Journal of the Mathematical Society of Japan Vol. 5, No. 1, April, 1953.

Generalized l^p spaces and the Schur property.

By I. HALPERIN and H. NAKANO

(Received Nov. 12, 1952)

1.—The following situation (essentially) was considered by H. Nakano [3]. This problem was considered by W. Orlicz [5] in a restricted form. Let J be a collection, not necessarily countable, of marks α. For given J-sequences $p=\{p(\alpha)\}$, $w=\{w(\alpha)\}$ with $p(\alpha)\geqq 1$ and $w(\alpha)>0$ for all α, let $l=l(p, w)$ denote the space of all real or complex valued J-sequences $x=\{x(\alpha)\}$ for which $\|x\|$ is finite; here, by definition,

(1.1) $\|x\|=\inf \eta$ for all $\eta>0$ with $\sum w(\alpha)\left|\frac{x(\alpha)}{\eta}\right|^{p(\alpha)}\leqq 1$ the symbol $\sum$ indicating that the non-zero addends are denumerable and have an absolutely convergent sum in the usual sense (if there are no such η then $\|x\|$ is defined to be ∞). The notation $l(p, w)$ may be replaced by $l(p)$ if $w(\alpha)=1$ for all α, and by l_p if, in addition, $p(\alpha)=p$ (a constant) for all α.

If R, S are two collections of J-sequences, $R\cong S$ shall mean that numbers $m(\alpha)$ exist such that the relations $y(\alpha)=m(\alpha)\,x(\alpha)$ set up a (1,1) correspondence between all x in R and all y in S.

A Banach space is said to have the Schur property if every weakly convergent sequence of its elements is necessarily convergent in norm (as shown by J. Schur [4], l^1, with J the set of all positive integers, has this property).

2.—The arguments used in [3] show:

(I): Every $l(p, w)$ is a Banach (i.e., linear, normed and complete) space.

(II): $l(p, w_1)\cong l(q, w_2)$ if and only if

(2.1) $\sum \theta^{\frac{p(\alpha)\,q(\alpha)}{[p(\alpha)-q(\alpha)]}}<\infty$ for some $0<\theta<1$, the sum to be taken over all α for which $p(\alpha)\neq q(\alpha)$.

(III): $l(p, w)$ has the Schur property if

(2.2) for every $\varepsilon>0$ the α for which $p(\alpha)>1+\varepsilon$ are finite in number.

(IV): There are $l(p, w)$ with the Schur property for which $l(p, w)\cong l^1$ is false.

The proofs given in [3] depend on results from the theory of modulared semi-ordered linear spaces. In the present note this dependence will be avoided by a simplification in the proofs. It will also be shown that the condition (2.2) is necessary as well as sufficient for (III) and some generalizations of (I), (II) and (III) will be given.

3.—

LEMMA 1. *If* $k \geqq 1, 0 \leqq t \leqq 1$ *and* $u, v \geqq 0$, *then* $(tu+(1-t)v)^k \leqq tu^k+(1-t)v^k$.

PROOF. See [2, page 77, Example (3)].

LEMMA 2. *If* $k>1$, $1/k+1/k'=1$ *and* $u, v \geqq 0$, *then*

$$\min(u^k, v^{k'}) \leqq uv \leqq \max(u^k, v^{k'}) .$$

PROOF. Suppose $u^k \leqq v^{k'}$. Then $u \leqq v^{k'/k}$ so that $uv \leqq v^{(k'/k)+1}=v^{k'}$. Similarly $v \geqq u^{k/k'}$ implies $uv \geqq u^k$.

LEMMA 3. *If* $p(n)>q(n) \geqq 1$ *for* $n=1, 2, \cdots$, *and if* $\sum_{n=1}^{\infty} \theta^{\frac{p(n)q(n)}{p(n)-q(n)}} = \infty$ *for all* $0<\theta<1$, *then there are numbers* $r(n)$ *and* $x(n)$ *with* $r(n) \to 0$ *as* $n \to \infty$ *such that* $\sum_{n=1}^{\infty} |x(n)|^{p(n)} < \infty$ *but* $\sum_{n=1}^{\infty} |r(n)x(n)|^{q(n)} = \infty$.

PROOF. We need only show that for any $r>0$ and any $0<\varepsilon<2$ there is an integer N and numbers $x(1), x(2), \cdots, x(N)$ such that $\sum_{n=1}^{N} |x(n)|^{p(n)} \leqq \varepsilon$ but $\sum_{n=1}^{N} |rx(n)|^{q(n)} \geqq 1$. (Repetition of this step with $\varepsilon=r=2^{-m}$ then proves the lemma.) Now choose θ with $0<\theta<\min(\varepsilon r/2, 1)$, choose N so that

$$1 \leqq \sum_{n=1}^{N} \theta^{\frac{p(n)q(n)}{p(n)-q(n)}} \leqq 2$$

and set $x(n)=r^{-1}\theta^{\frac{p(n)}{p(n)-q(n)}}$ for $n \leqq N$. Then

$$\sum_{n=1}^{N} |x(n)|^{p(n)} = \sum_{n=1}^{N} r^{-p(n)} \theta^{p(n)+\frac{p(n)q(n)}{p(n)-q(n)}}$$

$$\leqq \sum_{n=1}^{N} (\varepsilon/2)^{p(n)} \theta^{\frac{p(n)q(n)}{p(n)-q(n)}}$$

$$\leqq (\varepsilon/2) \sum_{n=1}^{N} \theta^{\frac{p(n)q(n)}{p(n)-q(n)}} \leqq \varepsilon ,$$

and

$$\sum_{n=1}^{N} |rx(n)|^{q(n)} = \sum_{n=1}^{N} \theta^{\frac{p(n)q(n)}{p(n)-q(n)}} \geqq 1.$$

COROLLARY. *For the $x(n)$ of Lemma* 1, $\sum_{n=1}^{\infty}|x(n)|^{p(n)}<\infty$ *but* $\sum_{n=1}^{\infty}|rx(n)|^{q(n)}=\infty$ *for every* $r>0$.

THEOREM 1. *Suppose $p(\alpha)>q(\alpha)\geqq 1$ for all α. In order that $\sum|x(\alpha)|^{p(\alpha)}<\infty$ should imply that $\sum|rx(\alpha)|^{q(\alpha)}<\infty$ for some $r>0$, the condition*

(3.1) $\sum\theta^{\frac{p(\alpha)q(\alpha)}{p(\alpha)-q(\alpha)}}<\infty$ *for some $0<\theta<1$ is necessary and sufficient.*

PROOF. The necessity follows easily from the Corollary to Lemma 1. On the other hand, if θ is a number for which (3.1) holds, then, using Lemma 2 with $k=p(\alpha)/q(\alpha)$,

$$\sum|\theta x(\alpha)|^{q(\alpha)}=\sum\theta^{q(\alpha)}|x(\alpha)|^{q(\alpha)}$$

$$\leqq\sum\left(\theta^{q(\alpha)\frac{p(\alpha)}{p(\alpha)-q(\alpha)}}+|x(\alpha)|^{q(\alpha)\frac{p(\alpha)}{q(\alpha)}}\right)$$

$$\leqq\sum\theta^{\frac{p(\alpha)q(\alpha)}{p(\alpha)-q(\alpha)}}+\sum|x(\alpha)|^{p(\alpha)}.$$

COROLLARY 1. *Suppose $p(\alpha)\geqq 1, q(\alpha)\geqq 1$ for all α. In order that $\sum|x(\alpha)|^{p(\alpha)}<\infty$ should imply that $\sum|rx(\alpha)|^{q(\alpha)}<\infty$ for some $r>0$ the condition*

$$\sum\theta^{\frac{p(\alpha)q(\alpha)}{p(\alpha)-q(\alpha)}}<\infty \qquad \textit{for some } 0<\theta<1$$

(the sum to be taken over all α for which $p(\alpha)>q(\alpha)$) is necessary and sufficient.

PROOF. This follows from the fact that $|x(\alpha)|\geqq 1$ for at most a finite number of α, and for other α, if $q(\alpha)\geqq p(\alpha)$ then $|\theta x(\alpha)|^{q(\alpha)}\leqq |x(\alpha)|^{q(\alpha)}\leqq|x(\alpha)|^{p(\alpha)}$ for all $0<\theta<1$.

COROLLARY 2. *$l(p)$ and $l(q)$ contain the same J-sequences if and only if* (2.1) *holds.*

4. Proof of (I).

4.1. The identity $\|cx\|=|c|\,\|x\|$ is clear. Moreover if $\|x\|\leqq\eta$ and $\|y\|\leqq\delta$ for some $\eta,\delta>0$, then using Lemma 1,

$$\sum w(\alpha)\left|\frac{x(\alpha)+y(\alpha)}{\eta+\delta}\right|^{p(\alpha)}\leqq\sum w(\alpha)\left(\left|\frac{x(\alpha)}{\eta}\right|^{p(\alpha)}\frac{\eta}{\eta+\delta}+\left|\frac{y(\alpha)}{\delta}\right|^{p(\alpha)}\frac{\delta}{\eta+\delta}\right)$$

$$\leq \frac{\eta}{\eta+\delta}+\frac{\delta}{\eta+\delta}=1$$

implying the triangle inequality $\|x+y\| \leq \|x\|+\|y\|$ for all x, y. Thus $l(p, w)$ is a linear, normed space and $\|x\|=0$ implies $x(\alpha)=0$ for all α since we assume $w(\alpha)>0$ for all α.

4.2. If $0 \leq u_1(\alpha) \leq u_2(\alpha) \leq \cdots$ and $u(\alpha)=\lim_{n\to\infty} u_n(\alpha) \leq \infty$ then $\|u\|=\lim_{n\to\infty}\|u_n\|$. For clearly $\geq$ holds and on the other hand if $\|u_n\| \leq \delta$ for all n then for every $\varepsilon>0, \sum w(\alpha)\left|\frac{u_n(\alpha)}{\delta+\varepsilon}\right|^{p(\alpha)} \leq 1$ for all n which implies $\sum w(\alpha)\left|\frac{u(\alpha)}{\delta+\varepsilon}\right|^{p(\alpha)} \leq 1$ and hence $\|u\| \leq \delta+\varepsilon$. From this follows $\|u\| \leq \delta$ and hence $\|u\| \leq \lim_{n\to\infty}\|u_n\|$.

4.3. If $\sum_{m=1}^{\infty}\|x_m\|<\infty$ then $x(\alpha)=\sum_{m=1}^{\infty} x_m(\alpha)$ is absolutely convergent for each α and $\|x\| \leq \sum_{m=1}^{\infty}\|x_m\|$. For § 4.2 and the triangle inequality apply to $u_n(\alpha)=\sum_{m=1}^{n}|x_m(\alpha)|$ and $u(\alpha)=\sum_{m=1}^{\infty}|x_m(\alpha)|$ show that $\|u\| \leq \sum_{m=1}^{\infty}\|x_m\|$. The relations $|x(\alpha)| \leq u(\alpha)$ now gives the statement of § 4.3.

4.4. To show that l is complete we may suppose x_n to be a sequence with $\|x_n-x_m\| \to 0$ as $n, m \to \infty$ and need only show that for some x, $\|x-x_n\| \to 0$ as $n \to \infty$. (The triangle inequality will then show that x is in l.) We need only obtain this with some infinite subsequence in place of the given sequence since the triangle inequality will extend this result to the original sequence. Thus, by suitable selection of subsequence, we may assume that $\sum_{n=1}^{\infty}\|x_{n+1}-x_n\|<\infty$. § 4.3 then implies that $x(\alpha)=x_1(\alpha)+\sum_{n=1}^{\infty}(x_{n+1}(\alpha)-x_n(\alpha))$ is defined for each α and $\|x-x_n\| \leq \sum_{m=n}^{\infty}\|x_{m+1}-x_m\|$ and $\to 0$ as $n \to \infty$.

5. Proof of (II).

5.1. Suppose $\rho(x)$ defined for every x in l and satisfying (i) $0 \leq \rho(x)<\infty$, (ii) $\rho(cx)=|c|\rho(x)$, and (iii) $\rho(x) \leq \rho(y)$ whenever $|x(\alpha)| \leq |y(\alpha)|$ for all α. Then, for some $M<\infty$, $\rho(x) \leq M\|x\|$ for all x. For otherwise there would be a sequence x_n with $\|x_n\|<2^{-n}$ and $\rho(x_n)>n$; then by § 4.3, $x(\alpha)=\sum_{n=1}^{\infty}|x_n(\alpha)|$ is in l but $\rho(x)>n$ for every n contradicting (i).

5.2. The relations $x(\alpha)=w(\alpha)^{1/p(\alpha)} y(\alpha)$ set up a (1,1) correspondence between all x in $l(p)$ and all y in $l(pw)$ (the mapping is actually isometric) so that $l(p, w) \cong l(p)$. Since the relation $\cong$ is transitive, $l(p, w_1) \cong l(q, w_2)$ if and only if $l(p) \cong l(q)$.

5.3. Suppose the relations $y(\alpha)=m(\alpha)\,x(\alpha)$ do set up a (1,1) correspondence between all x in $l(p)$ and all y in $l(q)$. Then § 5.1 applies to the function $\rho(x)=\|y\|$ showing that $\|y\| \leqq M \|x\|$ for some finite M. Choosing $|x(\alpha)|$ to have the value 1 for a particular α and the value 0 for all other α yields $|m(\alpha)| \leqq M$ for all α. Similarly $0 < m \leqq |m(\alpha)| \leqq M < \infty$ for all α. This means that $l(p) \cong l(q)$ if and only if $l(p)$ and $l(q)$ contain the same J-sequences ($l(p)$ and $l(q)$ will then be norm isomorphic although not necessarily norm equivalent as defined in [1, page 180]). Corollary 2 to Theorem 1 completes the proof of (II).

6. Proof of (III).

6.1. Clearly, if two Banach spaces are norm isomorphic then both or neither have the Schur property. In particular, $l(p, w)$ has the Schur property if and only if $l(p)$ has it.

6.2. Let $(1/p(\alpha))+(1/p'(\alpha))=1$ for each α for which $p(\alpha)>1$. If $|a(\alpha)| \leqq 1$ for every α and $\sum |a(\alpha)|^{p'(\alpha)}=A<\infty$ (the sum to be taken over all α for which $p(\alpha)>1$) then the linear functional

$$\varphi(x)=\sum a(\alpha)\,x(\alpha)$$

is bounded on $l(p)$ with $|\varphi| \leqq A+1$. For, using Lemma 2,

$$\begin{aligned} |a(\alpha)\,x(\alpha)| &\leqq |x(\alpha)|^{p(\alpha)}+|a(\alpha)|^{p'(\alpha)} && \text{if } p(\alpha)>1\,, \\ &\leqq |x(\alpha)|^{p(\alpha)} && \text{if } p(\alpha)=1\,, \end{aligned}$$

and so, for all $\|x\| \leqq 1$, $|\varphi(x)| \leqq 1+A$.

6.3. We suppose now that (2.2) holds, that x_n is weakly convergent, but not norm convergent, to 0 and we derive a contradiction. We may suppose $0<\varepsilon<\|x_n\| \leqq 1$ for all n (and hence $|x_n(\alpha)| \leqq 1$ for all α and all n) since weak convergence implies norm boundedness (see [1, page 80, Théorème 6]). § 6.2, with $|a(\alpha)|=1$ for a particular α and $=0$ for all other, α, shows that $x_n(\alpha) \rightarrow 0$ for every fixed α.

Since (2.2) implies that $p(\alpha) \leqq K$ for all α for some finite K, it follows that for all n,

$$\sum \left| \frac{x_n(\alpha)}{\varepsilon} \right|^{p(\alpha)} \leqq 1, \ 1 \geqq \sum | x_n(\alpha) |^{p(\alpha)} \geqq \eta$$

with $0 < \eta = \varepsilon^K < 1$. By induction on m we may choose $I(m)$ as disjoint finite collections of the α, and an $x_{n(m)}$ from the given sequence, so that

$$p(\alpha) < \frac{m}{m-1} \text{ for all } \alpha \text{ in } I(m),$$

$$\sum_{\alpha \in I(m)} | x_{n(m)}(\alpha) |^{p(\alpha)} > \frac{\eta}{2},$$

$$\sum_{\alpha \notin I(m)} | x_{n(m)}(\alpha) |^{p(\alpha)} < \left(\frac{\eta}{16}\right)^K$$

and so

$$\sum_{\alpha \notin I(m)} \left| \frac{x_{n(m)}(\alpha)}{\eta/16} \right|^{p(\alpha)} < 1.$$

Now for $\alpha \in I(m)$ with $x_{n(m)}(\alpha) \neq 0$ set

$$a(\alpha) = \frac{1}{2} \overline{x_{n(m)}(\alpha)} | x_{n(m)}(\alpha) |^{p(\alpha)-2},$$

and for all other α set $a(\alpha)=0$. Then $| a(\alpha) | \leqq 1$ for all α and

$$\sum | a(\alpha) |^{p'(\alpha)} \leqq \sum_{m=1}^{\infty} \sum_{\alpha \in I(m)} \left(\frac{1}{2}\right)^{p'(\alpha)} | x_{n(m)}(\alpha) |^{p(\alpha)}$$

$$\leqq \sum_{m=1}^{\infty} 2^{-m} = 1$$

so that, by § 6.2, $\sum a(\alpha) x_n(\alpha) \to 0$ as $n \to \infty$. But this is contradicted by:

$$| \sum a(\alpha) x_{n(m)}(\alpha) | \geqq \frac{1}{2} \frac{\eta}{2} - \sum_{\alpha \notin I(m)} | a(\alpha) x_{n(m)}(\alpha) | \geqq \frac{\eta}{4} - (1+1) \frac{\eta}{16}$$

for all m.

6.4. Suppose for some countable subset of J, which we may denote as 1,2, ..., that contrary to (2.2), $p(n) > 1+\varepsilon$ for all n for some $\varepsilon > 0$. For every bounded linear functional $\varphi(x)$ on l let $\varphi(e_\beta) = a_\beta$, where $|e_\beta(\alpha)| = 1$ if $\alpha=\beta$ and $=0$ if $\alpha \neq \beta$. Then $\| e_\beta \| = 1$ and for any finite sum $\| \sum_\beta c_\beta e_\beta \| \leq 1$ implies $| \sum_\beta c_\beta a_\beta | \leq | \varphi |$. But $\| \sum_{n=1}^\infty c_n e_n \| \leq 1$ whenever $\sum_{n=1}^\infty |c_n|^{p(n)} \leq 1$, in particular, whenever $\sum_{n=1}^\infty |c_n|^{1+\varepsilon} \leq 1$. Hence, by the converse to Holder's inequality [1, page 26, Theorem 15]

$$\sum_{n=1}^\infty |a_n|^{1-\frac{1}{1+\varepsilon}} \leq |\varphi|^{1-\frac{1}{1+\varepsilon}} < \infty$$

implying that $\varphi(e_n) = a_n \to 0$ as $n \to \infty$. Hence e_n is weakly convergent to 0 but $\| e_n \| = 1$ for all n. Thus l does not have the property of Schur if (2.2) fails to hold.

7. Proof of (IV). We need only find a sequence $p(n) = 1+\varepsilon(n)$ with $\varepsilon > 0$ for all n and $\varepsilon(n) \to 0$ as $n \to \infty$ and such that

$$\sum_{n=1}^\infty \left(\frac{1}{m}\right)^{1+\frac{1}{\varepsilon(n)}} = \infty$$

for all $m = 1, 2, \ldots$.

We may take $\varepsilon(n) = m^{-1}$ for all $N(m) < n \leq N(m+1)$ where $N(0)=0$ and $N(m)$ is defined by induction so that

$$\sum_{n=N(m)+1}^{N(m+1)} \left(\frac{1}{m}\right)^{1+m} \geq 1$$

i.e., so that $N(m+1) \geq N(m) + m^{m+1}$.

8. Generalization to the $l(p, w, B)$ spaces. With J, p and as before, let B be a family of Banach spaces $\{B(\alpha)\}$ and let $l(p, w, B)$ denote the space of J-sequences x with finite $\| x \|$ and with the value of $x(\alpha)$ an element of $B(\alpha)$ in place of a real or complex number. The preceding §§ may be read as they stand for this more general situation except for the following minor adjustments.

In § 4.2 the $u_n(\alpha)$ and $u(\alpha)$ continue to be real-valued J-sequences.

In § 5.1 the x_n should be selected so that for every fixed α all $x_n(\alpha)$ are of the form $c_n e_\alpha$ with e_α a fixed element in $B(\alpha)$; $x(\alpha)$ should then be taken to be $(\sum_{n=1}^\infty |c_n|) e_\alpha$.

(III) should be read: $l(p, w, B)$ has the property of Schur if and only if every $B(\alpha)$ has it and (2.2) holds.

In § 6.2, $a(\alpha)$ should be taken as an element of the conjugate Banach space $B(\alpha)^*$.

In § 6.3 application of the adjusted § 6.2 shows that for fixed α, $x_n(\alpha) \rightarrow 0$ weakly (in $B(\alpha)$) and hence $\rightarrow 0$ in norm in $B(\alpha)$ since $B(\alpha)$ is assumed to have the Schur property. The $I(m)$, $x_{n(m)}(\alpha)$ are defined as before but for each $\alpha \in I(m)$ with $x_{n(m)}(\alpha) \neq 0$, $a(\alpha)$ should be taken as

$$\frac{1}{2} | x_{n(m)}(\alpha) |^{t(\alpha)-1} u(\alpha)$$

with $u(\alpha)$ in $B(\alpha)^*$, $| u(\alpha) | = 1$, and $u(\alpha) x_{n(m)}(\alpha) = | x_{n(m)}(\alpha) |$ (such $u(\alpha)$ exist as shown in [1, page 55, Théorème 3]).

In § 6.4 the $e_\beta(\alpha)$ should be elements in $B(\alpha)$ and a_β should be the element in $B(\beta)^*$ obtained from φ by restricting x to have $x(\alpha)=0$ for $\alpha \neq \beta$. The argument of § 6.4 then shows that for finite sums, $\sum_\beta | c_\beta || a_\beta | \leqq | \varphi |$ whenever $|| \sum_\beta c_\beta e_\beta || \leqq 1$ for all such e_β. Hence $\sum_{n=1}^{\infty} | a_n |^{1-\frac{1}{1+\varepsilon}} < \infty$ and so $| a_n | \rightarrow 0$ as $n \rightarrow \infty$. Thus for any such e_n we have a sequence which is weakly convergent but not convergent in norm.

It is clear that if v_n were a sequence of element in a particular $B(\alpha)$ which converged weakly, but not in norm, to 0 then the same would be true in $l(p, w, B)$ for x_n which have $x_n(\alpha) = v_n$ for the particular α and $=0$ for all other α. Thus in order that $l(p, w, B)$ should have the property of Schur it is necessary that each $B(\alpha)$ have it.

9. Generalization to the $l(P, W, B)$ spaces. We mention briefly another possible generalization. Let the pair p, w be replaced by a family of pairs $P = \{p_\lambda\}$, $W = \{w_\lambda\}$ where λ varies over an arbitrary set Λ, $p_\lambda(\alpha) \geqq 1$ for all α, λ, $w_\lambda(\alpha) \geqq 0$ for all α, λ and for each α, $w_\lambda(\alpha) > 0$ for at least one λ. Define $|| x ||_\lambda$ for each pair p_λ, w_λ as in (1.1) and define $|| x || = \sup || x ||_\lambda$ for all $\lambda \in \Lambda$.

The arguments of § 4 extend easily to show that each $l(P, W, B)$ is a Banach space. We propose to discuss these spaces more fully in a later note.

Queen's University and Hokkaido University.

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