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On the divisors of differential forms on algebraic varieties.

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The theory of differential forms on compact kählerian manifolds has been much developed by the theory of harmonic integrals. But in algebraic geometry it is desirable to construct the theory, following A. Weil, independently of the characteristic of its universal domain in purely algebro-geometric way. We study here the divisors of differential forms on algebraic varieties.

First, we define the divisors of differential form ω on an abstract variety, reconstructing differential forms independently of its reference field. Then we consider a generic hyperplane section W of its ambient projective model V with reference to the field of definition for ω. Our interest lies in the relation between the divisors (ω) and $(\bar{\omega})$, where $\bar{\omega}$ is a differential form on W induced by ω. We shall obtain the following theorem: Let p be the degree of ω and r the dimension of V. Then, if $p \leqq r-2$ we have $(\bar{\omega})=(\omega) \cdot W$, and if $p=r-1$ we have $(\bar{\omega})=(\omega) \cdot W+X$, where X is a positive W-divisor. In the proof, the notion of generating subvariety of V with reference to a field of definition for V plays an essential rôle. Some relations might hold, as it seems to me, between our theorem and Lefschetz's theorem concerning invariant cycles and vanishing cycles on algebraic varieties.[1]

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1) Cf. Lefschetz [4]. Numbers in brackets refer to the bibliography at the end of the paper.

§ 1. Differential forms and the divisors of differential forms.

1. Let V be a Variety[2], k a field of definition for V and P a generic Point of V over k. Then the space of linear differential forms is defined as the dual module of derivations of $k(P)$ over k (cf. W–F, IX, 2)[3]. But in this definition the notion of differential forms are connected closely with its reference field. To avoid this inconvenience we shall define an equivalence relation of differential forms belonging to the different fields as follows[4].

(1) Let P and P' be two generic Points of V over k, then there exists an isomorphism σ of $k(P)$ onto $k(P')$ leaving every element of k invariant, and the space of linear differential forms belonging to the extension $k(P)$ over k are mapped isomorphically onto the space of linear differential forms belonging to the extension $k(P')$ over k, by the isomorphism σ^* induced by σ. In this case we shall write $\sigma^*(\omega) \approx \omega$.

(2) Let k and P be as above, K a field containing k such that K and $k(P)$ are linearly disjoint over k. Then there exists a unique differential form Ω belonging to the extension $K(P)$ over K which is an extension of ω. In this case we also write $\omega \approx \Omega$.

Let k be a field of definition for V, P a generic Point of V over k. We shall now consider all such pairs (k, P)[5] and linear differential forms $\omega_k(P)$ belonging to these pairs. Now we shall define an equivalence relation among $\omega_k(P)$ as follows.

Two differential forms $\omega_k(P)$ and $\Omega_{k'}(P')$ belonging respectively to the pairs (k, P) and (k, P') are said to be equivalent, if they can be connected by a finite number of relations $\approx$ defined in (1) and (2).

This is clearly an equivalence relation and we shall call the class thus defined "*linear differential forms on V*", and any member in this class will be called its *representative* and denoted as $\omega_k(P)$. Differential forms of higher degree can then be defined in a natural way. The justification of the above definition can be assured step by

2) In this paper we shall adopt the notation and terminology used in Weil [6].
3) This means Chapter IX, 2 of Weil [6].
4) This idea is due to a valuable suggestion by Prof. Igusa.
5) It is precisely the way to define a variety, cf. W–F, IV.

step but we do not go into the details. A differential form is said to be defined over k if it has a representative belonging to some pair such as (k, P).

2. Let us denote by $\Omega(V)$ the abstract field of functions on V^r, where Ω is the universal domain. Then since $\Omega(V)$ is a regular extension of dimension r over Ω, we can define differential forms in $\Omega(V)$ also from the dual of the derivations in $\Omega(V)$ over Ω and they can be expressed in the form

$$(\alpha) \qquad \omega = \sum_{i_1<\cdots<i_p} \varphi_{i_1\cdots i_p} d\tau_{i_1}\cdots d\tau_{i_p},$$

where $\varphi_{i_1\cdots i_p}$ and τ_i are functions on V. Let k be a common field of definition for $\varphi_{i_1\cdots i_p}$ and τ_i, and P a generic Point of V over k, then the local expression of ω

$$\omega_k(P) = \sum_{i_1<\cdots<i_p} \varphi_{i_1\cdots i_p}(P)\, d\tau_{i_1}(P)\cdots d\tau_{i_p}(P)$$

determines uniquely a differential form belonging to the pair (k, P) and they are equivalent in the sense of n°1. Conversely, equivalent differential forms determine uniquely a differential form in $\Omega(V)$ such that its local expressions are the representatives of a differential form defined in n°1. Hence it is quite natural to identify these two notions. In the following we shall adopt the expression (α) for differential form ω.

REMARK. From the definitions, the results stated in Koizumi (2, 3) can then be applied directly to our differential forms.

For convenience we shall extend the notion of uniformizing parameters at a simple Point on a Variety as follows.

DEFINITION 1. *Let $\tau_1, \cdots, \tau_r$ be r-functions on V^r, k a common field of definition for τ_i, P a generic Point of V over k and P' a simple Point of V. Then (τ) will be called uniformizing parameters at P' on V if $(\tau_i(P))$ are uniformizing parameters at P' on V in the sense of* W-F, IX, 2.

As it can be seen easily the above definition does not depend on the choice of k and P, and our terminologies are reasonable. In the following we shall always use the word uniformizing parameters in the sense of DEF. 1. When we use it in the sense of Weil [6], i.e. when considered as *quantities*, we shall call them *uniformizing Q-parameters.*

Thus using a uniformizing parameters at a simple Point of V, any differential form can be expressed uniquely in the form

$$\omega = \sum_{i_1 < \cdots < i_p} \varphi_{i_1 \cdots i_p} d\tau_{i_1} \cdots d\sigma_{i_p}.$$

3. Let ω be a differential form on V^r of degree p, A^{r-1} any simple Subvariety of V and $\tau_1, \cdots, \tau_r$ uniformizing parameters along A on V (it means that (τ) are uniformizing parameters at some Point of A on V, hence also at any generic Point of A over its field of definition). Then ω can be expressed uniquely in the form

$$\omega = \sum_{i_1 < \cdots < i_p} \varphi_{i_1 \cdots i_p} d\tau_{i_1} \cdots d\tau_{i_p}$$

and $v_A\big((\omega)\big)$ is defined as the minimum value of $v_A\big((\varphi_{i_1 \cdots i_p})\big)$. To justify this definition it is necessary and sufficient to show that the number $v_A\big((\omega)\big)$ is independent of the choice of the uniformizing parameters (τ) used to define it. For that purpose we shall introduce the notion of partial derivatives.

DEFINITION 2. *Let f be a function on V and $\tau_1, \cdots, \tau_r$ separating transcendence basis of $\Omega(V)$ over Ω, then the differential df can be expressed uniquely in the form $df = \varphi_1 d\tau_1 + \cdots + \varphi_r d\tau_r$. Then we shall put $\partial f / \partial \tau_i = \varphi_i$.*

Now the proposition 2 of Koizumi [2] can be restated with a slight modification as follows.

PROPOSITION 1. *Let f be a function on V, P' a simple Point of V such that f is defined and finite at P' and $\tau_1, \cdots, \tau_r$ uniformizing parameters on V at P'. Then $\partial f / \partial \tau_i$ $(1 \leqq i \leqq r)$ are defined and finite at P'.*

Let $\tau'_1, \cdots, \tau'_r$ be another set of uniformizing parameters on V along A, then we have

$$\omega = \sum_{j_1 < \cdots < j_p} \varphi'_{j_1 \cdots j_p} d\tau'_{j_1} \cdots d\tau'_{j_p}$$

where $\varphi'_{j_1 \cdots j_p}$ are linear combinations of $\varphi_{i_1 \cdots i_p}$ with the coefficients of the form $\partial \tau_{i_1} / \partial \tau'_{j_1} \cdots \partial \tau_{i_p} / \partial \tau_{j_p}$ which are defined and finite along A by the above proposition. Hence we must have

$$v_A\big((\varphi'_{j_1 \cdots j_p})\big) \geqq \underset{(i)}{\mathrm{Min.}} \big(v_A\,((\varphi_{i_1 \cdots i_p}))\big),$$

where $\{i_1,\cdots,i_p\}$'s are sequences of indices such that they appeared in the expression of $\varphi'_{j_1\cdots j_p}$ as the linear combinations of $\varphi_{i_1\cdots i_p}$. Then

$$\operatorname*{Min.}_{(j)} v_A\big((\varphi'_{j_1\cdots j_p})\big) \geqq \operatorname*{Min.}_{(i)} \big(v_A((\varphi_{i_1\cdots i_p}))\big)\,.$$

where (i) and (j) denote the sets of all sequences of indices such that $i_1<\cdots<i_p$, $j_1<\cdots<j_p$. In the same way we have the converse inequality and our assertion is proved.

Now we can define the divisors of a differential form ω as

$$(\omega)=\sum_A v_A\big((\omega)\big)\cdot A\,,$$

where $\sum$ denotes the sum over all simple Subvarieties of dimension $r-1$ of V^r. It is to be noted that any component of (ω) is algebraic over the field of definition for ω.

REMARK. For the notion of partial derivatives we see easily that the usual rules hold also, e. g. $\partial/\partial\tau_i(\partial\varphi/\partial\tau_j)=\partial/\partial\tau_j(\partial\varphi/\partial\tau_i)$, hence $d^2=0$.

4. The following propositions are immediate consequences of the definition of (ω) and W-F, VIII, Th. 6, and the proofs will be omitted.

PROPOSITION 2. *Let ω be a differential form and f a function on V, then we have*

$$(f.\,\omega)=(f)+(\omega)\,.$$

PROPOSITION 3. *Let ω_1, ω_2 be two differential forms of degree p on V, then we have*

$$v_A\big((\omega_1+\omega_2)\big)\geqq \text{Min.}\big(v_A((\omega_1))\,,\ v_A((\omega_2))\big)$$

for any simple Svbvariety A^{r-1} of V^r, and the equality holds if we have $v_A\big((\omega_1)\big)\neq v_A\big((\omega_2)\big)$.

PROPOSITION 4. *Let ω_1, ω_2 be two differential forms on V of degrees p and q respectively such that $p+q\leqq r$, and A^{r-1} any simple Subvariety of V^r, then we have*

$$v_A\big((\omega_1\cdot\omega_2)\big)\geqq v_A\big((\omega_1)\big)+v_A\big((\omega_2)\big)\,.$$

THEOREM 1. *Let V be a Variety, ω a differential form of the first kind, then we have $(\omega)>0$. When V is a complete non-singular Variety the converse is also true*

This is an immediate consequence of the proposition 5 of Koizumi [2] and the proof will be omitted.

5. Let V be a Variety, U its simple Subvariety and ω a differential form on V finite along U. Then using the uniformizing parameters $\tau_1, \cdots, \tau_r$ along U on V, ω can be expressed uniquely in the form

$$\omega = \sum_{i_1 < \cdots < i_p} \varphi_{i_1 \cdots i_p} d\tau_{i_1} \cdots d\tau_{i_p} .$$

Then the differential form $\bar{\omega}$ on U induced by ω can be defined as in Koizumi [2] and has the expression

$$\bar{\omega} = \sum_{i_1 < \cdots < i_p} \bar{\varphi}_{i_1 \cdots i_p} d\bar{\tau}_{i_1} \cdots d\bar{\tau}_{i_p} ,$$

where $\bar{\varphi}_{i_1 \cdots i_p}$ and $\bar{\tau}_i$ are functions on U induced by $\varphi_{i_1 \cdots i_p}$ and τ_i respectively. It is to be noted that this expression is not necessarily reduced.

§ 2. Some lemmas on uniformizing parameters.

PROPOSITION 5. *Let V be an algebraic variety in S^N, $P=(x_1, \cdots, x_N)$ a generic point of V over a field of definition k for V, and P' a simple point of V. Then we can choose among (x_i) $(1 \leq i \leq N)$ uniformizing Q-parameters on V at P.*

PROOF. Let $\mathfrak{P}$ be the defining ideal of V in $k[X]$. Since P' is a simple point of V, we can find $N-r$ polynomials $F_j(X)$ $(1 \leq j \leq N-r)$ in $\mathfrak{P}$ such that the rank of the matrix

$$(\partial F_j / \partial x'_i) \qquad \begin{pmatrix} 1 \leq j \leq N-r \\ 1 \leq i \leq N \end{pmatrix}$$

is $N-r$. Suppose that we have

$$|\partial F_j / \partial x'_i| \neq 0; \qquad \text{for} \qquad 1 \leq j \leq N-r, \qquad r+1 \leq i \leq N,$$

then r-polynomials X_i $(1 \leq i \leq r)$ constitute a set of uniformizing linear forms at P' on V, hence $x_1, \cdots, x_r$ are uniformizing Q-parameters on V at P' by W-F, IX, 2. Q. E. D.

Let V^r be a projective model of an algebraic variety immersed in a projective space L^N and $P=(\xi_0, \cdots, \xi_N)$ a generic Point of V over k. Set $x_i=\xi_i/\xi_0$ and let $u_1, \cdots, u_N$ be N-independent variables over $k(x)$. Then if we put

$$u_0=-(u_1 x_1+\cdots+u_N x_N)$$

and

$$K=k(u_0, u_1, \cdots, u_N)$$

$K(x)$ is a regular extension of K[6] of dimension $r-1$ and P has a locus W over K. Then it is seen that we have

$$W=V\cdot H,$$

where H is defined by the equation

$$u_0 X_0+u_1 X_1+\cdots+u_N X_N=0.$$

We shall remark that any Point of W which is simple on V is simple on W and vice versa. Hence if V has no singular Subvariety of dimension $r-1$, then W has no singular Subvariety of dimension $r-2$, especially when V has no singular Point, then W is also a non-singular Variety.[7]

Let P' be any Point on W and $(\xi'_0, \xi'_1, \cdots, \xi'_N)$ be its homogeneous coordinates and suppose that $\xi_0\neq 0$, then from prop. 5 we can select r-quantities among $x_1, \cdots, x_N$ such that $x_{i_1}, \cdots, x_{i_r}$ are uniformizing Q-parameters on V at P', and we have the following

PROPOSITION 6.[8] *There exist $(r-1)$ quantities $x_{j_1}, \cdots, x_{j_{r-1}}$ among $x_{i_1}, \cdots, x_{i_r}$ such that they are uniformizing Q-parameters on W at P'.*

PROOF. Without loss of generalities we can suppose that $x_1, \cdots, x_r$ are uniformizing Q-parameters at P' on V. Then there exist $N-r$ polynomials $F_j(X)$ in $\mathfrak{P}$ $(1\leq j\leq N-r)$ such that we have

6) Cf. J. Igusa [1]
7) Cf. Y. Nakai [5]
8) This proposition can be generalized as follows;
Let V^r be a Variety U^s a simple Subvariety of V, P' a Point of U simple on both of U and V, and $\tau_1, \cdots, \tau_r$ uniformizing parameters on V at P'. Then we can choose among $(\bar{\tau}_i)$ s-functions such that they form a set of uniformizing parameters on U at P', where $\bar{\tau}_i$'s are functions on U induced by the functions τ_i.

$$|\partial F_j/\partial x'_i| \neq 0 \quad \begin{pmatrix} 1 \leq j \leq N-r \\ r+1 \leq i \leq N \end{pmatrix},$$

where $x'_i = \xi'_i/\xi'_0$. Then the tangential linear variety to $\boldsymbol{W}$ at $\boldsymbol{P}'$ is given by the equations

$$\sum_{i=1}^{N} \partial F_j/\partial x'_i \cdot (X_i - x'_i \cdot X_0) = 0 \qquad (1 \leq j \leq N-r),$$

$$\sum_{i=0}^{N} u_i X_i = 0 .$$

Then to prove the assertion it is necessary and sufficient to show that at least one of the determinants

$$\begin{vmatrix} \partial F_1/\partial x'_j, & \partial F_1/\partial x'_{r+1}, \cdots\cdots, \partial F_1/\partial x'_N \\ \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots \\ \partial F_{N-r}/\partial x'_j, & \partial F_{N-r}/\partial x'_{r+1}, \cdots, \partial F_{N-r}/\partial x'_N \\ u_j, & u_{r+1}, \cdots\cdots, u_N \end{vmatrix} \quad (j=1, \cdots, r)$$

is different from 0. But this is shown already in Nakai [5].

§ 3. Generating subvarieties.

7. DEFINITION 3. *Let V^r be a Variety defined over k, $\boldsymbol{B}^s$ be any Subvariety of V defined over a field K containing k. Then we shall say that $\boldsymbol{B}$ is a generating Subvariety of V with reference to k if $\boldsymbol{B}$ contains a Point $\boldsymbol{P}$ which is r-dimensional over k.*

Let V, W, k and K be as in n°6 and let A be a Subvariety of V algebraic over k, then we see easily that any component of $A \cdot \boldsymbol{W}$ is not a generating Subvariety of V with reference to k. Now we have the

PROPOSITION 7. *Let $\overline{A}^{r-2}$ be any simple Subvariety of W^{r-1} algebraic over K, and $\boldsymbol{Q}$ a generic Point of $\overline{A}$ over $\overline{K}$, then we have* $\dim_{k(Q)}(u_0, u_1, \cdots, u_N) \geq N-1$. *The equality hold if and only if $\overline{A}$ is a generating Subvariety of V with reference to k. Moreover when $\overline{A}$ is not a generating Subvariety of V with reference to k and $r \geq 3$, there exists a unique Subvariety A^{r-1} of V algebraic over k such that we have $\overline{\boldsymbol{A}} = \boldsymbol{A} \cdot \boldsymbol{W}$.*

PROOF. Let $\eta_0, \eta_1, \cdots, \eta_N$ be the homogeneous coordinates of Q and suppose that $\eta_0 \neq 0$, and put $K_1 = k(u_1, \cdots, u_N)$. Since $\bar{A}$ lies on W we have $K_1(Q) \supset K$, and hence $\dim_{K_1}(Q) = \dim_{K_1}(u_0) + \dim_K(Q) = r-1$, and $r \geqq \dim_k(Q) \geqq r-1$. Moreover $\dim_k(Q) + \dim_{k(Q)}(u_1, \cdots, u_N) = \dim_k(u_1, \cdots, u_N) + \dim_{K_1}(Q) = N+r-1$, then we have $\dim_{k(Q)}(u_1, \cdots, u_N) \geqq N+r-1-r = N-1$ and we have the equality if and only if $\dim_k(Q) = r$. Hence if $\bar{A}$ is not a generating Subvariety of V with reference to k we have $\dim_k(Q) = r-1$, and the locus of Q over $\bar{k}$ determines a Subvariety A of V algebraic over k. Moreover if $r \geqq 3$, $A \cdot W$ is irreducible and clearly $A \frown W \supset \bar{A}$, then we must have $\bar{A} = A \cdot W$. Since $[A \cdot W]_V = [A \cdot H]_L$ and H is a generic hyperplane with reference to k, such a Variety is determined uniquely.

When $r=2$ the last half of the above proof fails in showing the existance of A such that $\bar{A} = A \cdot W$. But in this case we can see easily that we have $i(A \cdot W, \bar{A}; V) = 1$ for A defined in the proof. The existence and uniqueness of such a Variety will be used in the next paragraph.

§ 4. Main theorem.

8. THEOREM 2. *Let V^r be a projective model of an algebraic Variety, ω a differential form of degree $p\,(\leqq r-1)$ on V, k a field of definition for ω and W a generic hyperplane section of V with reference to k. Then ω induces on W a non-zero differential form $\bar{\omega}$ of the same degree and we have*

$$(\bar{\omega}) = (\omega) \cdot W \qquad \textit{for} \quad p \leqq r-2$$

and

$$(\bar{\omega}) = (\omega) \cdot W + \bar{X} \qquad \textit{for} \quad p = r-1,$$

where $\bar{X} > 0$ and every component of $\bar{X}$ is a generating Subvariety of V with reference to k.

PROOF. Since the notions of a Variety and differential forms are independent of its reference field, we can suppose without loss of generalities that k is algebraically closed. Let

$$u_0 X_0 + u_1 X_1 + \cdots + u_N X_N = 0$$

be the defining equation for a generic hyperplane in the ambient projective space with reference to k, and put $K=k(u)$. Then $W=V\cdot H$ is defined over K. Let P be a generic Point of W over K, then P is also a generic Point of V over k and we have $\dim_{k(P)}(u)=N$. Let A be a component of (ω), then since ω is defined over k, A is also defined over k. Then $\overline{A}=A\cdot W$ is defined and irreducible for $r>3$, prime rational cycle for $r=2$, over K. Put $a=v_A\left((\omega)\right)$. Let A_0 be a representative of A, V_0 the representative of V on which A_0 lies and $P_0=(1, x_1,\cdots, x_N)$ be the representative of P in V_0. Then we can suppose from propositions 5 and 6 that $x_1,\cdots, x_r$ are uniformizing Q-parameters on V along $\overline{A}$ and $x_1,\cdots, x_{r-1}$ are uniformizing Q-parameters on W along $\overline{A}$. Let τ_i be the functions on V defined over k by $\tau_i(P)=x_i$ $(1\leq i\leq r)$, then ω can be expressed uniquely in the form

$$\omega=\sum_{i_1<\cdots<i_p}\varphi_{i_1\cdots i_p}\,d\tau_{i_1}\cdots d\tau_{i_p}\,, \tag{1}$$

where $\varphi_{i_1\cdots i_p}$ are functions on V defined over k. Then by the definition we have

$$a=\operatorname*{Min}_{(i)}\left(v_A((\varphi_{i_1\cdots i_p}))\right),$$

We shall denote by $^{-}$ the functions on W induced by the functions on V, then $\overline{\omega}$ can be expressed in the form

$$\overline{\omega}=\sum_{i_1<\cdots<i_p}^{1}\overline{\varphi}_{i_1\cdots i_p}\,d\overline{\tau}_{i_1}\cdots d\overline{\tau}_{i_p}+\sum_{j_1<\cdots<j_{p-1}}^{2}\overline{\varphi}_{j_1\cdots j_{p-1}r}\,d\overline{\tau}_{j_1}\cdots d\overline{\tau}_{j_{p-1}}\,d\overline{\tau}_r\,, \tag{2}$$

where

$$d\overline{\tau}_r=-1/\overline{\alpha}_r\cdot\sum_{j=1}^{r-1}\overline{\alpha}_j\cdot d\overline{\tau}_j\,, \tag{3}$$

$$\overline{\alpha}_j=u_j+\sum_{s=r+1}^{N}u_s\cdot\overline{\psi}_{sj}\,,\qquad \psi_{sj}=\partial\tau_s/\partial\tau_j\,,\qquad (1\leq j\leq r)\,, \tag{4}$$

and $\sum_1$ denotes the sum over all sequences of indices $i_1<\cdots<i_p$, taken from $1,\cdots,r$, $\sum_2$ the sum over all sequences of indices $j_1<\cdots<j_{p-1}$ taken from $1,\cdots,r-1$ respectively. Substituting (3) in (2) we have the reduced expression

$$(2') \qquad \bar{\omega} = \sum_{i_1<\cdots<i_p} (\bar{\varphi}_{i_1\cdots i_p} - \bar{\alpha}_{i_p}/\bar{\alpha}_r \cdot \bar{\varphi}_{i_1\cdots i_{p-1}r} + \cdots$$
$$\cdots + (-1)^p\, \bar{\alpha}_{i_1}/\bar{\alpha}_r \cdot \bar{\varphi}_{i_2\cdots i_p r})\, d\bar{\tau}_{i_1}\cdots d\bar{\tau}_{i_p}\,.$$

From this expression of $\bar{\omega}$ and the fact that $u_1,\cdots,u_N$ are independent variables over $k(P)$ we see at once $\bar{\omega}=0$ if and only if $\omega=0$. Let Q be a generic Point of $\bar{A}$[9] over $\bar{K}$, then since V is defined over k, the quantities $\bar{\varphi}_{i_1\cdots i_p}(Q)$ and $\bar{\psi}_{sj}(Q)$ are contained in $k(Q)$. By proposition 1, $\bar{\alpha}_j$ are finite along $\bar{A}$; moreover $\bar{\alpha}_j$ cannot be zero along $\bar{A}$. In fact, if it is not true we have $\bar{\alpha}_j(Q)=0$, and this is a non-identically zero relation among $u_1,\cdots,u_N$ with coefficients in $k(Q)$ (u_j is contained in $\bar{\alpha}_j(Q)$ with coefficient 1). But since $\bar{A}$ is not a generating Sub-variety of V with reference to k it is impossible by prop. 7. We shall now show

$$(5) \qquad v_{\bar{A}}\big((\bar{\varphi}_{i_1\cdots i_p} - \bar{\alpha}_{i_p}/\bar{\alpha}_r \cdot \bar{\varphi}_{i_1\cdots i_{p-1}r} + \cdots + (-1)^p\, \bar{\alpha}_{i_1}/\bar{\alpha}_r \cdot \bar{\varphi}_{i_2\cdots i_p r})\big)$$
$$= \mathrm{Min}\,\big(v_{\bar{A}}((\bar{\varphi}_{i_1\cdots i_p})),\cdots, v_{\bar{A}}((\bar{\varphi}_{i_2\cdots i_p r}))\big)\,.$$

In fact, let the right hand side be equal to b and f a function on V defined over k such that we have $v_A\big((f)\big)=1$. Then to prove the assertion it is sufficient to show that the function

$$\frac{\bar{\varphi}_{i_1\cdots i_p}}{\bar{f}^b} - \frac{\bar{\alpha}_{i_p}}{\bar{\alpha}_r}\cdot\frac{\bar{\varphi}_{i_1\cdots i_{p-1}r}}{\bar{f}^b} + \cdots + (-1)_p\frac{\bar{\alpha}_{i_1}}{\bar{\alpha}_r}\cdot\frac{\bar{\varphi}_{i_2\cdots i_p r}}{\bar{f}^b}$$

cannot be zero along $\bar{A}$. But in the above expression there exists at least one term which is not zero along $\bar{A}$, hence it is impossible by the same reasoning as above and we get the equality (5). Combining the equality (5) and the fact $v_A\big((\varphi_{i_1\cdots i_p})\big)=v_{\bar{A}}\big((\bar{\varphi}_{i_1\cdots i_p})\big)$ we see at once $v_{\bar{A}}\big((\bar{\omega})\big)=a$. Finally we see that we have

$$(6) \qquad (\bar{\omega})=(\omega)\cdot W+\bar{X}\,.$$

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9) When $r=2$ it should be understood that Q be any Point of $\bar{A}$, and in the following $\bar{A}$ must be replaced by Q.

REMARK. The equality (5) holds for any Subvariety of W which is not a generating Subvariety of V with reference to k, we shall use this later.

Before going into the rest of the proof, we shall prove a

LEMMA. *Using the same notations as above, let* $\overline{B}^{r-2}$ *be any Subvariety of* W, $\tau_1,\cdots,\tau_r$ *uniformizing parameters on* V *along* $\overline{B}$ *and* $\tau_1,\cdots,\tau_{r-1}$ *uniformizing parameters on* W *along* $\overline{B}$, *then* $\overline{B}$ *cannot be a component of* $(\overline{\alpha}_r)_0$, *where* α_r *is defined as in the above proof.*

PROOF. Let $M=(1, y_1,\cdots, y_N)$ be a generic Point of $\overline{B}$ over its field of definition, then from prop. 5 and 6 our hypothesis tells us the existence of polynomials $F_j(X)$ in $k[X]$ such that

$$|\partial F_j/\partial y_s| \neq 0 \qquad (1\leq j\leq N-r,\quad r+1\leq s\leq N),$$

$$\begin{vmatrix} \partial F_j/\partial y_t \\ u_t \end{vmatrix} \neq 0 \qquad (1\leq j\leq N-r,\quad r\leq t\leq N),$$

$$F_j(x)=0 \qquad (1\leq j\leq N-r).$$

Multiplying the $(i+1)$-th column of the second determinant $\overline{\varphi}_{r+i,r}(M)$ $(1\leqq i\leqq N-r)$ and adding to the first column, we have

$$\begin{vmatrix} 0, & \partial F_j/\partial y_s \\ \overline{\alpha}_r(M), & u_s \end{vmatrix} \neq 0 \qquad \begin{pmatrix} 1\leq j\leq N-r, \\ r+1\leq s\leq N \end{pmatrix},$$

i. e.

$$\overline{\alpha}_r(M)\cdot|\partial F_j/\partial y_s| \neq 0 .$$

Thus we have $\overline{\alpha}_r(M)\neq 0$. Q. E. D.

We are now in the position to prove that $\overline{X}$ appeared in (6) is positive. Let $\overline{B}$ be any component of $\overline{X}$ which is not contained in $(\omega)\cdot W$, and suppose that $\tau_1,\cdots,\tau_r$ are uniformizing parameters on V along $\overline{B}$ and $\overline{\tau}_1,\cdots,\overline{\tau}_{r-1}$ uniformizing parameters on W along $\overline{B}$. Then we can express ω and $\overline{\omega}$ in the form (1) and (2') respectively. First we shall show that $\overline{B}$ is a generating Subvariety of V with reference to k. For, in the contrary case there exists a Variety B^{r-1} defined over k such that $\mathrm{i}(B\cdot W, \overline{B}; V)=1$ by prop. 7 and its remark, and it is unique. Hence B must be a component of (ω), which contradicts to our assumption that $\overline{B}$ is not a component of $(\omega)\cdot W$. Thus $\overline{B}$ is a generating Subvariety of V with reference to k and is not a component of $(\overline{\varphi}_{i_1\cdots i_p})=(\varphi_{i_1\cdots i_p})\cdot W$ for any set of indices $i_1,\cdots,i_p$. Now suppose that we have $v_{\overline{B}}((\overline{\omega}))<0$. By the previous lemma, $\overline{B}$ is not a component of $(\overline{\alpha}_r)_0$ and $\overline{\alpha}_1,\cdots,\overline{\alpha}_{r-1}$ are finite along $\overline{B}$, hence if we have $v_{\overline{B}}((\overline{\omega}))<0$ some of the functions $\overline{\varphi}_{i_1\cdots i_p}$ must be infinite along $\overline{B}$, but it is impossible by the above considerations. Thus we must have $v_{\overline{B}}((\overline{\omega}))\geq 0$, and the last assertion of our theorem is proved.

Now suppose that we have $p\leq r-2$ and $v_{\overline{B}}((\overline{\omega}))>0$, then the functions

$$\overline{\alpha}_r\cdot\overline{\varphi}_{i_1\cdots i_p}-\overline{\alpha}_{i_p}\cdot\overline{\varphi}_{i_1\cdots i_{p-1}r}+\cdots+(-1)^p\,\overline{\alpha}_{i_1}\cdot\overline{\varphi}_{i_2\cdots i_p r}$$

vanish along $\overline{B}$ for all combinations of indices $i_1,\cdots,i_p$ taken from $1,\cdots,r-1$, i.e.

$$\overline{\alpha}_r(M)\cdot\overline{\varphi}_{i_1\cdots i_p}(M)-\cdots+(-1)^p\,\overline{\alpha}_{i_1}(M)\cdot\overline{\varphi}_{i_2\cdots i_p r}(M)=0\,, \tag{7}$$

where M is a generic Point of $\overline{B}$ over its field of definition. But since $p\leq r-2$ there exists at least one more relation of the form (7), e.g. taking $i_{p+1}\,(<r)$ different from $i_1,\cdots,i_p,r$, we have a relation

$$\begin{aligned}\overline{\alpha}_r(M)\cdot\overline{\varphi}_{i_2\cdots i_{p+1}}(M)-\overline{\alpha}_{i_{p+1}}(M)\cdot\overline{\varphi}_{i_2\cdots i_p r}(M)+\cdots\\ +(-1)^p\,\overline{\alpha}_{i_2}(M)\,\overline{\varphi}_{i_3\cdots i_{p+1}r}(M)=0\,.\end{aligned} \tag{8}$$

Since u_j appeared only in $\overline{\alpha}_j$, for $1\leq j\leq r$, u_{i_1} appears in (7) but not in (8), $u_{i_{p+1}}$ appears in (8) but not in (7), therefore these two relations are independent, hence we have

$$\dim_{k(M)}(u_1,\cdots,u_N)\leq N-2\,.$$

Since u_0 is contained in $k(M,u_1,\cdots,u_N)$ this contradicts to prop. 7. Then we must have $v_{\overline{B}}((\overline{\omega}))=0$, and $\overline{X}=0$. Thus the theorem is completely proved.

It seems to be desirable that even when $p=r-1$ we have $\overline{X}=0$. But it is not true in general as will be shown in the following example.

EXAMPLE. Let V be a projective space L^2, $P=(1, x, y)$ be a generic Point of L^2 over Π (prime field of characteristic p) and ω a linear differential form defined over Π such as

$$\omega = d\xi + \xi\, d\eta,$$

where ξ and η are functions on L defined over Π by $\xi(P)=x$ and $\eta(P)=y$ respectively. Then we see easily that we have $(\omega)=-3A$, where A denotes the line at infinity. On the other hand let $uX+vY+w=0$ be the defining equation for a generic line in L with reference to Π, then we have

$$\bar{\omega} = (1-\bar{\xi}\cdot u/v)\cdot d\bar{\xi}.$$

Hence we have $(\bar{\omega})=-3\mathfrak{A}+M$, where $\mathfrak{A}=A\cdot W$ and M is a Point with inhomogeneous coordinates $\big(v/u, -(1+w/v)\big)$. M is certainly a generating Point of L with reference to Π.

Repeating the above process we obtain immediately the following

COROLLARY 1. *Let V^r be a projective model of an algebraic Variety, ω a differential form of degree p defined over k and W be a generic s-section of V, i.e. the intersection product of V with $(r-s)$ independent generic hyperplanes over k. Then we have*

$$(\bar{\omega})=(\omega)\cdot W^s \qquad \textit{if} \quad s\geqq p+1,$$

$$(\bar{\omega})=(\omega)\cdot W^p+\bar{X} \qquad \bar{X}\succ 0,$$

where $\bar{\omega}$ is the differential form on W induced by ω, and any component of $\bar{X}$ is a generating Subvariety of V with reference to k.

COROLLARY 2. *Let V be a non-singular projective model defined over k, W a generic hyperplane section of V with reference to k and ω a differential form on V defined over k. Then ω is of the first kind if and only if the induced differential form $\bar{\omega}$ on W is of the first kind.*

PROOF. By Theorem 1, ω is of the first kind if and only if $(\omega)\succ 0$ on non-singular Variety. Moreover if V has no singular Point then W is also a non-singular model. By Theorem 2 $(\omega)\succ 0$ implies $(\bar{\omega})\succ 0$. Conversely let $(\bar{\omega})=(\omega)\cdot W+\bar{X}\succ 0$. Then, since any component of $(\omega)\cdot W$ is not in $\bar{X}$, we have $(\omega)\cdot W\succ 0$. Moreover any component of (ω) is algebraic over k and W is generic over k, hence we must have $(\omega)>0$. This completes the proof.

9. As an application of the Theorem 2 we have the following proposition well known in the case of curves.

PROPOSITION 8. *Let V be a projective model, f a function on V, and A^{r-1} any simple Subvariety of V^r, such that $v_A((f))=a$. Then we have*

$$v_A((df))=a-1 \qquad \textit{if} \quad a \not\equiv 0 \quad (\mathrm{mod}\, p),$$

$$v_A((df))\geqq a-1 \qquad \textit{if} \quad a \equiv 0 \quad (\mathrm{mod}\, p),$$

where p is the characteristic of the universal domain.

PROOF. Let k be a field of definition for f and C a generic 1-section of V with reference to k, and put

$$\sum P_i = A \cdot C.$$

Then any one of P_i appears in $A \cdot C$ with coefficient 1 and not a generating Point of V with reference to k. Hence we have $v_{P_i}((\bar{f}))=a$ and

$$v_{P_i}((d\bar{f}))=a-1 \qquad \text{if} \quad (a \not\equiv 0 \quad (\mathrm{mod}\, p),$$

$$v_{P_i}((d\bar{f}))\geqq a-1 \qquad \text{if} \quad (a \equiv 0 \quad (\mathrm{mod}\, p)$$

from the curve theory. But we have by Cor. 1 of Th. 2

$$(d\bar{f})=(df)\cdot C+\bar{\bar{X}}$$

and any component of $\bar{\bar{X}}$ is a generating Point of V with reference to k, hence we must have the assertion. Q. E. D.

We shall now add one remark. Let V^r, and W^{r-1} be as in Th. 2, and denote by $^{-}$ the differential forms on W induced by the differential forms on V. Then we can easily see that $d\omega = d\bar{\omega}$. Then by the first part of the Th. 2, we have $d\omega=0$ if and only if $d\bar{\omega}=0$. Hence to show that every differential form of the first kind on V is closed it is sufficient to treat only the case when the degree of ω is equal to $r-1$. This remark may be useful in the future investigations.

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