Journal of the Mathematical Society of Japan Vol. 6, Nos. 3-4, December, 1954

Isometric imbedding of Riemann manifolds in a Riemann manifold.

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(Received, Oct. 26, 1953)

1. Introduction. S. S. Chern and N. H. Kuiper [11] [1] obtained some theorems concerned with estimates on the lower bound of the dimension of the Euclidean space in which a compact Rieman manifold with some properties can be imbedded isometrically. The object of this paper is to generalize these results to the problem on the isometric imbedding of Riemann manifolds in another Riemann manifold.

The author will also make use of the methods in [11] in certain open sets which S. B. Myers [9] investigated in connection with non-existence of compact minimal subvarieties of dimension $n-1$ in Riemann manifolds of dimension n with some additional properties.

2. μ-domains. Let V_n be a Riemann manifold of dimension $n\geqq 2$ and class C^r [2]. Let O be any point of V_n, let $x^1, x^2, \cdots, x^n$ be geodesic normal coordinates with respect to a rectangular frame (R_0) at O. Let U be a neighborhood of O on which the coordinates are introduced. Let us denote the open set U considering together with the coordinates by $U(O, x)$, put $U=|U(O, x)|$ and call it a *geodesic coordinate neighborhood.* Let us attach to each point $P\in U$ that frame $(R)=\{P, e_i\}, i=1, 2, \cdots, n,$ which we obtain from (R_0) by parallel displacement along the geodesic arc $OP\subset U(O, x)$ [3]. Then, by means of the adapted family of frames [4] to the coordinates, let the connexion of V_n and the structure of the space be given by the following equations

1) Numbers in brackets refer to the list of references at the end of the paper.

2) $r\geqq 4$ is sufficient for all purposes in this paper.

3) By "a geodesic arc $OP\subset U(O, x)$", we shall mean that if $P=(x_0^i)$, the geodesic is given by the equations $x^i=tx_0^i$, $0\leqq t\leqq 1$.

4) See Cartan [1], p. 235.

$$\begin{cases} dP=\omega^i e_i, \quad de_i=\omega_i^k e_k \text{ [5]}, \\ \omega_i^j=\omega_{ij}=-\omega_{ji} \end{cases}$$

and

$$(1) \qquad \begin{cases} d\omega^i=\omega^k\wedge\omega_{ki} \\ d\omega_{ij}=\omega_{ik}\wedge\omega_{kj}+\dfrac{1}{2}R_{ijkh}\omega^k\wedge\omega^h, \end{cases}$$

where R_{ijkh} are the components of the Riemann-Christoffel tensor of the space. Now, let us put $x^i=a^i r$, $a^i a^i=1$, then we can give ω^i, ω_{ij} by the formulas

$$(2) \qquad \begin{cases} \omega^i(r, a;\ dr, da)=a^i dr+\omega^{*i}(r, a;\ da), \\ \omega_{ij}(r, a;\ dr, da)=\omega^*_{ij}(r, a;\ da) \end{cases}$$

as is well known. From (1), (2), we get the equations

$$(3) \qquad \begin{cases} \dfrac{\partial\omega^{*i}}{\partial r}=da^i+a^k\omega^*_{ki}=da^i+a^k\omega_{ki}=Da^i, \\ \dfrac{\partial\omega^*_{ij}}{\partial r}=R_{ijkh}a^k\omega^{*h} \end{cases}$$

where D denotes the covariant differentiation of the space. For $r=0$, we have

$$(4) \qquad \omega^{*i}(0, a;\ da)=0, \qquad \omega^*_{ij}(0, a;\ da)=0.$$

We get easily from (3), (4) the equations

$$(5) \qquad a^i\omega^{*i}=0,$$

$$(6) \qquad \frac{\partial^2\omega^{*i}}{\partial r^2}=R_{kihj}a^k a^h\omega^{*j}.$$

Then, in U the line element of the space is given by

$$ds^2=\omega^i\omega^i=dr\,dr+\omega^{*i}\omega^{*i}.$$

Now let us consider the following quadratic differential form in da^i

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5) The summation convention of tensor analysis is used throughout.

(7) $$\frac{1}{2}\frac{\partial}{\partial r}\omega^{*i}\omega^{*i}=\varphi(r, a; da, da)=\varphi_{ij}(r, a)da^{i}da^{i},$$

$$(\varphi_{ij}=\varphi_{ji}).$$

We get by (3)

(8) $$\varphi=da^{i}\omega^{*i}+a_{i}\omega^{*j}\omega^{*}_{ij}.$$

Furthermore, we get by (3), (6) the equation

(9) $$\frac{\partial\varphi}{\partial r}=(da^{i}+a^{k}\omega^{*}_{ki})(da^{i}+a^{h}\omega^{*}_{hi})+R_{ijkh}a^{i}\omega^{*j}a^{k}\omega^{*h}.$$

For $r=0$, we have from (4), (8), (9)

$$\varphi(0, a; da, da)=0,$$

$$\frac{\partial\varphi}{\partial r}(0, a; da, da)=da^{i}da^{i}.$$

It follows that φ is positive for sufficiently small positive r and any a^{i}, da^{i} such that $a^{i}a^{i}=1$, $a^{i}da^{i}=0$. By $U^{+}(O, x)(U^{-}(O, x))$ let us denote the open subset of U at any point of which φ is positive (negative) definite for all directions orthogonal to the tangent direction to the geodesic joining O to the point at it. If $U=U^{+}(O, x)+O$, we call it a μ-domain with center at O.

Let us denote the plane element spanned by the directions a^{i} and ω^{*i} with respect to the frame (R) by $\pi=\pi(P, da)$, then (9) is written, by means of $a^{i}a^{i}=1$ and (5), as

(9′) $$\frac{\partial\varphi}{\partial r}=\frac{1}{2}\frac{\partial^{2}}{\partial r^{2}}\omega^{*i}\omega^{*i}$$

$$=(da^{i}+a^{k}\omega^{*}_{ki})(da^{i}+a^{h}\omega^{*}_{hi})-K(P, \pi)\omega^{*i}\omega^{*i}$$

where $K(P, \pi)$ denotes the sectional curvature for $\pi(P, da)$.

If $K(P, \pi)\leqq 0$ for all π at any point P, we have $\partial\varphi/\partial r\geqq 0$. Then we have the following lemma.

LEMMA 1. *If V_n is complete*[6] *and has non-positive sectional curvatures for all plane elements at any point, any geodesic coordinate*

6) That is, from any point on each geodesic, we can take measure of any length on it both sides.

neighborhood $U(O, x)$ *is a* μ*-domain with center at* O, *where* O *denotes a point of* V_n.

If V_n is complete, every pair of points P, $Q \in V_n$ can be joined by a geodesic which is a shortest curve joining P to Q.[7] Furthermore, if it has nowhere positive sectional curvature, then on each geodesic through any fixed point O of V_n, there exists no conjugate point of the point. Hence we have a simple lemma.

LEMMA 2. *If* V_n *is complete and has non-positive sectional curvature for all plane elements at any point, then for any two points* O *and* P, *there exists a* μ*-domain with center at* O *containing any geodesic arc* OP *which joins* O *to* P *and has no double point.*

If V_n is a space of constant curvature K, then

$$R_{ijkh} = -K(\delta_{ik}\delta_{jh} - \delta_{ih}\delta_{ik})$$

as is well known, where δ_{ij} denote the Kronecker deltas. Then (6) becomes

$$\frac{\partial^2 \omega^{*i}}{\partial r^2} = -K\omega^{*i}.$$

It follows that $\omega^{*i} = \sin(r\sqrt{K})da^i/\sqrt{K}$ by (4). Hence we have

$$\omega^{*i}\omega^{*i} = \frac{1}{K}\sin^2(r\sqrt{K})da^i da^i,$$

$$\varphi = \frac{1}{\sqrt{K}}\sin^2(r\sqrt{K})\cos(r\sqrt{K})da^i da^i.$$

When $K > 0$, φ is positive definite for $0 < r < \pi/2\sqrt{K}$.

LEMMA 3. *Let* V_n *be a Riemann manifold of positive constant curvature* K, *for any two points* O *and* P *such that* $\mathrm{dist}(O, P) < \pi/2\sqrt{K}$, *there exists a* μ*-domain with center at* O *containing any geodesic arc* OP *which joins* O *to* P *and whose length* $= \mathrm{dist}(O, P)$. *Furthermore, if* V_n *is complete and simply connected, any open spherical neighborhood* $U(O, \pi/2\sqrt{K})$ *with center at* O *and of radius* $\pi/2\sqrt{K}$ *is a* μ*-domain with center at* O.

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7) See Hopf and Rinow [4] or Rinow [5].

In general, from (4), (8), (9), we see that there exist μ-domains with center at O.

3. Subamanifolds. Let V_{n+N} be a Riemann manifold of dimension $n+N(n, N\geqq 1)$ and class C^r. Let M be a differentiable submanifold of dimension n and class $C^t(3\leqq t\leqq r)$. Let V_n be the Riemann manifold defined on M with the induced metric from V_{n+N}. Let P be any point of V_{n+N} and let $(\bar{R})=\{P, \bar{e}_A\}$, $A=1, 2, \cdots, n, n+1, \cdots, n+N$ be a frame at P. Then, let the connexion of V_{n+N} and the structure of it be given by the equations

$$dP=\bar{\omega}^A\bar{e}_A\,, \qquad de_A=\bar{\omega}_A^B\bar{e}_B\,, \tag{10}$$

$$\begin{cases} d\bar{\omega}_A=\bar{\omega}^B\wedge\bar{\omega}_B^A\,, \\ d\bar{\omega}_B^A=\bar{\omega}_B^C\wedge\bar{\omega}_C^A+\dfrac{1}{2}\bar{R}_{BCE}^A\,\bar{\omega}^C\wedge\bar{\omega}^E\,, \end{cases} \tag{11}$$

where $\bar{R}_{BCE}^A$ are the components of the Riemann-Christoffel tensor of V_{n+N}.

On M, let

$$e_i=\bar{e}_AP_i^A\,, \qquad i=1, 2, \cdots, n \tag{12}$$

be n linearly independent tangent vectors to M at P and let be

$$e_\alpha=\bar{e}_AQ_\alpha^A\,, \qquad \alpha=n+1, \cdots, n+N \text{ [8]} \tag{13}$$

be N mutually orthogonal and normal unit vectors to M at P. Let us put

$$\bar{e}_A=e_iP_A^i+e_\alpha Q_A^\alpha\,. \tag{14}$$

On M let

$$dP=\omega^ie_i\,, \qquad de_A=\omega_A^Be_B\,, \qquad g_{AB}=e_Ae_B\,.$$

Since $g_{i\alpha}=e_i\cdot e_\alpha=0$, $g_{\alpha\beta}=\delta_{\alpha\beta}$, we get

$$0=dg_{i\alpha}=\omega_i^Ag_{A\alpha}+\omega_\alpha^Ag_{iA}=\omega_i^\alpha+\omega_\alpha^jg_{ij}\,,$$

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8) Let us agree on the following ranges of indices throughout:
$i, j, k, \cdots=1, 2, \cdots, n$
$\alpha, \beta, \gamma, \cdots=n+1,\ n+2, \cdots, n+N.$
$A, B, C, \cdots=\ ,2, \cdots, n+N.$

that is

(15) $$\omega_i^\alpha = -g_{ij}\omega_\alpha^j .$$

Furthermore, since $\omega^\alpha = 0$ on M, we get from the equations analogous to the first of (11) with respect to the frame $\{P, e_A\}$

$$0 = \omega^A \wedge \omega_A^\alpha = \omega^i \wedge \omega_i^\alpha ,$$

hence by a lemma of E. Cartan, we can write ω_i^α as

(16) $$\omega_i^\alpha = A_{\alpha ij}\omega^j , \qquad A_{\alpha ij} = A_{\alpha ji} .$$

The quadratic differential forms

(17) $$\Phi_\alpha(\omega, \omega) = A_{\alpha ij}\omega^i\omega^j$$

are the so-called second fundamental forms of M.

From the equations $d\omega^i = \omega^A \wedge \omega_A^i = \omega^j \wedge \omega_j^i$, the connexion of the Riemann space V_n must be given by the Pfaffian forms ω^i, ω_j^i with respect to the frame $(R) = \{P, e_i\}$. Hence we obtain by (15), (16)

$$\begin{aligned}
\frac{1}{2} R_j{}^i{}_{kh}\omega^k \wedge \omega^h &= d\omega_j^i - \omega_j^k \wedge \omega_k^i \\
&= d\omega_j^i - \omega_j^A \wedge \omega_A^i + \omega_j^\alpha \wedge \omega_\alpha^i \\
&= P_A^i P_j^B (d\bar{\omega}_B^A - \bar{\omega}_B^C \wedge \bar{\omega}_C^A) + \omega_j^\alpha \wedge \omega_\alpha^i \\
&= \frac{1}{2} P_A^i P_j^B \bar{R}_{BCE}^A \bar{\omega}^C \wedge \bar{\omega}^E - g^{im} A_{\alpha ik}\omega^k \wedge A_{\alpha mh}\omega^h ,
\end{aligned}$$

hence

(18) $$R_j{}^i{}_{kh} = \bar{R}_{BCE}^A P_j^B P_A^i P_k^C P_h^E - g^{im}(A_{\alpha jk}A_{\alpha mh} - A_{\alpha jh}A_{\alpha mk})$$

or

(18′) $$R_{ijkh} = \bar{R}_{ABCE} P_i^A P_j^B P_k^C P_h^E - (A_{\alpha ik}A_{\alpha jh} - A_{\alpha ih}A_{\alpha jk}) .$$

On the other hand, we have from (10), (12), (13), (14)

$$\begin{aligned}
de_i &= \omega_i^j e_j + \omega_i^\alpha e_\alpha = \omega_i^j P_j^A \bar{e}_A + \omega_i^\alpha Q_\alpha^A \bar{e}_A \\
&= dP_i^A \bar{e}_A + P_i^B d\bar{e}_B = dP_i^A e_A + P_i^B \bar{\omega}_B^A \bar{e}_A ,
\end{aligned}$$

9) See Chern and Kuiper [11].

hence

$$dP_i^A+\bar{\omega}_B^A P_i^B-\omega_i^j P_j^A=A_{\alpha ij}Q_\alpha^A\omega^j\,. \tag{19}$$

4. Indices of relative nullity. At any point $P\in M$ let $\nu(P)$ be the integer such that $n-\nu(P)$ is the minimum number of linearly independent linear differential forms in terms of which $\Phi_\alpha(\omega,\omega)$ can be expressed. According to S. S. Chern and N. H. Kuiper $\nu(P)$ is called the *index of relative nullity* at P. $n-\nu(P)$ is evidently the number of linearly independent equations in the system $\omega_i^\alpha=A_{\alpha ij}\omega^j=0$. Let us put

$$\nu(M)=\min_{P\in M}\nu(P)\,.$$

Now, in the following, we shall assume that V_{n+N} is complete. Let O be any point of V_{n+N} and let P_0 be a *locally maximum distance point* (*minimum distance point* $\neq O$) of M from O in V_{n+N}, so that there exists a relative open neighborhood of P_0 in M on which the distance from O to P_0 in V_{n+N} is maximum (minimum).

Let us suppose that there exists a geodesic coordinate neighborhood $U(O,x)$ in V_{n+N} containing P_0 such that the length of the geodesic arc joining O to P in $U(O,x)$[10] is equal to dist (O,P) at any point $P\in|U(O,x)|$.

With respect to the adapted family of frames to the coordinates x in U, as stated in the first section, we shall introduce the quantities $\bar{\omega}^{*A}, \bar{\omega}^*_{AB}, \bar{\varphi}$ such that

$$x^A=a^A r\,,\qquad a^A a^A=1\,, \tag{20}$$

$$\begin{cases} \bar{\omega}^A(r,a\,;\,dr,da)=a^A dr+\bar{\omega}^{*A}(r,a\,;\,da)\,, \\ \bar{\omega}_{AB}(r,a\,;\,dr,da)=\bar{\omega}^*_{AB}(r,a;\,da)=\bar{\omega}^{*B}_A \end{cases} \tag{21}$$

which satisfy

$$\bar{\omega}^{*A}(0,a;\,da)=0\,,\qquad \bar{\omega}^*_{AB}(0,a;\,da)=0\,, \tag{22}$$

$$\begin{cases} \dfrac{\partial\bar{\omega}^{*A}}{\partial r}=da^A+a^B\bar{\omega}^*_{BA}\,, \\ \dfrac{\partial\bar{\omega}^*_{AB}}{\partial r}=\bar{R}_{ABCE}a^C\bar{\omega}^{*E}\,, \end{cases} \tag{23}$$

10) See Foot-note 3).

(24) $$a^A\bar{\omega}^{*A}=0$$

and

(25) $$\bar{\varphi}(r, a, da, da)=\frac{1}{2}\frac{\partial}{\partial r}\bar{\omega}^{*A}\bar{\omega}^{*A}=da^A\bar{\omega}^{*A}+a^A\bar{\omega}^{*B}\bar{\omega}^{*}_{AB},$$

etc.

Now, on M we must have $dr=0$, $d^2r\leqq 0$ ($d^2r\geqq 0$) at the point P_0. Hence, at the point we have by means of (21), (24) the relations

(26) $$\bar{\omega}^A=\bar{\omega}^{*A}=P_i^A\omega_i,$$

(27) $$a^AP_i^A=0 \qquad \text{or} \qquad a^AP_A^i=0.$$

Furthermore, at P_0 we get from (21) the equations

$$dP_i^A\omega^i+P_i^Ad\omega^i=a^Ad^2r+d\bar{\omega}^{*A},$$

hence by (24), (27)

$$a^AdP_i^A\omega^i=d^2r-da^A\bar{\omega}^{*A}.$$

Accordingly, making use of (19), (22), (27), we have

$$\begin{aligned} d^2r&=a^AdP_i^A\omega^i+da^A\bar{\omega}^{*A} \\ &=a^A(-\bar{\omega}_B^AP_i^B+\omega_i^jP_j^A+A_{\alpha ij}Q_\alpha^A\omega^j)\omega^i+da^A\bar{\omega}^{*A} \\ &=da^A\bar{\omega}^{*A}+a^A\bar{\omega}^{*B}\bar{\omega}^{*}_{AB}+A_{\alpha ij}Q_\alpha^Aa^A\omega^i\omega^j, \end{aligned}$$

that is

$$\underset{(\leqq)}{0\geqq} d^2r=\bar{\varphi}(r, a, da, da)+\Phi_\alpha(\omega, \omega)Q_\alpha^Aa^A.$$

If $P_0\in U^+(O, x)$ ($U^-(O, x)$), we must have

(29) $$\Phi_\alpha(\omega, \omega)Q_\alpha^Aa^A\underset{(>)}{<}0.$$

This shows that there exists no tangent direction to M at P_0 such that $\Phi_\alpha(\omega, \omega)=0$ hold simultaneously. Hence, it follows $\nu(P_0)=0$.

On the other hand, if M is a minimal variety in V_{n+N}, it must hold that $g^{ij}A_{\alpha ij}=0$ at each point of M[11]. (29) implies $g^{ij}A_{\alpha ij}Q_\alpha^Aa^A\neq 0$

11) See Eisenhart [2], p. 178.

at P_0, accordingly we cannot have $g^{ij}A_{\alpha ij}=0$. This shows that a minimal variety has no such point as P_0. Thus we obtain the following theorem.[12]

THEOREM 1. *Let V_{n+N} be a complete Riemann manifold of dimension $n+N(n, N\geqq 1)$ and let M be a differentiable submanifold of V_n and dimension n. If there exists a point O, a geodesic coordinate neighborhood $U(O, x)$ in V_{n+N} and a locally maximum distance point P_0 (minimum distance point $P_0 \neq 0$) of M from O in V_{n+N} such that $P_0 \in U^+(O, x)(U^-(O, x))$ and the length of the geodesic arc joining O to P in $U(O, x)$ is equal to dist(O, P) at any point $P \in |U(O, x)|$, then $\nu(M)=0$. M cannot be a minimal variety in V_{n+N}.*

By virtue of Lemma 2, Theorem 1, we get easily the theorem:

THEOREM 2. *Let V_{n+N} be a complete Riemann manifold of dimension $n+N(n, N\geqq 1)$ with non-positive sectional curvatures for all plane elements at any point and let M be a compact differentiable submanifold of dimension n and disjointed from the minimum point locus[13] with respect to some point of V_{n+N}, then $\nu(M)=0$. There exists no compact minimal variety of dimension n and disjointed from the minimum point locus with respect to some point of V_{n+N}.*

By virtue of Lemma 3, Theorem 1, we have easily the theorem:

THEOREM 3. *Let V_{n+N} be a Riemann manifold of positive constant curvature K and dimension $n+N(n, N\geqq 1)$. Let M be a compact differentiable submanifold of dimension n. If M is contained in an open spherical neighborhood of radius $\pi/2\sqrt{K}$[14], especially the diameter of M in $V_{n+N}<\pi/2\sqrt{K}$, then $\nu(M)=0$ and M cannot be a minimal variety of V_{n+N}.*

Lastly, returning to the beginning of the section, we shall remark $\nu(P)$. Let us denote the $n\times N$-matrix whose (i, λ)-element, $i=1, 2, \cdots, n$; $\lambda=1, 2, \cdots, N$, is $A_{n+\lambda, ik}$ by M_k. Then, from the definition of $\nu(P)$,

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12) See Chern [10], p. 23 and Mayers [9], Theorem 4.

13) See Mayers [7], [8].

14) The "spherical neighborhood" in the sense used in metric spaces may not become a geodesic coordinate neighborhood as stated in Section 2, since the space V_{n+N} is is not always simply connected. But any point in the neighborhood and the center can be joined by a geodesic arc whose length is equal to the distance between the two points. Accordingly the arc is simple and there exists a μ-domain containing it by Lemma 3. This is sufficient in order to make use of the argument in this section.

$n-\nu(P)$ is the maximum number of linearly independent matrices of the matrices $M_1, M_2, \cdots, M_n$.

5. Indices of nullity. Let V_n be a Riemann manifold of dimension n. At any point $P \in V_n$ and for any constant K, let $\mu(P, K)$ be the integer such that $n-\mu(P, K)$ is the minimum number of linearly independent linear differential forms in terms of which $\Omega_{ij}+Kg_{ik}g_{jh} \times \omega^k \wedge \omega^h = \frac{1}{2}\{R_{ijkh}+K(g_{ik}g_{jh}-g_{ih}g_{jk})\}\omega^k \wedge \omega^h$ can be expressed. The number $\mu(P, K)$ will be called *the index of nullity relative to constant* K *at* P. When $K=0$, it turns to the index defined by S. S. Chern and N. H.Kuiper[15]. Let us put

$$\mu(V_n, K)=\min_{P \in V_n} \mu(P, K).$$

Let now be V_{n+N} a Riemann manifold of constant curvature K and dimension $n+N(n \geqq 2, N \geqq 1)$. Let M be a differentiable submanifold of dimension n and let V_n be the Riemann manifold defined on M with the induced metric from V_{n+N}. Then, from (18′) and the assumption on V_{n+N}, we obtain easily the equations

$$R_{ijkh}+K(g_{ik}g_{jh}-g_{ih}g_{jk})=-(A_{\alpha ik}A_{\alpha jh}-A_{\alpha ih}A_{\alpha jk}). \tag{30}$$

Accordingly we have $n-\mu(P, K) \leqq n-\nu(P)$ or $\nu(P) \leqq \mu(P, K)$.

On the other hand, $A_{\alpha ik}A_{\alpha jh}-A_{\alpha ih}A_{\alpha jk}$ is the (i, j)-element of the $n \times n$-matrix $N_{kh}=M_kM'_h-M_hM'_k$, where M'_h denotes the transposed matrix of M_h. Hence, $\mu(P, K)$ is the dimension of the linear space of the solutions of the equations

$$N_{ij}y^j=0$$

in n variables $y^1, \cdots, y^n$. Hence $n-\mu(P, K)$ is the minimum number of variables such that the quadratic exterior form $N_{ij}y^i \wedge y^j$ on the ring of $n \times n$-matrices can be expressed by them. By virtue of the remark at the end of the last section and Theorem 3 in Ôtsuki [13], we have

$$(n-\nu(P))-(n-\mu(P, K)) \leqq N,$$

that is $\mu(P, K)-\nu(P) \leqq N$. Thus we obtain the following inequalities

15) See Chern and Kuiper [11].

(31) $$\nu(P) \leqq \mu(P, K) \leqq N + \nu(P),$$

accordingly

(32) $$\nu(M) \leqq \mu(V_n, K) \leqq N + \nu(M).$$

Thus we obtain a theorem as follows:

THEOREM 4. *Let V_{n+N} be a Riemann manifold of constant curvature K and dimension $n+N(n \geqq 2, N \geqq 1)$ and let V_n be a subspace of V_{n+N} and dimension n. Then, between the indices of nullity relative to K and relative nullity the following inequalities hold*

$$\nu(P) \leqq \mu(P, K) \leqq N + \nu(P)$$

at any point $P \in V_n$.

Theorem 4 and Theorem 1 clearly give the theorem:

THEOREM 5. *If a compact Riemann manifold of dimension n has at every point an index of nullity relative to constant $K \geqq \mu_0$, it cannot be isometrically imbedded in a μ-domain of a Riemann manifold of constant curvature K of dimension $n+\mu_0-1$.*

In order to verify the theorem we need minor modifications of the argument on locally maximum distance points.

According to $K \leqq 0$ or >0, by means of Theorems 2, 3, 4, we obtain especially more detailed theorems as follows:

THEOREM 6. *If a compact Riemann manifold of dimension n has at every point an index of nullity relative to non-positive constant $K \geqq \mu$, it cannot be isometaically imbedded in a complete Riemann manifold of constant curvature K of dimension $n+\mu_0-1$ so that it is disjointed from the minimal point locus with respect to any point of the Riemann manifold.*

THEOREM 7. *If a compact Riemann manifold of dimension n has at every point an index of nullity relative to positive constant $K \geqq \mu_0$ and its diameter $< \pi/2\sqrt{K}$, it cannot be isometrically imbedded in a complete Riemann manifold of constant curvature K of dimension $n+\mu_0-1$.*

6. Some theorems on isometric imbedding. Let V_{n+N} be a Riemann manifold of dimension $n+N(n \geqq 2, N \geqq 1)$ and class C^r. Let M be a differentiable submanifold of dimension n and class C^t in V_{n+N}. Let V_n be the Riemand manifold defined on M with the induced metric from V_{n+N}. At each point $P \in V_{n+N}$, let $(\bar{R})=\{P, \bar{e}_A\}$ be a rectangular frame and let $\bar{R}_{ABCE}$ be the components of the

Riemann-Christoffel tensor of V_{n+N} with respect to $(\bar{R})$. For $P \in M$, let $\{P, e_i, e_\alpha\}$ be a frame such that $e_i = \bar{e}_A P_i^A$ are tangent to M at P and $e_\alpha = \bar{e}_A Q_\alpha^A$, $\alpha = n+1, \cdots, n+N$, are mutually orthogonal normal unit vectors to M at P and let R_{ijkh} be the components of the Riemann-Christoffel tensor on V_n with respect to the frame $(R) = \{P, e_i\}$.

For any tangent plane element π spanned by mutually orthogonal tangent unit vectors $e_i\xi^i$, $e_i\eta^i$ to M at P, we have from (18′)

$$-R_{ijkh}\xi^i\eta^j\xi^k\eta^h = -R_{ABCE}P_i^A\xi^i P_j^B\eta^j P_k^C\xi^k P_h^E\eta^h + (A_{\alpha ik}A_{\alpha jh} - A_{\alpha ih}A_{\alpha jk})\xi^i\eta^j\xi^k\eta^h,$$

hence

$$K(\pi) = \bar{K}(\pi) + \Phi_\alpha(\xi, \xi)\Phi_\alpha(\eta, \eta) - \Phi_\alpha(\xi, \eta)\Phi_\alpha(\xi, \eta), \tag{33}$$

where $\bar{K}(\pi)$, $K(\pi)$ denote the sectional curvatures[16] for π of the spaces V_n, V_{n+N} respectively. $K(\pi) - \bar{K}(\pi)$ is called the relative sectional curvature.

Now let us assume that at every point of M there is a q-dimensional linear subspace in the tangent space of M along whose plane elements the relative sectional curvatures are non positive.

At any point $P \in M$, let $\mathfrak{T}(P)$ be the $q(P)$-dimensional linear subspace of the tangent space of M at P. If $e_i\xi^i$, $e_i\eta^i \in \mathfrak{T}(P)$, from (33) we have

$$\Phi_\alpha(\xi, \xi)\Phi_\alpha(\eta, \eta) - \Phi_\alpha(\xi, \eta)\Phi_\alpha(\xi, \eta) \leqq 0.$$

If $N < q(P)$, by virtue of Theorem 2 in Ôtsuki [12], there exists a tangent unit vector $e_i\xi^i$ such that

$$\Phi_\alpha(\xi, \xi) = 0, \qquad \alpha = n+1, \cdots, n+N.$$

Combining this with the results obtained in Section 4, we have the following theorems.

THEOREM 8. *Let V_{n+N} be a complete Riemann manifold of dimension $n+N (n \geqq 2, N \geqq 1)$ and let M be a submanifold of dimension n. Let us suppose that $N < \min_{P \in M} q(P)$, then for any point $O \in V_{n+N}$ and any geodesic coordinate neighborhood $U(O, x)$ in V_{n+N}, there exists no*

16) See Cartan [1], p. 195.

17) In the sense stated in Section 2, see Foot-note 3).

locally maximum (minimum) distance point P_0 $(\neq O)$ *of* M *from* O *in* V_{n+N} *such that* $P_0 \in U^+(O, x)$ $(U^-(O, x))$ *and the length of the geodesic arc joining* O *to* P *in* $U(O, x)$[17] *is equal to* dist(O, P) *at any point* $P \in |U(O, x)|$.

THEOREM 9. *Let* V_n *be a compact Riemann manifold with the property that at every point there is a* q*-dimensional linear subspace in the tangent space along whose plane elements the sectional curvatures are non-positive. Then* M *cannot be isometrically imbedded in a* μ*-domain of a Riemann manifold of dimension* $n+q-1$ *whose sectional curvatures are non-negative. Especially, if the diameter of* $V_n < \pi/2\sqrt{K}$, *then it cannot be isometrically imbedded in a Riemann space of constant curvature* K *and of dimension* $n+q-1$.

Lastly, returning to the beginning of the section, let us assume that at every point of M, the relative sectional curvatures in all plane elements are positive. Then we have from (33)

$$\Phi_\alpha(\xi, \xi)\Phi_\alpha(\eta, \eta) - \Phi_\alpha(\xi, \eta)\Phi_\alpha(\xi, \eta) < 0$$

for any two linearly independent vectors $e_i\xi^i$, $e_i\eta^i$. It follows from this $N \geqq n-1$[18]. For suppose $N \leqq n-2$. By the same reason as above, there exists a tangent vector $e_i\xi^i (\neq 0)$ such that $\Phi_\alpha(\xi, \xi) = 0$, $\alpha = n+1, \cdots, n+N$. Let $e_i\eta^i$ be a tangent vector linearly independent of $e_i\xi^i$ such that

$$\Phi_\alpha(\xi, \eta) = 0\,, \qquad \alpha = n+1, \cdots, n+N\,.$$

Then the relative sectional curvature will be zero in the plane element spanned by $e_i\xi^i$, $e_i\eta^i$, which contradicts to the assumption that it is strictly negative. Hence we have the theorem:

THEOREM 10. *An* n*-dimensional Riemann manifold of negative (non-positive) sectional curvature cannot be isometrically imbedded in a* $(2n-2)$*-dimensional Riemann manifold of non-negative (positive) sectional curvature.*

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18) The method of the following verification was suggested to the author by Prof. S. S. Chern in the case V_{n+N} is an Euclidean space.

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