domain on the $w=\xi+i\eta=re^{i\theta}$-plane, which is bounded by a Jordan curve C, which passes through $w=0$ and touches the real axis at $w=0$ and its inner normal at $w=0$ coincides with the positive η-axis. We suppose that in a neighbourhood of $w=0$, C lies between two curves H and $\overline{H}$, each of which is symmetric to the imaginary axis and whose part on the right of the imaginary axis is*

$$H:\ \theta=\theta(r) \quad \text{and} \quad \overline{H}:\ \theta=-\theta(r) \quad (0\leqq r\leqq\delta), \quad \theta(0)=0,$$

where $\theta(r)>0$ is a continuous increasing function of $r>0$, such that

$$\int_0^\delta \frac{\theta(r)dr}{r}<\infty .$$

If we map D conformally on $\Im z>0$ by $w=w(z)$, $w(0)=0$, then

$$\lim_{z\to 0}\frac{w(z)}{z}=\gamma , \qquad 0<\gamma<\infty ,$$

uniformly, when $z\to 0$ in any Stolz domain, whose vertex is at $z=0$.

PROOF. We take a $(0<a<\delta)$, so small that $w_0=ia\in D$ and $K: |w-ia|=a$ be a circle. Let D_1 be the common part of the inside of K and the part of the w-plane, which lies above the curve H, and Γ be its boundary. We may assume that H has points in K and let $\Gamma(H)$ be the part of Γ, which belongs to H. Let $G_{D_1}(w, ia)$ be the Green's function of D_1, then

$$G_{D_1}(w, ia)=\log\frac{a}{|w-ia|}-v(w), \tag{1}$$

where $v(w)$ is harmonic in D_1 and as in the proof of Lemma 1, if δ is small,

$$v(w)\leqq\frac{2(a-|w-ia|)}{a} \quad \text{on} \quad \Gamma(H). \tag{2}$$

Let $w=re^{i\theta}\in\Gamma(H)$, then $\theta=\theta(r)$, so that

$$\frac{a-|w-ia|}{a}=\frac{a-\sqrt{a^2-2ra\sin\theta(r)+r^2}}{a}\leqq\frac{2r\sin\theta(r)}{a},$$

hence

$$v(w)\leqq\frac{4r\sin\theta(r)}{a} \quad \text{on} \quad \Gamma(H). \tag{3}$$

By taking a small, we may assume that the part of H, which lies on the right of the imaginary axis lies below a line L: $\arg w=\theta(\delta)$. Since the equation of K is $r=2a\sin\theta$, if $w=r_0e^{i\theta}$ be the common point of K and L, then $r_0=2a\sin\theta(\delta)$, so that if $w=re^{i\theta}\in\Gamma(H)$, then

$$r\leqq 2a\sin\theta(\delta).$$

We extend the definition of $\theta(r)$ for $-\delta\leqq r\leqq 0$, by putting $\theta(-r)=-\theta(r)$ $(0\leqq r\leqq\delta)$ and put

$$u(w)=\frac{1}{\pi}\int_{-a}^{a}t\sin\theta(t)\frac{\eta}{|w-t|^2}\,dt\,,\qquad w=\xi+i\eta\,, \tag{4}$$

then $u(w)$ is harmonic in $\Im w>0$.

Let $w_1=\xi_1+i\eta_1=r_1e^{i\theta_1}\in\Gamma(H)$, then $\theta_1=\theta(r_1)$, $\eta_1=r_1\sin\theta_1$, so that

$$u(w_1)\geqq\frac{r_1\sin\theta_1}{\pi}\int_{r_1}^{r_1+r_1\sin\theta_1}\frac{t\sin\theta(t)}{|w_1-t|^2}\,dt\,. \tag{5}$$

We can prove easily that for $r_1\leqq t\leqq r_1+r_1\sin\theta_1$, if r_1 is small, $|w_1-t|\leqq 3r_1\sin\theta_1$, so that

$$u(w_1)\geqq\frac{r_1\sin\theta_1}{9\pi r_1^2\sin^2\theta_1}\int_{r_1}^{r_1+r_1\sin\theta_1}t\sin\theta(t)dt\geqq\frac{r_1\sin\theta(r_1)}{9\pi}\,.$$

Hence by (3),

$$u(w)\geqq\frac{a}{36\pi}\,v(w)\quad\text{on}\quad\Gamma(H)\,, \tag{6}$$

so that by the maximum principle, (6) holds in D_1. As the proof of Lemma 1, if $w=\rho e^{i(\frac{\pi}{2}-\varphi)}\in K_{\rho_0}(\varphi_0)$, then

$$u(w)\leqq\frac{2\rho}{\pi\cos\varphi_0}\int_0^a\frac{\sin\theta(t)}{t}\,dt\,,$$

so that

$$v(w)\leqq\frac{72\rho}{a\cos\varphi_0}\int_0^a\frac{\sin\theta(t)dt}{t}\,,\qquad w\in K_{\rho_0}(\varphi_0)\,. \tag{7}$$

By means of (7), we can complete the proof by a suitable modification of the proof of Theorem 3.

2. If we assume only the existence of a tangent of C at $w=0$, then we have

THEOREM 6. *Let D be a domain on the $w=\xi+i\eta$-plane, which is bounded by a Jordan curve C, which passes through $w=0$ and touches the real axis at $w=0$ and its inner normal at $w=0$ coincides with the positive η-axis. If we map D conformally on $\Im z>0$ by $w=w(z)$, $w(0)=0$, then for any $\varepsilon>0$,*

$$0<A|z|^{1+\varepsilon}\leqq|w(z)|\leqq B|z|^{1-\varepsilon},$$

$$0<A|z|^{\varepsilon}\leqq|w'(z)|\leqq B|z|^{-\varepsilon}, \qquad z\in K_{\rho_0}(\varphi_0),$$

where $A>0$, $B>0$ are constants.

PROOF. By Lindelöf's theorem, the image of $K_{\rho_0}(\varphi_0)$ on the w-plane is contained in a sector $K_{\rho_1}(\varphi_1)$ ($\varphi_1=\varphi_0+\delta$, $\delta>0$), with $w=0$ as its vertex and is contained in D, where $\delta\to 0$ with $\rho_0\to 0$.

By

$$v=i\left(\frac{w}{i}\right)^{\frac{\pi}{2\varphi_1}}, \qquad \zeta=i\left(\frac{z}{i}\right)^{\frac{\pi}{2\varphi_0}}, \tag{1}$$

we map $\varDelta=K_{\rho_1}(\varphi_1)$ on a half-disc $\varDelta^*$: $|v|\leqq\rho_1^{\frac{\pi}{2\varphi_1}}$, $\Im v>0$ and $K=K_{\rho_0}(\varphi_0)$ on a half-disc K^*: $|\zeta|\leqq\rho_0^{\frac{\pi}{2\varphi_0}}$, $\Im\zeta>0$, then K^* is mapped on a domain $V\subset\varDelta^*$. Let $v=ia\in V$ ($a>0$) and $G_V(v, ia)$, $G_{\varDelta^*}(v, ia)$ be the Green's function of V and $\varDelta^*$ respectively, then $G_V(v, ia)\leqq G_{\varDelta^*}(v, ia)\leqq\text{const.}|v|$. Since V is mapped on K^*, we have as before, $G_V(v, ia)\geqq\text{const.}|\zeta|$, $\zeta\in K_{\rho_0}(\varphi_0)$, so that $|\zeta|\leqq\text{const.}|v|$, or $\text{const.}|z|^{1+\varepsilon}\leqq|w|$, $z\in K_{\rho_0}(\varphi_0)$ $\left(\varepsilon=\dfrac{\delta}{\varphi_0}\right)$. If we interchange z and w, we have $|w|\leqq\text{const.}|z|^{1-\varepsilon}$, so that

$$0<A|z|^{1+\varepsilon}\leqq|w(z)|\leqq B|z|^{1-\varepsilon}, \qquad z\in K_{\rho_0}(\varphi_0), \tag{2}$$

where $A>0$, $B>0$ are constants. By (2), if we apply the similar consideration as Lemma 2 on $w=w(z)$ and $z=z(w)$, we have

$$0<A|z|^{\varepsilon}\leqq|w'(z)|\leqq B|z|^{-\varepsilon}, \qquad z\in K_{\rho_0}(\varphi_0), \tag{3}$$

with suitable constants $A>0$, $B>0$.

REMARK. Hence a segment $z=re^{i\theta}$ $(0<r\leqq\rho_0, 0<\theta<\pi)$ is mapped on a rectifiable curve on the w-plane and vice versa a segment on the w-plane through $w=0$ is mapped on a rectifiable curve on the z-plane.

5. Kellogg's Theorem

1. Let D be a domain on the $w=\xi+i\eta$-plane, which is bounded by a Jordan curve C, which has continuous tangents and is represented by $w=w(s)=\xi(s)+i\eta(s)$, where s is the arc length of C, measured from a fixed point, such that

$$|w'(s+h)-w'(s)|\leqq\kappa|h|^{\alpha}, \qquad \kappa=\text{const.}, \ 0<\alpha<1. \tag{1}$$

We map D conformally on $|z|<1$ by $w=f(z)$, then Kellogg[7] proved the following theorem.

THEOREM 7. *$f'(z)$ is continuous and $\neq 0$ in $|z|\leqq 1$ and*

$$|f'(e^{i(\theta+h)})-f'(e^{i\theta})|\leqq\kappa_1|h|^{\alpha}, \qquad \kappa_1=\text{const.}\ .$$

Warschawski[8] gave a simple proof of this theorem. We shall simplify his proof a little by means of Green's functions.

PROOF. Let $z_0=e^{i\theta_0}$, $w_0=f(z_0)$. By a suitable linear transformation, we assume that $w_0=0$ and C touches the η-axis and the inner normal of C at $w=0$ coincides with the positive ξ-axis. Then we can prove easily that in a neighbourhood of $w=0$, C can be expressed in the form: $\xi=\xi(\eta)$ ($|\eta|\leqq\delta_0$), such that

$$|\xi|\leqq K|\eta|^{1+\alpha}, \qquad K=\text{const.}\ . \tag{1}$$

We can prove that K and δ_0 can be chosen independent of w_0. Now we consider

$$w=\xi^*+i\eta^*=\varphi(\zeta)=\zeta-\zeta^{1+\beta} \qquad (\beta=\alpha/2). \tag{2}$$

Then we can prove easily that $\left|\dfrac{\varphi(\zeta_1)-\varphi(\zeta_2)}{\zeta_1-\zeta_2}\right|>0$, if $|\zeta_i|\leqq\dfrac{1}{3}$, $\Re\zeta_i>0$ $(i=1,2)$, so that $\varphi(\zeta)$ is regular and univalent in a half-disc: $|\zeta|\leqq\dfrac{1}{3}$, $\Re\zeta>0$. If $\zeta=re^{i\theta}$, then

$$\xi^*=r\cos\theta-r^{1+\beta}\cos(1+\beta)\theta\,,$$

$$\eta^*=r\sin\theta-r^{1+\beta}\sin(1+\beta)\theta\,.$$

7) O. D. Kellogg: Harmonic functions and Green's integrals. Trans. Amer. Math. Soc. 13 (1912).

8) S. Warschawski: Über einen Satz von O. D. Kellogg. Göttinger Nachr. 1932.

Let $\Gamma: \left|\zeta-\frac{1}{6}\right|=\frac{1}{6}$ be a circle and Γ^* be its image on the w-plane. If ζ tends to $\zeta=0$ on Γ, then since $\frac{1}{3}\cos\theta=r$,

$$\xi^*=3r^2-r^{1+\beta}\cos(1+\beta)\theta\sim r^{1+\beta}\left|\cos(1+\beta)\frac{\pi}{2}\right|, \quad |\eta^*|\sim r,$$

so that

$$\xi^*\geqq \text{const.}\,|\eta^*|^{1+\beta}. \tag{3}$$

Hence by (1), $\xi^*>\xi$ for the same η, so that the part of Γ^*, which lies in a small neighbourhood of $w=0$ belongs to D. Hence if a $\left(0<a<\frac{1}{6}\right)$ is small, then the image K^* of the circle K: $|\zeta-a|=a$ and hence the image Δ^* of the disc Δ: $|\zeta-a|\leqq a$ is contained in D. It can be easily proved that a can be chosen independently of w_0. Let $w^*=\varphi(a)>0$ and $G_\Delta(\zeta, a)$, $G_{\Delta^*}(w, w^*)$ be the Green's function of Δ and Δ^* respectively, then

$$G_{\Delta^*}(w, w^*)=G_\Delta(\zeta, a)=\log\frac{a}{|\zeta-a|}.$$

Let U^* be the sector: $|w|\leqq\rho_0(\rho_0<w^*)$, $|\arg w|\leqq\varphi_0<\frac{\pi}{2}$, which is contained in Δ^* and U be its image on the ζ-plane, then for any $\zeta\in U$,

$$\log\frac{a}{|\zeta-a|}\geqq\text{const.}\,|\zeta|.$$

Since $|\zeta|\sim|w|$, we have

$$G_{\Delta^*}(w, w^*)\geqq\text{const.}\,|w|, \qquad w\in U^*. \tag{4}$$

Let $G_D(w, w^*)$ be the Green's function of D, then since $\Delta^*\subset D$,

$$G_D(w, w)\geqq G_{\Delta^*}(w, w)\geqq\text{const.}\,|w|, \qquad w\in U^*. \tag{5}$$

If $w^*=f(z^*)$, then

$$G_D(w, w^*)=\log\left|\frac{1-\bar{z}^*z}{z-z^*}\right|. \tag{6}$$

As Warschawski proved, the image of U^* in $|z|<1$ contains a sector

$$V:\ |z-z_0|\leqq\rho_1,\ \left|\arg\left(\frac{z_0-z}{z_0}\right)\right|\leqq\varphi_1=\varphi_0-\varepsilon,$$

where $\varepsilon\to0$ with $\rho_0\to0$ and ρ_1, φ_1 are independent of w_0. If $z\in V$, then

$$\log\left|\frac{1-\bar{z}^*z}{z-z^*}\right|\leqq\text{const.}\,|z-z_0|,$$

so that by (6), (5),

$$|w|=|f(z)|\leqq K|z-z_0|,\qquad K=\text{const.},\quad z\in V.$$

Hence by Lemma 2, if $z_0=e^{i\theta_0}$,

$$|f'(re^{i\theta_0})|\leqq K_1,\quad 1-\rho_1\leqq r<1,\quad K_1=\text{const.},\tag{7}$$

where K and K_1 are independent of z_0. From (7), by the maximum principle, we have

$$|f'(z)|\leqq K_1\qquad\text{in}\qquad |z|<1.\tag{8}$$

This being established, we can complete the proof as Warschawski.

2. As an application of Kellogg's theorem, we shall prove the following theorem. Let D be a domain on the w-plane, which is bounded by a Jordan curve C, which passes through $w=0$. A part of C, which lies in a small neighbourhood of $w=0$ is divided by $w=0$ into two parts C_1, C_2. We assume that $C_i\,(i=1,2)$ are analytic curves and make an inner angle $\alpha\pi\,(0<\alpha\leqq2)$ at $w=0$. We map D conformally on $\Im z>0$ by $w=w(z)$, $w(0)=0$. Then

THEOREM 8. *If $\rho>0$ is small,*

$$0<A|z|^\alpha\leqq|w(z)|\leqq B|z|^\alpha,$$

$$0<A|z|^{\alpha-1}\leqq|w'(z)|\leqq B|z|^{\alpha-1},\quad 0<|z|\leqq\rho,\quad \Im z\geqq0,$$

where $A>0$ $B>0$ are constants.

PROOF. We assume that C_1 touches the positive real axis. By $\zeta=w^{\frac{1}{\alpha}}$, we map D on a domain $\varDelta$ on the ζ-plane and let $C_i\,(i=1,2)$ become Γ_i, then Γ_i touches the real axis at $\zeta=0$.

Now on C_1, $w=w(\sigma)$, where σ is the arc length, measured from $w=0$, and let Γ_1: $\zeta=\zeta(s)$, where s is the arc length, measured from $\zeta=0$. From

$$d\zeta = \frac{1}{\alpha} w^{\frac{1}{\alpha}-1} dw, \quad \text{we have} \quad ds = \frac{1}{\alpha} |w|^{\frac{1}{\alpha}-1} d\sigma.$$

Hence, if we put $w(\sigma)=r(\sigma)e^{i\theta(\sigma)}$, then

$$\zeta'(s) = e^{i(\frac{1}{\alpha}-1)\theta(\sigma)} w'(\sigma),$$

so that

$$|\zeta'(s_1)-\zeta'(s_2)| \leqq \text{const.} |\sigma_1-\sigma_2|. \tag{1}$$

$$s = \frac{1}{\alpha}\int_0^\sigma |w|^{\frac{1}{\alpha}-1} d\sigma \sim \frac{1}{\alpha}\int_0^\sigma \sigma^{\frac{1}{\alpha}-1} d\sigma \sim \sigma^{\frac{1}{\alpha}} \sim |w|^{\frac{1}{\alpha}} \sim |\zeta|.$$

From $w=\zeta^\alpha$, we have $d\sigma=\alpha|\zeta|^{\alpha-1}ds \sim \alpha s^{\alpha-1}ds$, so that

$$|\sigma_1-\sigma_2| \leqq \text{const.} \left|\int_{s_1}^{s_2} s^{\alpha-1} ds\right| \leqq \text{const.} |s_1^\alpha - s_2^\alpha|. \tag{2}$$

If $\alpha \geqq 1$, then $|s_1^\alpha - s_2^\alpha| \leqq |s_1-s_2|$ and if $0<\alpha<1$, then $|s_1^\alpha - s_2^\alpha| \leqq |s_1-s_2|^\alpha$, so that in any case, $|\sigma_1-\sigma_2| \leqq \text{const.} |s_1-s_2|^\beta$, $(0<\beta<1)$, hence by (1), (2),

$$|\zeta'(s_1)-\zeta'(s_2)| \leqq \text{const.} |s_1-s_2|^\beta, \qquad 0<\beta<1. \tag{3}$$

We map Δ conformally on $\Im z>0$ by $\zeta=\zeta(z)$, $\zeta(0)=0$, then by (3) and Kellogg's theorem,

$$0<A_1 \leqq \left|\frac{\zeta(z)}{z}\right| \leqq B_1, \ 0<A_1 \leqq |\zeta'(z)| \leqq B_1, \ 0<|z| \leqq \rho, \ \Im z \geqq 0,$$

where $A_1>0$, $B_1>0$ are constants.

Since $\left|\frac{w}{z^\alpha}\right| = \left|\frac{\zeta}{z}\right|^\alpha$, $\left|\frac{dw}{dz}\right| = \alpha|\zeta|^{\alpha-1}\left|\frac{d\zeta}{dz}\right|$, we have

$$0<A|z|^\alpha \leqq |w| \leqq B|z|^\alpha, \ A|z|^{\alpha-1} \leqq \left|\frac{dw}{dz}\right| \leqq B|z|^{\alpha-1}, \ 0<|z| \leqq \rho, \ \Im z \geqq 0, \tag{4}$$

where $A>0$, $B>0$ are constants.

THEOREM 9. *Let D be a domain on the w-plane, which is bounded by a rectifiable Jordan curve C. We map D conformally on $|z|<1$ by $w=w(z)$. Then for almost all $e^{i\theta}$ on $|z|=1$,*

$$\lim_{z\to e^{i\theta}} \frac{w(z)-w(e^{i\theta})}{z-e^{i\theta}} = \lim_{z\to e^{i\theta}} w'(z) = \gamma, \qquad 0<|\gamma|<\infty,$$

uniformly, when $z \to e^{i\theta}$ from the inside of a Stolz domain, whose vertex is at $e^{i\theta}$.

PROOF. By F. and M. Riesz's theorem[9], $w(e^{i\theta})$ is absolutely continuous on $|z|=1$ and $w'(e^{i\theta}) = \frac{1}{ie^{i\theta}} \frac{\partial w(e^{i\theta})}{\partial\theta}$ exists almost everywhere on $|z|=1$ and is integrable, so that almost everywhere on $|z|=1$, $|w'(e^{i\theta})| < \infty$. Let a measurable set e on $|z|=1$ be mapped on a set E on C, then

$$mE = \int_e |w'(e^{i\theta})| d\theta\,, \tag{1}$$

where mE is the measure of E. Since by F. and M. Riesz's theorem, a set of positive measure on $|z|=1$ corresponds to a set of positive measure on C, we see from (1), that $w'(e^{i\theta}) \neq 0$ almost everywhere, so that $0 < |w'(e^{i\theta})| < \infty$ almost everywhere on $|z|=1$.

For such $e^{i\theta}$, by Fatou's theorem[10],

$$i\rho e^{i\varphi} w'(z) = \frac{\partial}{\partial\varphi} w(\rho e^{i\varphi}) \to \frac{\partial}{\partial\theta} w(e^{i\theta}) = ie^{i\theta} w'(e^{i\theta})$$

uniformly, when $z = \rho e^{i\varphi} \to e^{i\theta}$ from the inside of a Stolz domain, whose vertex is at $e^{i\theta}$, so that

$$\lim_{z \to e^{i\theta}} w'(z) = w'(e^{i\theta})\,, \qquad 0 < |w'(e^{i\theta})| < \infty \tag{2}$$

and hence

$$\lim_{z \to e^{i\theta}} \frac{w(z) - w(e^{i\theta})}{z - e^{i\theta}} = w'(e^{i\theta}) \tag{3}$$

uniformly, when $z \to e^{i\theta}$ from the inside of a Stolz domain, whose vertex is at $e^{i\theta}$.

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9) F. and M. Riesz: Über die Randwerte einer analytischen Funktion. 4. congr. scand. math. à Stockholm (1916).

10) P. Fatou: Séries trigonométriques et séries de Taylor. Acta Math. 30 (1906).