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On the three-dimensional cohomology group of Lie algebras.

By Mitsuya Mori

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Eilenberg and MacLane [1] have built up, by means of the cohomology theory, an analogue in the theory of groups to the theory of the Brauer group of normal simple algebras with a fixed splitting field over a given field: The theory of similarity classes of Q-kernels with a fixed abelian group G as center. They arrived at a remarkable result that the group of similarity classes of Q-kernels is isomorphic to the three-dimensional cohomology group $H^3(Q, G)$ of Q over the abelian coefficient group G, and gave an answer to the problem of Baer [2] on group extensions in terms of the two-dimensional cohomology theory. On the other hand, Chevalley and Eilenberg [3] have shown that the two-dimensional cohomology group $H^2(L, Z, P)$ of a Lie algebra L with respect to an abelian Lie algebra Z and a representation P of L over Z is isomorphic to the group of equivalent classes of extensions of L by (Z, P).

In the present paper, we shall try to develop further the theory of Chevalley and Eilenberg to obtain in the theory of Lie algebras an analogous result to that of Eilenberg and MacLane in the theory of groups. We shall introduce in § 1 the concept of L-kernels as an analogue of Q-kernels and define the similarity group of L-kernels for Lie algebras with a fixed center. We shall show in § 2 that this similarity group is isomorphic with a subgroup of the three-dimensional cohomology group $H^3(L, Z, P)$ whose meaning will be given later on, but we have not succeeded to decide whether these two groups are isomorphic with each other. In § 3 we shall deal with an analogue of the problem of Baer on group extensions for Lie algebras.

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§ **1.** In the following, we consider Lie algebras $E=\{e,\cdots\}$, $L=\{x,y,z,\cdots\}$ over a fixed field F, and a homomorphism ϕ from E onto L. Let $V=\{v,u,\cdots\}$ be the kernel of ϕ, and $D(V)$, $I(V)$ the derivation-algebra and inner derivation-algebra of V respectively. If we put $\sigma_e v=[v,e]$, $e\in E$, then $\sigma_e \in D(V)$, and if $e\equiv e'$ (mod V) then we have $\sigma_e\equiv\sigma_{e'}$ (mod $I(V)$). So if we put $\mathfrak{p}_x=\sigma+I(V)$, where σ is a derivation induced by an element e in E mapped by ϕ to an element $x\in L$, then the mapping $x\rightarrow\mathfrak{p}_x$ is a homomorphism from L into $D(V)/I(V)$. We call the pair (E,ϕ) an *extension of L by the kernel* $(V,\mathfrak{p})$. We shall say that an *L-kernel* $(V,\mathfrak{p})$ is given, when a Lie algebra V and a homomorphism $\mathfrak{p}$ from L into $D(V)/I(V)$ are given. Especially, if Z is the center of V, and $c\in Z$, then for all $\sigma\in\mathfrak{p}_x$, σc is a fixed element in Z, determined by c and x, and if we put $\sigma c=P_x c$, then the mapping $x\rightarrow P_x$ is a representation of L over Z. In the following, we fix a Lie algebra L and an abelian Lie algebra Z of a finite dimension over F and a representation P of L over Z, and consider the L-kernels $(V,\mathfrak{p})$ such that V has Z as center,[1] and $\sigma c=P_x c$, $\forall\sigma\in\mathfrak{p}_x$, $\forall x\in L$.

The product of two kernels $(V_1,\mathfrak{p}^{(1)})$, $(V_2,\mathfrak{p}^{(2)})$ is defined as follows: Let $V_1\times V_2=\{(v_1,v_2)\,;\,v_i\in V_i, i=1,2\}$ be the direct product of V_1 and V_2, S the ideal $\{(c,-c)\,;\,c\in Z\}$ and put $V=(V_1\times V_2)/S$. Then V is a Lie algebra with obvious addition and commutation, and has a center which is isomorphic to and will be identified with Z. (such identification will be done in the following without mentioning it especially.)

Put $\sigma_x=(\sigma_x^{(1)},\sigma_x^{(2)})$, $\sigma_x^{(i)}\in\mathfrak{p}_x^{(i)}$, $i=1,2$, and $\mathfrak{p}_x=\sigma_x+I(V)$, then the pair $(V,\mathfrak{p})$ is an L-kernel, which we shall call the product of $(V_1,\mathfrak{p})$ and $(V_2,\mathfrak{p}^{(2)})$ and denote by $(V_1,\mathfrak{p}^{(1)})\otimes(V_2,\mathfrak{p}^{(2)})$.

An L-kernel $(V,\mathfrak{p})$ is said to be *equivalent* to an L-kernel $(V',\mathfrak{p}')$, written $(V,\mathfrak{p})\cong(V',\mathfrak{p}')$, if there exists an isomorphism τ from V onto V' such that

$$\tau c=c \qquad \forall c\in Z,$$

$$\tau\sigma_x\tau^{-1}\in\mathfrak{p}'_x \qquad \forall\sigma_x\in\mathfrak{p}_x, \qquad \forall x\in L.$$

This equivalence is clearly reflexive, symmetric and transitive, and $(V_1,\mathfrak{p}^{(1)})\cong(V_1',\mathfrak{p}'^{(1)})$, $(V_2,\mathfrak{p}^{(2)})\cong(V_2',\mathfrak{p}'^{(2)})$ imply $(V_1,\mathfrak{p}^{(1)})\otimes(V_2,\mathfrak{p}^{(2)})$

1) We shall call (Z,P) the center of the L-kernel $(V,\mathfrak{p})$.

$\cong (V'_1, \mathfrak{p}'^{(1)}) \otimes (V'_2, \mathfrak{p}'^{(2)})$, so we can define the $\otimes$-multiplication of the equivatent classes of L-kernels, which is associative and commutative, the class containing $Z=(Z, P)$ being the unit.

An L-kernel $(V, \mathfrak{p})$ is said to be *extendible*, if there exists an extension (E, ϕ) of L by the kernel $(V, \mathfrak{p})$, such that $\sigma_e \in \mathfrak{p}_{\phi e}$, where $\sigma_e v=[v, e]$, for each e in E. Then we have the following lemma:

LEMMA 1. *The product of two extendible L-kernels is also extendible.*

PROOF. Let (E_1, ϕ_1) and (E_2, ϕ_2) be the extensions of L by the extendible L-kernels $(V_1, \mathfrak{p}^{(1)})$ and $(V_2, \mathfrak{p}^{(2)})$ with center (Z, P) respectively. Consider the subalgebra $R=\{(e_1, e_2)\,;\ \phi_1(e_1)=\phi_2(e_2)\}$ of the direct product $E_1 \times E_2$ and set $\phi'(e_1, e_2)=\phi_1(e_1)=\phi_2(e_2)$ for $(e_1, e_2) \in R$. Let S be the ideal $\{(c, -c)\,;\ c \in Z\}$ of R. Then ϕ' induces a homomorphism ϕ from $E=R/S$ onto L. We shall call the extension (E, ϕ) the product of (E_1, ϕ_1) with (E_2, ϕ_2), and denote by $(E_1, \phi_1) \otimes (E_2, \phi_2)$. The kernel of (E, ϕ) is then exactly the product $(V, \mathfrak{p})$ of $(V_1, \mathfrak{p}^{(1)})$ with $(V_2, \mathfrak{p}^{(2)})$.

Now we define the inverse $(V^*, \mathfrak{p}^*)$ of an L-kernel $(V, \mathfrak{p})$ as follwoing: Let V^* be the set of element v^*, which is in the one-to-one correspondence to $v \in V$. We define the following law of compositions in V^*: $(\alpha v)^*=\alpha v^*$, $v_1^*+v_2^*=(v_1+v_2)^*$, $[v_1^*, v_2^*]=[v_2, v_1]^*$, then V^* is a Lie algebra over F of the same dimension as V, and has Z as center. Put $\sigma^* v^*=(\sigma v)^*$ for each $\sigma \in D(V)$, then $\sigma^* \in D(V^*)$, and if we put $\mathfrak{p}_x^*=\{\sigma^*\,;\ \sigma \in \mathfrak{p}_x\}$, then $(V^*, \mathfrak{p}^*)$ is an L-kernel, which we shall call the inverse of $(V, \mathfrak{H})$. Now we have

LEMMA 2. *The product* $(V, \mathfrak{p}) \otimes (V^*, \mathfrak{p}^*)$ *of an L-kernel* $(V, \mathfrak{p})$ *and its inverse* $(V^*, \mathfrak{p}^*)$ *is extendible.*

In fact, $\tilde{E}=\{(v, \sigma, x)\,;\ v \in V, \sigma \in \mathfrak{p}_x, x \in L\}$ is a Lie algebra with respect to the law of compositions:

$$(v, \sigma, x)+(v, \tau, y)=(v+u, \sigma+\tau, x+y)\,,$$

$$[(v, \sigma, x), (u, \tau, y)]=([v, u]+\tau v-\sigma u, [\sigma, \tau], [x, y])\,.$$

and the mapping $\psi(v, \sigma, x)=x$ is a homomorphism from $\tilde{E}$ onto L. Then $(\tilde{E}, \psi)$ is an extension of L by the kernel $(\tilde{V}, \tilde{\mathfrak{p}})$, where $\tilde{V}=\{(v, \sigma, 0)\,;\ \sigma \in I(V)\}$ and $\tilde{\mathfrak{p}}$ is constructed as follows: The element of $\tilde{V}$ will be denoted generally by $\tilde{v}$. We put $\tilde{u}(x)=(u, \sigma_x, x)$ and $\tilde{\sigma}_x \tilde{v}$

$=[\tilde{v}, \tilde{u}(x)]$. Then $\tilde{\mathfrak{p}}$ is defined by $\tilde{\mathfrak{p}}_x=\tilde{\sigma}_x+I(\tilde{V})$. Thus $(\tilde{V}, \tilde{\mathfrak{p}})$ is an extendible L-kernel. Now we put $V_0=\{(v, 0, 0)\}$, and let $\sigma_x^{(0)}$ be the restriction of $\tilde{\sigma}_x$ to V_0, then we have an L-kernel $(V_0, \mathfrak{p}^{(0)})$ with the homomorphism $\mathfrak{p}^{(0)}: L \rightarrow D(V_0)/I(V_0)$, $\mathfrak{p}_x^{(0)}=\sigma_x^{(0)}+I(V_0)$, and here we have $(V_0, \mathfrak{H}^{(0)}) \cong (V, \mathfrak{p})$ by the correspondence $(v, 0, 0) \leftrightarrow v$. Put moreover $V_1=\{v, \sigma_{-v}, 0)\}$ and let $\sigma_x^{(1)}$ be the restriction of $\tilde{\sigma}_x$ to V_1, then we have an L-kernel $(V_1, \mathfrak{p}^{(1)})$ with the homomorphism $\mathfrak{p}^{(1)}: L \rightarrow D(V_1)/I(V_1)$, $\mathfrak{p}_x^{(1)}\sigma={}_x^{(1)}+J(V_1)$, and here we have $(V_1, \mathfrak{p}^{(1)}) \cong (V^*, \mathfrak{p}^*)$ by the correspondence $(v, \sigma_{-v}, 0) \leftrightarrow v^*$. Now every element of the form $(v, \sigma_u, 0)$ in $\tilde{V}$ is uniquely written as the sum $(v+u, 0, 0)+(-u, \sigma_u, 0)$, where $(v+u, 0, 0) \in V_0$ and $(-u, \sigma_u, 0) \in V_1$. Hence we conclude that $(\tilde{V}, \tilde{\mathfrak{H}}) \cong (V_0, \mathfrak{p}^{(0)}) \otimes (V_1, \mathfrak{H}^{(1)}) \cong (V, \mathfrak{p}) \otimes (V^*, \mathfrak{p}^*)$.

Now an L-kernel $(V, \mathfrak{p})$ is said to be *similar* to $(V', \mathfrak{p}')$, written $(V, \mathfrak{p}) \sim (V', \mathfrak{p}')$, if there exist two extendible L-kernels $(U, \mathfrak{q})$, $(U', \mathfrak{q}')$ such that

$$(V, \mathfrak{p}) \otimes (U, \mathfrak{q}) \cong (V', \mathfrak{p}') \otimes (U', \mathfrak{q}') .$$

Then this relation of similarity is reflexive,[2] symmetric and transitive by lemma 1. Moreover equivalent L-kernels are similar,[2] and $V_1 \sim V_1'$, $V_2 \sim V_2'$ imply $V_1 \otimes V_2 \sim V_1' \otimes V_2'$, as is easily seen. Now we classify the L-kernels by the similarity relation and denote the class containing the L-kernel $(V, \mathfrak{p})$ by $[V, \mathfrak{p}]$. By the above remark, we can define the $\otimes$-product $[(V, \mathfrak{p}) \otimes (V', \mathfrak{p}')]$ of $[V, \mathfrak{p}]$ and $[V', \mathfrak{p}']$, and this multiplication is associative and commutative, $Z=[Z, P]$ being the unit, and $[V^*, \mathfrak{p}^*]$ the inverse of $[V, \mathfrak{p}]$. Hence we have

THEOREM 1. *The similarity classes of L-kernels form an abelian group under the above defined $\otimes$-multiplication.*

We shall call this group the similarity group of (L, Z, P).

§ **2.** Now we proceed to consider the three-dimensional cohomology of Lie algebra. Let an L-kernel $(V, \mathfrak{p})$ be given. If a derivation $\sigma_x \in \mathfrak{p}_x$ is linear in x, i. e. if it has the property

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2) (Z, P) is extendible. In fact, we put E as the direct sum $L+Z$ with $\phi(x, c)=x$. Commutation in E is defined by the formula $[(x, c), (y, d)]=([x, y], P_y c-P_x d)$. Then E is a Lie algebra, which is an extension of L by the kernel (Z, P). Hence it follows immediatelly that the similarity relation is reflexive, and that equivalent L-kernels are similar.

$$(1)\qquad \sigma_{\alpha x+\beta y}=\alpha\,\sigma_x+\beta\,\sigma_y\,,$$

we shall call σ_x a *σ-function.* The existence of such a function is assured as follows: Let $x_1,\cdots,x_n$ be a base of L over F, and select an arbitrary derivation σ_{x_i}, fixed once for all, from $\mathfrak{p}_{x_i}$, $1\leq i\leq n$. Then put $\sigma_x=\sum\alpha_i\,\sigma_{x_i}$, for $x=\sum\alpha_i\,x_i$, $\alpha_i\in F$. σ_x is then obviously a σ-function. For a given σ-function σ_x, we call a *v-function related to σ_x, of x, $y\in L$,* a function $v(x,y)$ with values in V satisfying $[\sigma_x,\sigma_y]-\sigma_{[x,y]}=\sigma_{v(x,y)}$ and having the properties

$$(2)\qquad \begin{aligned} v(\alpha\,x+\beta\,y,z)&=\alpha\,v(x,z)+\beta\,v(y,z)\\ v(x,\alpha\,y+\beta\,z)&=\alpha\,v(x,y)+\beta\,v(x,z)\\ v(y,x)&=-v(x,y)\end{aligned}$$

i.e. when $v(x,y)$ is bilinear and alternative in x,y. The existence of such a v-function related to σ_x is assured as follows: Let as before $x_1,\cdots,x_n$ be a base of L over F. Then since $[\sigma_{x_i},\sigma_{x_j}]$, $\sigma_{[x_i,x_j]}$ belongs to the same class $\mathfrak{p}_{[x_i,x_j]}$, there exists an element $v(x_i,x_j)$ in V such that $[\sigma_{x_i},\sigma_{x_j}]-\sigma_{[x_i,x_j]}=\sigma_{v(xi,xj)}$. Here we can suppose that $v(x_j,x_i)=-v(x_i,x_j)$. We fix once for all such element $v(x_i,x_j)$ for each pair (i,j), $i,j=1,\cdots,n$, and put $v(x,y)=\sum\alpha_i\,\beta_j\,v(x_i,x_j)$ for $x=\sum\alpha_i\,x_i$, $y=\sum\beta_j\,x_j$, $\alpha_i,\beta_j\in F$. Then $v(x,y)$ is a v-function related to σ_x. Now we have

$$\begin{aligned}&[\,[\sigma_x,\sigma_y],\sigma_z]+[\,[\sigma_y,\sigma_z],\sigma_x]+[\,[\sigma_z,\sigma_x],\sigma_y]\\ &=[\sigma_{[x,y]},\sigma_z]+[\sigma_{[y,z]},\sigma_x]+[\sigma_{[z,x]},\sigma_y]+[\sigma_{v(x,y)},\sigma_z]+[\sigma_{v(y,z)},\sigma_x]+[\sigma_{v(z,x)},\sigma_y]\\ &=\sigma_{[[x,y],z]}+\sigma_{[[y,z],x]}+\sigma_{[[z,x],y]}+\sigma_{\sigma_x v(y,z)}+\sigma_{\sigma_y v(z,x)}+\sigma_{\sigma_z v(x,y)}\\ &+\sigma_{v([x,y],z)}+\sigma_{v([y,z],x)}+\sigma_{v([z,x],y)}=\sigma_{f(x,y,z)}=0\,,\end{aligned}$$

where

$$(3)\qquad \begin{aligned} f(x,y,z)=&\,\sigma_x\,v(y,z)+\sigma_y\,v(z,x)+\sigma_z\,v(x,y)\\ &+v([x,y],z)+v([y,z],x)+v([z,x],y)\,.\end{aligned}$$

Since $\sigma_{f(x,y,z)}=0$, $f(x,y,z)$ belongs to Z. We can easily verify that $f(x,y,z)$ is a 3-Z-cochain. Moreover we have

LEMMA 3. *$f(x,y,z)$ is a 3-Z-cocycle.*

PROOF. Let us compute the coboundary $\partial f(x,y,z,t)$ of f:

$$\begin{aligned}\partial f(x,y,z,t) &= P_x f(y,z,t) - P_y f(x,z,t) + P_z f(x,y,t) - P_t f(x,y,z)\\ &\quad + f([x,y],z,t) - f([x,z],y,t) + f([x,t],y,z) + f[(y,z],x,t)\\ &\quad - f[(y,t],x,z) + f([z,t],x,y)\,.\end{aligned} \tag{4}$$

In the right hand side of this formula, express each f in the form (3) and replace each P by σ. Then we have terms of the following types: $\sigma_* \sigma_* v(*,*)$, $\sigma_* v([*,*],*)$, $\sigma_{[*,*]} v(*,*)$, $\sigma_* v(*,[*,*])$, $v([[*,*],*],*)$, $v([*,*],[*,*])$ and $v([*,[*,*]],*)$. Many of these of terms cancel with each other as we have the relations such as $\sigma_x v([y,z],t)+\sigma_x v(t,[y,z])=0$, $v[[z,t],[x,y]]+v([x,y],[z,t])=0$ and $v([[x,y],z],t)-v([[x,z],y],t)+v([[y,z],x],t)=0$, in virtue of (2). Thus (4) is reduced to the following:

$$\begin{aligned}&(\sigma_x\sigma_y-\sigma_y\sigma_x)\,v(z,t)+\sigma_{[x,y]}v(z,t)+(\sigma_x\sigma_z-\sigma_z\sigma_x)\,v(t,y)+\sigma_{[x,z]}v(t,y)\\ &+(\sigma_x\sigma_t-\sigma_t\sigma_x)\,v(y,z)+\sigma_{[x,t]}v(y,z)+(\sigma_z\sigma_y-\sigma_y\sigma_z)\,v(t,x)-\sigma_{[y,z]}v(t,x)\\ &+(\sigma_y\sigma_t-\sigma_t\sigma_y)\,v(z,x)+\sigma_{[y,t]}v(z,x)+(\sigma_z\sigma_t-\sigma_t\sigma_z)\,v(x,y)+\sigma_{[z,t]}v(x,y)\,.\end{aligned}$$

Since $[\sigma_x,\sigma_y]-\sigma_{[x,y]}=\sigma_{v(y,y)}$, this is rewritten as follows:

$$\begin{aligned}&-[v(z,t),\ v(x,y)]-[v(t,y),\ v(x,z)]-[v(y,z,),\ v(x,t)]\\ &+[v(t,x),\ v(y,z)]-[v(z,x),\ v(y,t)]-[v(x,y),\ v(z,t)]\,,\end{aligned}$$

hence we have $\partial f(x,y,z,t)=0$. Thus if we choose a σ-function σ_x and a v-function $v(x,y)$, we obtain a 3-Z-cocycle $f(x,y,z)$ defined by (3). We shall call it the *f-cocycle related to σ_x and $v(x,y)$.*

We shall state here some more remarks. If we replace the v-function $v(x,y)$ by another v-function $v'(x,y)$ related to the same σ-function σ_x, then the element $c(x,y)=v'(x,y)-v(x,y)$ is a 2-Z-cochain. We shall call it a *c-element*. Obviously every 2-Z-cochain can be regarded as a c-element. The f-cocycle $f'(x,y,z)$ related to σ_x and $v'(x,y)$ is cohomologous to $f(x,y,z)$: $f'(x,y,z)=f(x,y,z)+\partial c(x,y,z)$. As we can take an arbitrary 2-Z-cochain as c-element $c(x,y)$, every cocycle $f(x, y, z)$ cohomologous to $f(x, y, z)$ can be regarded as f-cocycle. Now if we replace σ_x by another σ-function σ'_x, then there exists an element $v(x)$ in V such that $\sigma'_x = \sigma_x + \sigma_{v(x)}$. Here we can suppose that $v(x)$ is linear: $v(\alpha x + \beta y) = \alpha v(x) + \beta v(y)$. Put $v'(x, y) = v(x, y) + \sigma_y v(x) - \sigma_x v(y) + [v(x), v(y)]$, then the f-cocycle related to σ'_x and $v'(x, y)$ is equal to $f(x, y, z)$. Thus we have the following

LEMMA 4. *To each L-kernel* $(V, \mathfrak{p})$ *corresponds a 3-Z-cohomology class, i.e. the class of an f-cocycle related to* σ_x *and* $v(x, y)$, σ_x *being an arbitrary* σ*-function and* $v(x, y)$ *a v-function related to* σ_x. *To equivalent kernels corresponds the same class.*

We shall denote the cohomology class corresponding to $(V, \mathfrak{p})$ by $F(V, \mathfrak{p})$.

LEMMA 5. *An L-kernel* $(V, \mathfrak{p})$ *is extendible if and only if* $F(V, \mathfrak{p})$ *is 0-class.*

PROOF. 1) Let us assume that $(V, \mathfrak{p})$ is extendible. let (E, ϕ) be an extension of L by $(V, \mathfrak{p})$, and $x_1, \cdots, x_n$ a base of L over F, and $u(x_i)$ an element in E mapped by ϕ to x_i, fixed once for all, $1 \leq i \leq n$, then put $u(x) = \sum \alpha_i u(x_i)$ for $x = \sum \alpha_i x_i$, $\alpha_i \in F$. For the construction of f-cocycle corresponding to $(V, \mathfrak{p})$, choose a σ-function σ_x, defined by $\sigma_x v = [v, u(x)]$. Then $v(x, y) = [u(x), u(y)] - u([x, y])$ is a v-function related to σ_x. We have then

$$[[u(x), u(y)], u(z)] + [[u(y), u(z)], u(x)] + [[u(z), u(x)], u(y)]$$

$$= [u([x, y]), u(z)] + [u([y, z]), u(x)] + [u([z, x]), u(y)]$$

$$+ [v(x, y), u(z)] + [v(y, z), u(x)] + [v(z, x), u(y)]$$

$$= u([[x, y], z]) + u([[y, z], x]) + u([[z, x], y]) + \sigma_x v(y, z) + \sigma_y v(z, x)$$

$$+ \sigma_z v(x, y) + v([x, y], z) + v([y, z], x) + v([z, x], y)$$

$$= f(x, y, z) = 0,$$

where $f(x, y, z)$ is the f-cocycle related to σ_x and $v(x, y)$, which proves that $F(V, \mathfrak{p})$ is 0-class.

2) Conversely, let us assume that $F(V, \mathfrak{p})$ is 0-class. Select now any σ-function σ_x. By the above remark we may then select a v-function $v(x, y)$ so that the f-cocycle related to σ_x and $v(x, y)$ is identically 0.

Then $E=\{(v, x)\,;\ v\in V,\ x\in L\}$ is a Lie algebra with law of compositions:

$$(v, x)+(u, y)=(v+u, x+y)$$

$$[(v, x), (u, y)]=([v, u]+\sigma_y v-\sigma_x u+v(x, y), [x, y])\,.$$

The correspondence $\phi(v, x)=x$ is a homomorphism from E onto L, and the kernel consists of the pair $(v, 0)$ which may be identified with the element v in V. The element $(0, x)$ in E is thereby mapped by ϕ to $x\in L$. If we take this element $(0, x)$ as $u(x)$, then we have $[(v, 0), u(x)]=[(v, 0), (0, x)]=(\sigma_x v, 0)$, so that $u(x)$ induces the derivation $\sigma_x\in\mathfrak{p}_x$. Thus $(V, \mathfrak{p})$ can be realized as the kernel of an extension E.

LEMMA 6.

$$F[(V_1, \mathfrak{p}^{(1)})\otimes(V_2, \mathfrak{p}^{(2)})]=F(V_1, \mathfrak{p}^{(1)})+F(V_2, \mathfrak{p}^{(2)})\,.$$

PROOF. Choose first $\sigma_x^{(i)}\in\mathfrak{p}_x^{(i)}$, $v_{(x,y)}^{(i)}\in V_i$, $i=1, 2$, such that $[\sigma_x^{(i)}, \sigma_y^{(i)}]-\sigma_{[x,y]}^{(i)}=\sigma_{v(x,y)}^{(i)}$, and denote the f-cocycle related to $\sigma_x^{(i)}$ and $v_{(x,y)}^{(i)}$ by $f_{(x,y,z)}^{(i)}$. In the product $(V, \mathfrak{p})=(V_1, \mathfrak{p}^{(1)})\otimes(V_2, \mathfrak{p}^{(2)})$, where $V=(V_1\times V_2)/S$, we shall put $\sigma_x=(\sigma_x^{(1)}, \sigma_x^{(2)})$, $v(x, y)=\big(v^{(1)}(x, y), v^{(2)}(x, y)\big)+S$. Then the f-cocycle related to σ_x and $v(x, y)$ is $(f^{(1)}(x, y, z), f^{(2)}(x, y, z))+S=(f^{(1)}(x, y, z)+f^{(2)}(x, y, z), 0)+S$. Since $f^{(i)}\in Z$, $i=1, 2$, it may be identified with $f^{(1)}+f^{(2)}\in Z$.

Now if $V_1\sim V_2$, then $V_1\otimes U_1\cong V_2\otimes U_2$ for extendible L-kernel U_1 and U_2. Since $F(U_1)=F(U_2)=0$ by lemma 5, it follows that $F(V_1)=F(V_2)$ by lemma 4 and 6. Conversely, suppose that $F(V_1)=F(V_2)$. If V_1^* is the inverse of V_1, then $V_1\otimes V_1^*$ is extendible and we have $-F(V_1)=F(V_1^*)$. Hence $F(V_2\otimes V_1^*)=0$, so that $V_2\otimes V_1^*$ is extendible. Now in $(V_1\otimes V_1^*)\otimes V_2\cong V_1\otimes(V_2\otimes V_1^*)$, $V_1\otimes V_1^*$ and $V_2\otimes V_1^*$ are both extendible, so we have $V_1\sim V_2$. In other words,

$$V_1\sim V_2 \quad \text{if and only if} \quad F(V_1)=F(V_2)\,.$$

Thus we have

THEOREM 2. *The similarity group of (L, Z, P) is isomorphic to a subgroup of the three-dimensional cohomology group $H^3(L, Z, P)$.*

§ 3. Now we consider the relation between the extensions of L by a given extendible L-kernel $(V, \mathfrak{p})$ and the two-dimensional cohomology group $H^2(L, Z, P)$. We shall thus solve the problem to construct the extensions by the center of $(V, \mathfrak{p})$ and to determine all extensions of L by $(V, \mathfrak{p})$.

Two extensions (E, ϕ), (E', ϕ') of L by the L-kernel $(V, \mathfrak{p})$ are called *equivalent*, written $(E, \phi) \approx (E', \phi')$, if there exists an isomorphism τ from E onto E' such that $\tau v = v$, $\forall\, v \in V$, and $\phi' \tau = \phi$. The equivalence of extensions of L by (Z, P) is similarly defined. We shall prove the following two lemmas:

LEMMA 7. *Every extension (E', ϕ') of L by an extendible L-kernel $(V, \mathfrak{p})$ is equivalent with the product $E \otimes D$ of a fixed extension (E, ϕ) of L by $(V, \mathfrak{p})$ with some extension (D, η) of L by the center (Z, P).*

LEMMA 8. *$E \otimes D_1 \approx E \otimes D_2$ if and only if $D_1 \approx D_2$.*

To prepare the proof of these lemmas, we give first a remark on the product of extensions. Let (E, ϕ) be any extension of L by $(V, \mathfrak{p})$, and as before $x_1, \cdots, x_n$ a base of L over F, and $u(x_i)$ an element in E with $\phi\, u(x_i) = x_i$, $1 \leq i \leq n$, fixed once for all. Put $u(x) = \sum \alpha_i\, u(x_i)$ for $x = \sum \alpha_i x_i$, $\alpha_i \in F$. The derivation $\sigma_x \in \mathfrak{D}_x$ defined by $\sigma_x v = [v, u(x)]$ is a σ-function, and the element $v(x, y)$ in V with $[u(x), u(y)] - u([x, y]) = v(x, y)$ is a v-function related to σ_y, which we shall call a factor set of (E, ϕ) corresponding to the set of representatives $\{u(x)\}$. Then Jacobian identity in E gives

$$\text{(5)} \qquad \sigma_x v(y, z) + \sigma_y v(z, x) + \sigma_z v(x, y) + v([(x, y], z) + v([y, z], x) + v([z, x], y) = 0 .$$

In an extension (D, η) of L by (Z, P), we shall select for each $x \in L$ a representative $d(x)$ in D with $\eta\, d(x) = x$ and $d(\alpha x) = \alpha\, d(x)$. We have thereby $[c, d(x)] = P_x c$. Then the element $c(x, y)$ defined by $[d(x), d(y)] - d([x, y]) = c(x, y)$ is a 2-Z-cocycle, as is easily seen. We shall call $c(x, y)$ a factor set of (D, η) corresponding to the set of representatives $\{d(x)\}$[3]. Now the element $(u(x), d(x))\, S$ in the product $E \otimes D = (E_1, \phi_1)$ is mapped by ϕ_1 to $x \in L$. If we take this element $(u(x), d(x))\, S$ as representative $u_1(x)$ in E_1 with $\phi_1 u_1(x) = x$, then we have

$$\text{(6)} \qquad [u_1(x), u_1(y)] - u_1([x, y]) = (v(x, y), c(x, y))\, S = (v(x, y) + c(x, y), 0)\, S$$

where the element $(v + c, 0)\, S$ may be identified with $v + c$ in V.[3)] In

3) Every element of the form $(v, c)\, S$ in the product $(V', \mathfrak{p}') = (V, \mathfrak{p}) \otimes (Z, P)$ may be identified with the element $v + c$ in V. In the following such an identification will be done without mentioning it especially.

other words, to the product of extensions (E, ϕ) and (D, η) corresponds the sum of the factor sets v and c.

PROOF OF LEMMA 7. Let (E', ϕ') be any extension of L by L-kernel $(V, \mathfrak{p})$. Select representatives $u'(x)$ in E' with $\phi' u'(x)=x$, $[v, u'(x)]=\sigma_x v$ and $u'(\alpha x)=\alpha u'(x)$, and a v-function $v'(x,y)$ related to σ_x such that $[u'(x), u'(y)]-u'([x,y])=v'(x,y)$, where $v'(x,y)$ is the factor set of (E', ϕ') corresponding to the set of representatives $\{u'(x)\}$. Since $[\sigma_x, \sigma_y]-\sigma_{[x,y]}=\sigma_{v(x,y)}=\sigma_{v'(x,y)}$, $v'(x,y)-v(x,y)$ lies in Z. We shall put $c(x,y)=v'(x,y)-v(x,y)$. Now the equation (5) holds also when we replace v by v'. The function $c(x,y)$ is thus a 2-Z-cocycle, and so determines an extension (D, η) of L by (Z, P). It follows immediately that $E \otimes D \approx E'$ by (6). This proves lemma 7.

PROOF OF LEMMA 8. Let (E_1, ϕ_1), (E_2, ϕ) be equivalent extensions with isomorphism τ: $E_1 \rightarrow E_2$ and $E_i=E \otimes D_i$, $i=1,2$, where D_i is an extension with a set of representatives $\{d_i(x)\}$ and the factor set $\{c_i(x,y)\}$ corresponding to $\{d_i(x)\}$. Let $u(x)$ be as before the representatives of E. Then the element $u_i(x)=(u(x), d_i(x))\,S$ may be taken as a representative of E_i, and we have $[u_i(x), u_i(y)]-u_i([x,y])=v(x,y)+c_i(x,y)$. Since $\phi_2 \tau=\phi_1$, $\tau u_1(x)$ is an element of the form $u_2(x)+b(x)$ for some $b(x) \in V$. Now we have $\tau[v, u_1(x)]=[v, \tau u_1(x)]=[v, u_2(x)+b(x)]=\sigma_x v+\sigma_{b(x)} v$, and on the other hand $\tau[v, u_1(x)]=\tau(\sigma_x v)=\sigma_x v$, hence it follows that $b(x) \in Z$ and $b(x)$ is a 1-Z-cochain. Moreover we have $\tau\{[u_1(x), u_1(y)]-u_1([x,y])\}=\tau(v+c_1)=v+c_1$, and on the other hand

$$
\begin{aligned}
&\tau\{[u_1(x), u_1(y)]-u_1([x,y])\}=[\tau u_1(x), \tau u_1(y)]-\tau u_1([x,y]) \\
&=[u_2(x)+b(x), u_2(y)+b(y)]-\{u_2([x,y])+b([x,y])\} \\
&=[u_2(x), u_2(y)]-u_2([x,y])-P_x b(y)+P_y b(x) \\
&-b([x,y])=v+c_2-\partial b(x,y),
\end{aligned}
$$

since $\tau u_1(x)=u_2(x)+b(x)$, $b(x) \in Z$ and $[b(x), u_2(y)]=P_y b(x)$. This shows $c_2-c_1=\partial b$, so that the cocycle c_i defining D_i, $i=1,2$, are cohomologous to each other: therefore D_1 and D_2 are equivalent.[3] This proves lemma 8. From lemma 7 and 8 follows

THEOREM 3. *Let $(V, \mathfrak{p})$ be an extendible L-kernel, then the set of equivalent classes of extensions (E', ϕ') of L by this kernel $(V, \mathfrak{p})$ may* *be put into one-to-one correspondence with the cohomology classes of* $H^2(L, Z, P)$.[4]

Now we construct the extensions by the center (Z, P). We shall call the graph of a given L-kernel $(V, \mathfrak{p})$, extendible or not, the Lie algebra $\Gamma=\{(x, \sigma); \sigma \in \mathfrak{p}_x\}$ contained in the direct product $L \times D(V)$. The mapping $\rho(x, \sigma)=x$ is then a homomorphism of Γ onto L with kernel isomorphic to $I(V)$. We define now $[v, (x, \sigma)]$ to be σv for $v \in V$, $(x, \sigma) \in \Gamma$. Then each element (x, σ) of Γ induces a derivation in V, and we have

$$(7) \qquad [c, (x, \sigma)]=P_x c, \qquad \forall\, c \in Z, \qquad \forall\,(x, \sigma) \in \Gamma .$$

Now let (E, ϕ) be an extension of L by the L-kernel $(V, \mathfrak{p})$. The correspondence $\psi e=(\phi e, \sigma_e)$ is then a homomorphism from E onto L with kernel Z, and we have $\rho \psi=\phi$. Thus the Lie algebra E may be regarded either as an extension of L by $(V, \mathfrak{p})$ or as an extension of Γ by (Z, P): in the latter case, Γ has a representation P over Z defined by (7). Now we have

THEOREM 4. (REDUCTION THEOREM). *Every extension* (E, ϕ) *of* L *by* $(V, \mathfrak{p})$ *induces an extension* (E, ψ) *of* Γ *by* (Z, P), Γ *being the graph of* $(V, \mathfrak{p})$ *defined as above, and* $(E, \phi) \approx (E', \phi')$ *implies* $(E, \psi) \approx (E', \psi')$. *It may happen that inequivalent extensions of* L *by* $(V, \mathfrak{p})$ *induce equivalent extensions of* Γ *by* (Z, P).

Now we shall introduce a subgroup $H^2_V(L, Z, P) \subset H_2(L, Z, P)$ determined by the given L-kernel $(V, \mathfrak{p})$ as follows: The lift correspondence $\Lambda c((x, \sigma), (y, \tau))=c(x, y)$ of $c \in C^2(L, Z, P)$ to $\Lambda c \in C^2(\Gamma, Z, P)$ provides a homomorphism from $H^2(L, Z, P)$ onto a subgroup $\Lambda H^2(L, Z, P)$ of $H^2(\Gamma, Z, P)$, since $\partial \Lambda c=\Lambda \partial c$, as is easily seen. $H^2_V(L, Z, P)$ is now defined to be the kernel of this homomorphism.

We shall call an extension (E, ψ) of Γ by (Z, P) *admissible,* if it is induced by some extension (E, ϕ) of L by $(V, \mathfrak{p})$.

Now let (E', ϕ') an extension of L by $(V, \mathfrak{p})$, and $E'=E \otimes D$,

4) We shall denote by $\{E\}$, $\{D\}$ the class of extensions equivalent with E, D respectively. Let E' be any extension and $E' \approx E \otimes D$, E being an extension, fixed once for all. Then the correspondence $\{E'\}=\{E \otimes D\} \rightarrow \{D\}$ provides a one-to-one correspondence between the set of classes of extensions of L by $(V, \mathfrak{p})$ and the set of extensions of L by (Z, P). The later set forms a group isomorphic to $H^2(L, Z, P)$. Cf. Chevalley and Eilenberg, l. c.

(E, ϕ) being a fixed extension of L by $(V, \mathfrak{p})$, and D an extension of L by (Z, P). Let $\{d(x)\}$ be a set of representatives and $c(x, y)$ the factor set corresponding to $\{d(x)\}$, and (E', ψ') the extension of Γ by (Z, P) induced by (E', ϕ'). We shall denote generally by $\tilde{x}$ the element (x, σ), $\sigma \in \mathfrak{p}_x$, in Γ. The element $\sigma \in \mathfrak{p}_x$ may be written in the form $\sigma_{u(x)+v}$, $v \in V$, and the element in E' has the from $(u(x)+v,\ d(x)+c)\,S,\ v \in V$, $c \in Z$, $u(x)$ being as before the representative of E with $\phi\, u(x)=x$. Then the element $(u(x)+v, d(x))\,S$ is mapped by ψ' to $\tilde{x}=(x, \sigma_{u(x)+v})$ in Γ. If we take this element $(u(x)+v,\ d(x))\,S$ as $\tilde{u}(\tilde{x})$, then we have $[\tilde{u}(\tilde{x}), \tilde{u}(\tilde{y})]-\tilde{u}([\tilde{x}, \tilde{y}])=c(x, y)$, where the left hand side is a factor set $\tilde{c}(\tilde{x}, \tilde{y})$ of (E', ψ') corresponding to the set of representative $\{\tilde{u}(\tilde{x})\}$. It follows that the factor set of (E', ψ') is cohomologous to $\Lambda\, c$. Conversely, let (E_1, ψ_1), (E_2, ψ_2) be equivalent extensions of Γ by (Z, P), induced by (E_1, ϕ_1), (E_2, ϕ_2) respectively. We may put $E_1 \approx E_1 \times D_1$, $E_2 \approx E_1 \times D_2$, where D_1 is a trivial extension of L by (Z, P), so that the factor set c_1 of D_1 is cohomologous to 0, and D_2 is an extension of L by (Z, P) with the factor set c_2. By the previous result, the factor set of (E_i, ψ_i), $i=1, 2$, is cohomologous to $\Lambda\, c_i$, respectively. Since (E_1, ψ_1), (E_2, ψ_2) are equivalent to each other we have $\Lambda\, c_1 \sim \Lambda\, c_2$, and moreover $\Lambda\, c_2 \sim 0$, since $c_1 \sim 0$. It follows that the cohomology class $\{c_2\}$ belongs to $H^2_V(,L\, Z, P)$. Thus we have the following

THEOREM 5. *Let $(V, \mathfrak{p})$ be an extendible L-kernel. Then the set of equivalent classes of admissible extensions of Γ by (Z,P) may be put into one-to-one correspondence with* $H^2(L, Z, P)/H^2_V(L, Z, P)$.[5] *If moreover (E, ψ) is an admissible extension of Γ by (Z, P), then the set of equivalent classes of extensions (E', ϕ') of L by $(V, \mathfrak{p})$ which induce the extensions (E', ψ') equivalent with (E, ψ) may be put into one-to-one correspondence with* $H^2_V(L, Z, P)$.[6]

By means of this theorem, we can enumerate all extensions of L by a given L-kernel.

Mathematical Institute, University of Tokyo.

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5) The set of equivalent classes of admissible extensions of Γ by (Z, P) may be put into one-to-one correspnodence with the image $\Lambda\, H^2(L, Z, P)$ of $H^2(L, Z, P)$ by Λ.

6) Let $\{c'\}$ be the class of cocycles (factor sets) corresponding to (E', ϕ'), where (E', ϕ') induces (E', ψ') quivalent with (E, ψ). The correspnodence $\{E'\} \to \{c'\}$, where $\{c'\}$ belongs to $H^2_V(L, Z, P)$ provides the one-to-one correspondence between the set of classes of extensions (E', ϕ') and $H^2_V(L, Z, P)$.

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