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On lattice points in an n-dimensional ellipsoid.

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1. Main theorems.

1. Let $\sum_{i,k=1}^{n} a_{ik}x_i x_k$ $(n \geqq 2)$ be a positive definite quadratic form with the determinant $D=|a_{ik}|>0$, then

$$\sum_{i,k=1}^{n} a_{ik}x_i x_k \leqq r^2 \tag{1}$$

is the inside of an n-dimensional ellipsoid and $V(r)$ be its volume:

$$V(r)=\frac{\pi^{\frac{n}{2}} r^n}{\sqrt{D}\,\Gamma\left(\frac{n}{2}+1\right)}. \tag{2}$$

Let $n(r)$ be the number of lattice points contained in (1) and put

$$n(r)=V(r)+\Omega(r). \tag{3}$$

Then Landau[1] proved that

$$\Omega(r)=O\left(r^{n-\frac{2n}{n+1}}\right) \quad (n \geqq 4) \tag{4}$$

and many researches are made concerning the order of $\Omega(r)$ by Landau, Walfisz, Jarnik and others. We shall prove

THEOREM 1. $\int_1^r \frac{\Omega(r)}{r^{n-1}}\,dr=O(1) \qquad (n \geqq 2)$.

We remark that the integral diverges, if we put the value of $\Omega(r)$ of (4) in it.

Let $a_i, k_i > 0$ $(i=1,2,\cdots,n)$ be integers and consider lattice points $(x_1,\cdots,x_n)$ contained in (1), such that

$$x_i \equiv a_i \pmod{k_i} \quad (i=1,2,\cdots,n) \tag{5}$$

1) E. Landau: Zur analytischen Zahlentheorie der definiten quadratischen Formen. Berliner Akademieber. 1915.

and $n(r; a, k)$ be the number of such lattice points and put

$$n(r; a, k)=\frac{1}{k_1\cdots k_n}V(r)+\Omega(r; a, k). \tag{6}$$

Then

THEOREM 2. $\int_1^r \frac{\Omega(r; a, k)}{r^{n-1}}dr=O(1) \quad (n\geqq 2).$

2. Let $\mathfrak{a}_1,\cdots,\mathfrak{a}_n$ be n linearly independent vectors in an n-dimensional space R_n through the origin O, then they span an n-dimensional parallelopiped D_0. Let G be the group of translations, which is generated by these vectors, then D_0 is its fundamental domain. Let Q be a point of D_0 and $Q^{(\nu)}$ $(\nu=0, 1, 2,\cdots)$ be its equivalents by G and $n(r, Q)$ be the number of $Q^{(\nu)}$ contained in a sphere S_r of radius r about the origin O and $v(r)$ be the volume of the inside of S_r. Then Theorem 1 and 2 can be deduced easily from the following theorem.

THEOREM 3. $\int_1^r \frac{n(r,Q)}{r^{n-1}}dr=\frac{1}{v(D_0)}\int_0^r \frac{v(r)}{r^{n-1}}dr+O(1) \quad (n\geqq 2),$

where $v(D_0)$ is the volume of D_0.

To prove Theorem 3, we shall use a potential function on a torus. First we shall prove its existence.

2. Existence of a potential function on an n-dimensional torus.

1. If we identify the equivalent points of the opposite faces of D_0, then we obtain an n-dimensional torus Ω. A harmonic function $u(x_1,\cdots,x_n)$ on Ω is, by definition, a harmonic function in the $(x_1,\cdots, x_n)$-space, such that

$$\Delta u=\frac{\partial^2 u}{\partial x_1^2}+\cdots+\frac{\partial^2 u}{\partial x_n^2}=0.$$

If we put $r=\sqrt{(x_1-x_1^0)^2+\cdots+(x_n-x_n^0)^2}$, then

$$u=\log\frac{1}{r} \quad (n=2), \qquad u=\frac{1}{r^{n-2}} \quad (n\geqq 3)$$

are the simplest harmonic functions.

Let $P\in D_0$, $Q\in D_0$ and $Q^{(\nu)}$ be the equivalents of Q, then we define the distance $r=PQ$ by $r=\underset{\nu}{\text{Min.}}\, PQ^{(\nu)}$, thus we define the metric on Ω.

We shall prove

THEOREM 4. *Let Q_1, Q_2 be two points of Ω, then there exists a potential function $v(P; Q_1, Q_2)$ on Ω, which is harmonic, except at Q_1, Q_2, where if $n \geqq 3$,*

(i)
$$v(P; Q_1, Q_2)-\frac{1}{PQ_1^{n-2}} \text{ is harmonic and vanishes at } P=Q_1,$$
$$v(P; Q_1, Q_2)+\frac{1}{PQ_2^{n-2}} \text{ is harmonic and vanishes at } P=Q_2.$$

(ii) *Let Q_2 be fixed and $U(Q_2)$ be its neighbourhood and Q_1 vary in $\Omega-U(Q_2)$, then there exist constants $\rho>0$, $K>0$, which are independent of Q_1, such that if P lies in a ρ-neighourhood of Q_1, then*

$$\left|v(P; Q_1, Q_2)-\frac{1}{PQ_1^{n-2}}\right| \leqq K.$$

A similar relation holds at Q_2 with $\frac{1}{PQ_2^{n-2}}$ instead of $\frac{-1}{PQ_1^{n-2}}$, if Q_2 varies in $\Omega-U(Q_1)$.

If $n=2$, then $\frac{1}{PQ_1^{n-2}}$, $\frac{1}{PQ_2^{n-2}}$ are replaced by $\log\frac{1}{PQ_1}$, $\log\frac{1}{PQ_2}$ respectively.

PROOF. We assume that $n \geqq 3$, the case $n=2$ can be proved similarly, if we take $\log\frac{1}{PQ}$ instead of $\frac{1}{PQ^{n-2}}$.

Let k be a positive integer and $S_k (k \geqq k_0)$ be a sphere of radius $\frac{1}{k}$ about Q_2 and (S_k) be its inside, where k_0 is taken so large that $(\bar{S}_{k_0})$ does not contain Q_1.

We put $\Omega_k=\Omega-(\bar{S}_k)$. Then by Parreau's method,[2] we can prove that, there exists a Green's function $g_k(P; Q_1)$ on Ω_k with Q_1 as its pole, such that $g_k(P; Q_1)$ is harmonic on Ω_k, except at Q_1, where $g(P; Q_1)-\frac{1}{PQ_1^{n-2}}$ is harmonic at $P=Q_1$ and $g_k(P; Q_1)=0$ on S_k.

We draw about Q_1 a sphere σ_0 of radius ρ_0 and a sphere σ_1 of radius ρ_1 $(\rho_0<\rho_1)$, where ρ_1 is taken so small that σ_1 is contained in Ω_{k_0}

2) M. Parreau: Sur les moyennes des fonctions harmoniques et analytiques et la classification des surfaces de Riemann. Thèse. Paris. 1952.

and let (σ_0), (σ_1) be the inside of σ_0, σ_1 respectively.

Let

$$M_k = \operatorname*{Max}_{P\in\sigma_0} g_k(P; Q_1) \quad (k \geqq k_0)\,, \tag{1}$$

then by the maximum principle,

$$u_k(P) = M_k - g_k(P; Q_1) \geqq 0 \text{ in } \Omega_k - (\bar{\sigma}_0)\,.$$

Since $u_{k_0}(P) - u_k(P)$ is harmonic in Ω_{k_0} and $u_{k_0}(P) \geqq 0$ on σ_0, and $u_k(P_0)=0$ at some point P_0 on σ_0, we have

$$\operatorname*{Max}_{P\in\sigma_0} \left(u_{k_0}(P) - u_k(P)\right) \geqq 0\,,$$

so that by the maximum principle, $\operatorname*{Max}_{P\in\sigma_1} \left(u_{k_0}(P) - u_k(P)\right) \geqq 0$, or

$$\operatorname*{Min}_{P\in\sigma_1} u_k(P) \leqq \operatorname*{Max}_{P\in\sigma_1} u_{k_0}(P)\,. \tag{2}$$

Since $u_k(P) \geqq 0$ in $\Omega_k - (\sigma_0)$, by Harnack's theorem, for any compact domain $\Delta \subset \Omega - (\bar{\sigma}_0) - (Q_2)$, which has a positive distance from σ_0, we have from (2), if $k \geqq k_1$,

$$u_k(P) = |M_k - g_k(P; Q_1)| \leqq K(\Delta)\,, \quad P\in\Delta\,, \quad (k \geqq k_1)\,, \tag{3}$$

where k_1 is taken so large than $\Delta \subset \Omega_{k_1}$ and $K(\Delta)$ is a constant depending on Δ only.

Hence

$$\left| g_k(P; Q_1) - M_k - \frac{1}{PQ_1^{n-2}} \right| \leqq \text{const. on } \sigma_1\,(k \geqq k_0)\,. \tag{4}$$

Since the left hand side of (4) is harmonic in (σ_1), the same relation holds in (σ_1), so that if we put

$$\lim_{P\to Q_1} \left(g_k(P; Q_1) - \frac{1}{PQ_1^{n-2}} \right) = \gamma_k\,, \tag{5}$$

then $|\gamma_k - M_k| \leqq$ const. $(k \geqq k_0)$, hence by (3),

$$|g_k(P; Q_1) - \gamma_k| \leqq K(\Delta)\,, \quad P\in\Delta\,, \quad (k \geqq k_2)\,, \tag{6}$$

where Δ is any compact domain in $\Omega - (Q_1) - (Q_2)$.

Hence we can find k_ν, such that

$$\lim_{\nu} \left(g_{k_\nu}(P; Q_1) - \gamma_{k_\nu} \right) = v(P; Q_1, Q_2) \tag{7}$$

converges uniformly in the wider sense in $\Omega-(Q_1)-(Q_2)$, so that $v(P; Q_1, Q_2)$ is harmonic on Ω, except at Q_1, Q_2.

Since

$$\left| g_k(P; Q_1)-\gamma_k-\frac{1}{PQ_1^{n-2}} \right| \leq \text{const.} \quad \text{on } \sigma_1 \, (k \geq k_0)\,,$$

the same relation holds in (σ_1), so that

$$\left| v(P; Q_1, Q_2)-\frac{1}{PQ_1^{n-2}} \right| \leq \text{const.} \quad \text{in } (\sigma_1)\,,$$

hence

$$v(P; Q_1, Q_2)-\frac{1}{PQ_1^{n-2}} \text{ is harmonic and vanishes at } Q_1\,. \tag{8}$$

Next we shall prove that $v(P; Q_1, Q_2)+\frac{1}{PQ_2^{n-2}}$ is harmonic at Q_2. We put

$$v_k(P)=g_k(P; Q_1)-\gamma_k\,, \tag{9}$$

then since $v_k(P)$ is harmonic in a ring domain $\varDelta(k, k_0)$, which is bounded by S_k anc S_{k_0}, we have for $P\in\varDelta(k, k_0)$,

$$\begin{aligned} v_k(P)= & \frac{1}{(n-2)A_n}\int_{S_{k_0}}\left(v_k \frac{\partial}{\partial \nu}\left(\frac{1}{r^{n-2}}\right)-\frac{1}{r^{n-2}}\cdot\frac{\partial v_k}{\partial \nu}\right) d\sigma_Q \\ & +\frac{1}{(n-2)A_n}\int_{S_k}\left(v_k \frac{\partial}{\partial \nu}\left(\frac{1}{r^{n-2}}\right)-\frac{1}{r^{n-2}}\cdot\frac{\partial v_k}{\partial \nu}\right) d\sigma_Q\,, \end{aligned} \tag{10}$$

$$v_k=v_k(Q)\,, \quad r=PQ\,,$$

where A_n is the area of a unit sphere, ν is the inner normal and $d\sigma_Q$ is the surface element.

Since $v_k=-\gamma_k$ on S_k,

$$\int_{S_k} v_k \frac{\partial}{\partial \nu}\left(\frac{1}{r^{n-2}}\right) d\sigma_Q=-\gamma_k\int_{S_k}\frac{\partial}{\partial \nu}\left(\frac{1}{r^{n-2}}\right) d\sigma_Q=0\,,$$

and since

$$\int_{S_k}\frac{\partial v_k}{\partial \nu}\, d\sigma_Q=\int_{\sigma_0}\frac{\partial v_k}{\partial \nu}\, d\sigma_Q=(n-2)A_n\,,$$

we have

$$\int_{S_k} \frac{1}{r^{n-2}} \cdot \frac{\partial v_k}{\partial \nu} d\sigma_Q \to \frac{(n-2)A_n}{PQ_2^{n-2}} \quad (k \to \infty).$$

Hence we have from (10), for $P \in (S_{k_0})$

$$v(P; Q_1, Q_2) = \frac{1}{(n-2)A_n} \int_{S_{k_0}} \left(v \frac{\partial}{\partial \nu} \left(\frac{1}{r^{n-2}} \right) - \frac{1}{r^{n-2}} \cdot \frac{\partial v}{\partial \nu} \right) d\sigma_Q - \frac{1}{PQ_2^{n-2}},$$

so that

$$v(P; Q_1, Q_2) + \frac{1}{PQ_2^{n-2}} \text{ is harmonic at } Q_2. \tag{11}$$

If we put $P=Q_2$ in the integral and make $k_0 \to \infty$, then we see than $v(P; Q_1, Q_2) + \frac{1}{PQ_2^{n-2}}$ vanishes at $P=Q_2$.

Hence the part (i) is proved. The part (ii) can be proved easily from the above proof.

REMARK. We have taken a partial sequence k_ν in (7), but we see easily that

$$\lim_k \big(g_k(P; Q) - \gamma_k\big) = v(P; Q_1, Q_2)$$

converges uniformly in the wider sense in $\Omega-(Q_1)-(Q_2)$.

2. Let $\mathfrak{a}$ be a vector through a point Q $(\neq Q_2)$ and Q_1 be a point on $\mathfrak{a}$, such that $\overline{QQ_1} = \Delta\nu$, then in

$$v(P; Q_1, Q_2) - v(P; Q, Q_2)$$

the singularity at Q_2 vanishes, so that

$$\lim_{\Delta\nu \to 0} \frac{v(P; Q_1, Q_2) - v(P; Q, Q_2)}{\Delta\nu} = \frac{\partial v(P; Q, Q_2)}{\partial \nu} = v_1(P; Q) \tag{12}$$

is harmonic on Ω except at Q, where

$$v_1(P; Q) - \frac{(n-2)\cos\theta}{r^{n-1}}, \qquad r = \overline{PQ} \tag{13}$$

is harmonic, θ being the angle subtained by two vectors $\mathfrak{a}$ and $\overrightarrow{QP}$. Hence we have

THEOREM 5. *There exists a potential function $v_1(P; Q)$ on Ω, which is harmonic except at Q, where*

$$v(P;Q)-\frac{\cos\theta}{r^{n-1}}, \qquad r=\overline{QP}$$

is harmonic.

By differentiating $v_1(P;Q)$ with Q, we obtain a potential function on Ω with a polar singularity of any order $\geqq n-1$ at Q.

3. Proof of Main theorems.

1. RROOF of THEOREM 3.

We follows the same idea as I have used in the former paper on Fuchsian groups.[3] We assume that $n\geqq 3$, the case $n=2$ can be proved similarly.

Let D_0 be the n-dimensional parallelopiped, which is spanned by n vectors $\mathfrak{a}_1,\cdots,\mathfrak{a}_n$ through the origin O. By identifying the opposite faces of D_0, we obtain an n-dimensional torus Ω and let $v(P;Q,Q_1)$ be the potential function on Ω, which is defined by Theorem 4.
We put

$$u(P;Q,Q_1)=\frac{1}{(n-2)}v(P;Q,Q_1), \tag{1}$$

then $u(P;Q,Q_1)$ has singularities $\frac{1}{(n-2)}\cdot\frac{1}{PQ^{n-2}}$, $\frac{-1}{(n-2)}\cdot\frac{1}{PQ_1^{n-2}}$ at Q and Q_1 respectively.

$u(P;Q,Q_1)$ is invariant by the group G of translations, which is generated by $\mathfrak{a}_1,\cdots,\mathfrak{a}_n$.

Let S_r be a sphere of radius r about the origin O. We assume that there are no equivalents $Q^{(\nu)},Q_1^{(\nu)}$ of Q,Q_1 on S_1 and $S_R\,(R>1)$. Then applying Green's formula:

$$\int_S\left(u\frac{\partial v}{\partial\nu}-v\frac{\partial u}{\partial\nu}\right)d\sigma=0,$$

where S is the boundary, $d\sigma$ the surface element and ν the inner normal of S, to harmonic functions:

$$u(P)=u(P;Q,Q_1), \qquad v(P)=\frac{1}{r^{n-2}}-\frac{1}{R^{n-2}}, \qquad r=\overline{OP}$$

3) M. Tsuji: Theory of Fuchsian groups. Jap. Journ. Math. 21 (1951).

for the domain, which is obtained from the ring domain $\varDelta=\varDelta(1, R)$: $1<r<R$, by taking off the insides of small spheres about $Q^{(\nu)}, Q_1^{(\nu)}$ contained in $\varDelta$ and then making the radii of these spheres tend to zero, we have

$$\frac{(n-2)}{R^{n-1}}\int_{S_R} u(P; Q, Q_1)\, d\sigma_P + A_n \sum_\nu \left(\frac{1}{r_\nu^{n-2}} - \frac{1}{R^{n-2}}\right)$$
$$-A_n \sum_\nu \left(\frac{1}{r_\nu'^{n-2}} - \frac{1}{R^{n-2}}\right) = (n-2)\int_{S_1} u(P; Q, Q_1)\, d\sigma_P + \text{const.}, \qquad (2)$$

where A_n is the area of a unit sphere, $r_\nu = OQ^{(\nu)}$, $r_\nu' = OQ_1^{(\nu)}$ and $d\sigma_P$ is the surface element and we sum up for all $Q^{(\nu)}, Q_1^{(\nu)}$ contained in $\varDelta$.

Let $n(r, Q)$ be the number of $Q^{(\nu)}$ contained in S_r, then

$$\sum_\nu \left(\frac{1}{r_\nu^{n-2}} - \frac{1}{R^{n-2}}\right) = \int_1^R \left(\frac{1}{r^{n-2}} - \frac{1}{R^{n-2}}\right) dn(r, Q)$$
$$= \left[\left(\frac{1}{r^{n-2}} - \frac{1}{R^{n-2}}\right) n(r, Q)\right]_1^R + (n-2)\int_1^R \frac{n(r, Q)}{r^{n-1}}\, dr$$
$$= (n-2)\int_1^R \frac{n(r, Q)\, dr}{r^{n-1}} + O(1),$$

so that if we put $d\omega_P = \dfrac{d\sigma_P}{R^{n-1}}$ and writing r instead of R, we have from (2),

$$\frac{1}{A_n}\int_{S_r} u(P; Q, Q_1)\, d\omega_P + \int_1^r \frac{n(r, Q)\, dr}{r^{n-1}} = \int_1^r \frac{n(r, Q_1)\, dr}{r^{n-1}} + O(1).$$

We put $u^+ = u$, if $u \geq 0$ and $u^+ = 0$, if $u < 0$, then $u = u^+ - (-u)^+$, hence

$$\frac{1}{A_n}\int_{S_r} u^+(P; Q, Q_1)\, d\omega_P + \int_1^r \frac{n(r, Q)\, dr}{r^{n-1}} = \frac{1}{A_n}\int_{S_r} \left(-u(P; Q, Q_1)\right)^+ d\omega_P$$
$$+ \int_1^r \frac{n(r, Q_1)\, dr}{r^{n-1}} + O(1). \qquad (3)$$

We assumed that there are no $Q^{(\nu)}, Q_1^{(\nu)}$ on S_1 and S_R, but we see easily that (3) holds, if there are $Q^{(\nu)}, Q_1^{(\nu)}$ on S_1 and S_R, hence (3) holds in general.

As Nevanlinna, we put

$$m(r, Q)=\frac{1}{A_n}\int_{S_r} u^+(P; Q, Q_1)\, d\omega_P ,$$

$$N(r, Q)=\int_1^r \frac{n(r, Q)\, dr}{r^{n-1}} , \tag{4}$$

$$T(r, Q)=m(r, Q)+N(r, Q) ,$$

then from (3),

$$T(r, Q)=\frac{1}{A_n}\int_{S_r}\big(-u(P; Q, Q_1)\big)^+ d\omega_P+\int_1^r \frac{n(r, Q_1)dr}{r^{n-1}}+O(1) . \tag{5}$$

Let $U(Q_1)$ be a neighbourhood of Q_1. We consider Q_1 as fixed and Q vary in $D_0-U(Q_1)$, then by the part (ii) of Theorem 4, the term $O(1)$ in (5) is uniformly bounded. Hence for any Q, Q_0 in $D_0-U(Q_1)$, we have

$$T(r, Q)=T(r, Q_0)+O(1) . \tag{6}$$

Let dv_Q be the volume element, then for any $P\in D_0$,

$$\int_{D_0-U(Q_1)} u^+(P; Q, Q_1)\, dv_Q < \text{const.} ,$$

so that from (4) and (6)

$$\int_{D_0-U(Q_1)} N(r, Q)\, dv_Q+O(1)=v\big(D_0-U(Q_1)\big)\big(T(r, Q_0)+O(1)\big) . \tag{7}$$

For $Q\in U(Q_1)$, we put

$$T_1(r, Q)=\frac{1}{A_n}\int_{S_r}\big(-u(P; Q_0, Q)\big)^+ d\omega_P+N(r, Q) , \tag{8}$$

then from (5),

$$T_1(r, Q_1)=T(r, Q_0)+O(1) .$$

If we consider $-u$ instead of u, we have similarly as (6), $T_1(r, Q)=T_1(r, Q_1)+O(1)$ for $Q\in U(Q_1)$, so that

$$T_1(r, Q)=T(r, Q_0)+O(1) , \quad Q\in U(Q_1) . \tag{9}$$

Since for any $P\in D_0$,

$$\int_{U(Q_1)}\big(-u(P; Q_0, Q)\big)^+ dv_Q < \text{const.} ,$$

we have from (8), (9)

$$\int_{U(Q_1)} N(r, Q)\, dv_Q + O(1) = v\big(U(Q_1)\big)\,\big(T(r, Q_0) + O(1)\big). \tag{10}$$

Hence from (7), (10),

$$\begin{aligned} T(r, Q_0) &= \frac{1}{v(D_0)} \int_{D_0} N(r, Q)\, dv_Q + O(1) \\ &= \frac{1}{v(D_0)} \int_1^r \frac{dt}{t^{n-1}} \int_{D_0} n(t, Q)\, dv_Q + O(1) \\ &= \frac{1}{v(D_0)} \int_1^r \frac{v(t)\, dt}{t^{n-1}} + O(1), \end{aligned} \tag{11}$$

where $v(t)$ is the volume of the inside of S_t.

Hence from (6), we have

$$T(r, Q) = T(r) + O(1), \tag{12}$$

where

$$T(r) = \frac{1}{v(D_0)} \int_0^r \frac{v(r)\, dr}{r^{n-1}}. \tag{13}$$

This is an anlogue of R. Nevanlinna's first fundamental theorem for meromorphic functions.

We shall prove that $m(r, Q) = O(1)$.

Let $r_1 = r - d$, $r_2 = r + d\, (d > 0)$ and $Q^{(\nu)}$ be equivalents of Q and $U(Q^{(\nu)})$ be a neighbourhood of $Q^{(\nu)}$ of radius d, then $u^+(P; Q, Q_1)$ is bounded outside of $U(Q^{(\nu)})$ $(\nu = 0, 1, 2, \cdots)$, hence

$$\int_{S_r} u^+(P; Q, Q_1)\, d\omega_P = O(1) + \sum_\nu \int_{S_r \cdot U(Q^{(\nu)})} u^+(P; Q, Q_1)\, d\omega_P, \tag{14}$$

where we sum up for all $Q^{(\nu)}$, contained between S_{r_1} and S_{r_2}. Now

$$\int_{S_r \cdot U(Q^{(\nu)})} u^+(P; Q, Q_1)\, d\sigma_P \leqq K\,(=\text{const.})\quad (\nu = 0, 1, 2, \cdots),$$

where $d\sigma_P$ is the surface element, so that

$$\int_{S_r \cdot U(Q^{(\nu)})} u^+(P; Q, Q_1)\, d\omega_P \leqq \frac{K}{r^{n-1}}.$$

Since $n(r_2, Q) - n(r_1, Q) = O(r^{n-1})$, we have

$$\sum_\nu \int_{S_r \cdot U(Q^{(\nu)})} u^+(P; Q, Q_1)\, d\omega_P \leqq \frac{K}{r^{n-1}}\big(n(r_2, Q) - n(r_1, Q)\big) = O(1), \tag{15}$$

so that from (14), (15),

$$m(r, Q) = \frac{1}{A_n} \int_{S_r} u^*(P; Q, Q_1)\, d\omega_P = O(1). \tag{16}$$

Hence from (12), $N(r, Q) = T(r) + O(1)$, or

$$\int_1^r \frac{n(r, Q)\, dr}{r^{n-1}} = \frac{1}{v(D_0)} \int_0^r \frac{v(r)}{r^{n-1}} dr + O(1). \tag{17}$$

We assumed that Q lies outside of $U(Q_1)$, but if we consider $-u$ instead of u, we see that (17) holds, if Q lies in $U(Q_1)$. Hence (17) holds for any $Q \in D_0$. Hence our theorem is proved.

2. PROOF of THEOREM 1.

By an orthogonal transformation, we transform

$$\sum_{i,k=1}^{n} a_{ik}\, x_i\, x_k < r^2 \tag{1}$$

into

$$\frac{\xi_1^2}{a_1^2} + \cdots + \frac{\xi_n^2}{a_n^2} < r^2 \tag{2}$$

and then by $\xi_1 = a_1 X_1, \cdots, \xi_n = a_n X_n$, into

$$X_1^2 + \cdots + X_n^2 < r^2. \tag{3}$$

Let a unit cube: $0 \leq x_1 \leq 1, \cdots, 0 \leq x_n \leq 1$ be transformed into a parallelopiped D_0 in the $(X_1, \cdots, X_n)$-space, then $v(D_0) = \dfrac{1}{a_1 \cdots a_n}$. The number $n(r)$ of lattice points contained in (1) is equal to the number $n(r, O)$ of equivalents of the origin O contained in (3). Let $v(r)$ be the volume of (3).

Since $\dfrac{v(r)}{v(D_0)} = a_1 \cdots a_n\ v(r) = V(r)$, where $V(r)$ is the volume of (1), we have by Theorem 3,

$$\int_1^r \frac{n(r)}{r^{n-1}} dr = \int_1^r \frac{n(r, O)}{r^{n-1}} dr = \frac{1}{v(D_0)} \int_0^r \frac{v(r)}{r^{n-1}} dr + O(1) = \int_1^r \frac{V(r)\, dr}{r^{n-1}} + O(1),$$

or

$$\int_1^r \frac{\Omega(r)}{r^{n-1}} dr = O(1). \tag{4}$$

Hence Theorem 1 is proved.

Similarly we can prove Theorem 2.

Remark. Let $\lambda > 1$, then by (4) for any $r > 1$,

$$\int_r^{\lambda r} \frac{\Omega(r)}{r^{n-1}}\, dr < \text{const.}\,. \tag{5}$$

If $\Omega(r)$ is of constant sign in $[r, \lambda r]$, then considering $\inf|\Omega(r)|$ we see that there exists τ $(r < \tau < \lambda r)$, such that

$$|\Omega(\tau)| < \text{const.}\, \tau^{n-2}. \tag{6}$$

Now $[r, \lambda r]$ can be divided into a finite number of disjoint intervals, in each of which $\Omega(r)$ is continuous and decreasing, so that, if $\Omega(r)$ changes its sign in $[r, \lambda r]$, then there exists τ, such that $\Omega(\tau) = 0$ or in one of the intervals $[r, \frac{1+\lambda}{2} r]$, $[\frac{1+\lambda}{2} r, \lambda r]$, $\Omega(r)$ is of constant sign, hence there exists τ, which satisfies (6). Hence we have

Theorem 6. *For any $r > 1$, there exists τ $(r < \tau < \lambda r)$, such that*

$$|\Omega(\tau)| < \text{const.}\, \tau^{n-2} \qquad (n \geq 2).$$

Hence if $n = 2$, $|\Omega(\tau)| < \text{const.}$. We remark that in Landau's estimation $\Omega(r) = O\left(r^{n - \frac{2n}{n+1}}\right)$, $n - \dfrac{2n}{n+1} > n - 2$.

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