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# On the extension property of normed spaces over fields with non-archimedean valuations.

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If we wish to develop a theory of normed spaces over non-archimedean fields after the model of the usual theory in archimedean cases, the first thing to do would be to establish an analogue of the Hahn-Banach theorem on linear functionals. We shall examine in the present note in which case this is possible. We shall prove a simple theorem, which answers completely the question when the ground field is e. g. the $p$-adic field. The idea of the "binary intersection property" given by L. Nachbin in his paper[1] was very useful to our purpose.

Let $k$ be a complete field with a non-trivial discrete valuation $|\ |$. This field $k$ will be fixed throughout this paper.

Suppose a vector space $S$ over $k$ is *normed*[2]; i. e. to each element $x \in S$ corresponds a real number $\|x\|$, which has the properties:

1. $\|x\| \geqq 0$; $\|x\|=0$ *if and only if* $x=0$
2. $\|x+y\| \leqq \|x\| + \|y\|$
3. $\|\alpha x\|=|\alpha|\,\|x\|$ *for all* $\alpha \in k$.

A space $S$ in which the stronger form of the triangular inequality

2′. $\|x+y\| \leqq \mathrm{Max}(\|x\|, \|y\|)$

holds is called *non-archimedean*[3].

A linear and continuous mapping $f$ from a normed space $S$ into $k$ is called a *linear functional*. The set of all such functionals is written by $S^*$.

As in the ordinary case, we call a linear mapping $f$ from $S$ into $k$ bounded if there exists a real number $c$ such that $|f(x)| \leqq c\|x\|$ for all $x \in S$.

*A linear mapping f is continuous if and only if it is bounded.* In fact, first, suppose $f$ is bounded. Letting $x_n \to x$, we have

$$|f(x)-f(x_n)|=|f(x-x_n)| \leqq c\,\|x-x_n\|$$

and so $f(x_n) \to f(x)$, which proves the sufficiency. Next, suppose $f$ is not bounded. Since $|\ \ |$ is non-trivial, we can select a sequence $\{\beta_n\}$ such that $|\beta_n| \to \infty$. To each $\beta_n$ there exists $x_n \in S$ such that $|f(x_n)| > |\beta_n^2|\,\|x_n\|$. Put $y_n = (\beta_n/f(x_n))x_n$, then $\|y_n\| = (|\beta_n|/|f(x_n)|)\|x_n\| < 1/|\beta_n|$; hence $y_n \to 0$. On the other hand, $|f(y_n)|=|(\beta_n/f(x_n))f(x_n)|=|\beta_n| \to \infty$, and so $f$ is not continuous, which proves the necessity.

Therefore, we may define *norm* for $f \in S^*$ as usual: $\|f\|=\sup\limits_{x \neq 0} (|f(x)|/\|x\|)$. $S^*$ becomes thus a normed space over $k$: the *conjugate space* of $S$. From the definition of $\|f\|$, $S^*$ is non-archimedean, and it is complete.

Let $S$ be a normed space over $k$. $S$ is said to have the *extension property* if for any subspace $S_0$ of $S$ and for every $f_0 \in S_0^*$, there exists $f \in S^*$ such that $f$ is identical with $f_0$ on $S_0$ (written $f \to f_0$) and $\|f\|=\|f_0\|$.

We aim at the following

THEOREM. *A normed space S over k has the extension property if and only if S is non-archimedean.*

PROOF. Suppose $S$ is non-archimedean, and $f_0 \in S_0^*$, where $S_0$ is a subspace of $S$. If $f_0=0$, $f=0$ is the only norm-preserving extension. So we may suppose that $f_0 \neq 0$. Put $\mathfrak{M}=\{f_\lambda;\ f_\lambda \in S_\lambda^*,\ S \supset S_\lambda \supset S_0,\ f_\lambda \to f_0,\ \|f_\lambda\|=\|f_0\|\}$. Since $f_0 \in \mathfrak{M}$, $\mathfrak{M}$ is non-empty. For $f_\lambda, f_\mu \in \mathfrak{M}$, we shall write $f_\lambda > f_\mu$, if $f_\lambda \to f_\mu$, i.e. if $f_\lambda$ is a norm-preserving extension of $f_\mu$. $\mathfrak{M}$ is inductively ordered by this relation $>$, i.e. any non-empty linearly ordered subset $\mathfrak{L}$ in $\mathfrak{M}$ has a supremum in $\mathfrak{M}$. In fact, put $S_{\mathfrak{L}}=\bigcup\limits_{f_\lambda \in \mathfrak{L}} S_\lambda$, $f_{\mathfrak{L}}(x)=f_\lambda(x)$, $x \in S_\lambda$, then $f_{\mathfrak{L}} \in S_{\mathfrak{L}}^*$ and $\|f_{\mathfrak{L}}\|=\|f_0\|$, hence $f_{\mathfrak{L}} \in \mathfrak{M}$. Obviously $f_{\mathfrak{L}}=\sup\limits_{f_\lambda \in \mathfrak{L}} f_\lambda$. By Zorn's lemma there exists at least one maximal $f_{\mathfrak{M}} \in \mathfrak{M}$. The 'if'-part of our theorem will be proved, if we show that the domain $S_{\mathfrak{M}}$ of $f_{\mathfrak{M}}$ is identical with $S$, or that the $f_\mu$ whose domain $S_\mu$ is not identical with $S$, is not maximal.

For this purpose we prove the following

LEMMA. *Let $\{C_\alpha\}$ be a set of circles*[4] *in k, and suppose that for* *any* $\alpha$, $\beta$, $C_\alpha \frown C_\beta \neq 0$. *Then the total intersection* $\bigcap_\alpha C_\alpha \neq 0$[5].

PROOF OF LEMMA. As the valuation of $k$ is non-archimedean, every point of the circle may be considered as a center[6]. Hence, $C_\alpha \frown C_\beta \neq 0$ implies that two circles are concentric. So $\{C_\alpha\}$ is linearly ordered with respect to the inclusion relation.

Now, it is to be noted that a circle may have different radii, i.e. we may have $C=\{\eta;\ \eta \in k, |\eta-\xi| \leqq r\}=\{\eta;\ \eta \in k, |\eta-\xi| \leqq r'\}$ for $r \neq r'$. We shall call *the radius* of the circle $C$ the infimum of all such $r'$'s, and denote with $r_\alpha$ the radius of $C_\alpha$.

Then, $C_\alpha \supseteq C_\beta$ if and only if $r_\alpha \geqq r_\beta$, particularly $C_\alpha = C_\beta$ if and only if $r_\alpha = r_\beta$.

If $r_\alpha = 0$ for some $\alpha$, then $C_\alpha$ consists of a single point, and the lemma is trivial. So we may exclude this case, and suppose $r_\alpha$ are all $> 0$. We consider separately the following two cases.

First, let $\inf_\alpha r_\alpha = 0$. Then we can select a decreasing sequence of the radii $\{r_n\}$ such that $r_n \to 0$. We take a point $\gamma_n \in C_n - C_{n+1}$ for $n=1, 2, \cdots$, then $\{\gamma_n\}$ forms a Cauchy-sequence in $k$. The limit $\gamma$ of this sequence belongs to the total intersection.

Next, let $\inf_\alpha r_\alpha > 0$, and $\beta$ be an arbitrarily fixed index. Then, according to the discreteness of $k$, we have only a finite number of $r_\alpha$, such that $r_\alpha \leqq r_\beta$. So we have only a finite number of $C_\alpha$, such that $C_\alpha \subseteq C_\beta$. Hence the total intersection $\bigcap_\alpha C_\alpha = \bigcap_{r_\alpha \leqq r_\beta} C_\alpha \neq 0$, q.e.d.

We return to our $f_\mu$, whose domain $S_\mu$ is not identical with $S$. As $f_\mu \neq 0$, we have $f_\mu(S_\mu) = k$. Since we can select $z \in S - S_\mu$, $\rho(\beta) = \|f_0\|\,\mathrm{dist}(z, f_\mu^{-1}(\beta))$ is defined for all $\beta \in k$. We consider the set of circles $\{C_\beta;\ \beta \in k\}$, where $C_\beta = \{\alpha;\ |\alpha - \beta| \leqq \rho(\beta)\}$. It follows that

$$|\beta-\beta'| = |f_\mu(x) - f_\mu(x')| = |f_\mu(x-x')| \leqq \|f_\mu\|\,\|x-x'\| \quad .$$

$$= \|f_0\|\,\|x-x'\| \leqq \mathrm{Max}(\|f_0\|\,\|z-x\|,\ \|f_0\|\,\|z-x'\|).$$

So we get

$$|\beta-\beta'| \leqq \mathrm{Max}(\rho(\beta),\ \rho(\beta')).$$

This means that $C_\beta \frown C_{\beta'} \neq 0$. From the lemma there exists $\gamma \in \bigcap_{\beta \in k} C_\beta$. Namely, $|\gamma - \beta| \leqq \rho(\beta)$ for all $\beta \in k$, or

$$|\gamma - f_\mu(x)| \leqq \|f_0\|\,\|z-x\| \qquad (*)$$

for all $x \in S_\mu$. Let $S'$ be the space spanned by $S_\mu$ and $z$. Then, $x' \in S'$ can be written as $x'=x+\alpha z$, $x \in S_\mu$, $\alpha \in k$, uniquely. Put $f'(x')=f_\mu(x)+\alpha\gamma$, then obviously $f'$ is linear and $f' \to f_0$. If $\alpha \neq 0$, $|\gamma - f_\mu(-x/\alpha)| \leqq \|f_0\| \, \|z-(-x/\alpha)\|$ from (*). And so $|\alpha\gamma + f_\mu(x)| \leqq \|f_0\| \, \|\alpha z + x\|$. This inequality holds even for $\alpha=0$. Hence, for all $x' \in S'$, we get $|f'(x')| \leqq \|f_0\| \, \|x'\|$. So $\|f'\| \leqq \|f_0\|$ (bounded), and $f'$ is continuous. Thus, $f' \in S'^*$. Since $f' \to f_0$, we have $\|f'\| \geqq \|f_0\|$. Therefore $f' \in \mathfrak{M}$. Since $S'$ is not identical with $S_\mu$, $f_\mu$ can not be maximal. Thus the 'if'-part of the theorem is proved.

To prove the converse, suppose that $S$ has the extension property. As the conjugate $S^*$ is non-archimedean, $S^{**}$ is also non-archimedean. Thus, it is sufficient to show that $S$ can be imbedded in $S^{**}$.

Defining $X(f)=f(x)$, $x \in S, f \in S^*$, $X$ may be considered as $\in S^{**}$. The mapping: $x \to X$ is a $k$-homorphism and we have $\|X\| \leqq \|x\|$. Moreover, we shall show that $x \to X$ is a norm-preserving $k$-isomorphism: $\|X\|=\|x\|$. Suppose, namely, $x \neq 0$. The functional $f_0$ for subspace $S_0=\{\alpha x;\ \alpha \in k\}$, defined by $f_0(\alpha x)=\alpha$ is $\in S_0^*$, and $f_0(x)=1$, $\|f_0\|=1/\|x\|$. According to the extension property of $S$, there exists at least one $f \in S^*$ such that $f \to f_0$, $\|f\|=\|f_0\|$. Then, $f(x)=f_0(x)=1 \neq 0$. Hence $x \to X$ is a $k$-isomorphism. Lastly, for above $f$ we have $|X(f)|/\|f\|=|f(x)|/\|f\|=1/\|f\|=\|x\|$ and so $\|X\|=\sup_{f \neq 0}(|X(f)|/\|f\|) \geqq \|x\|$, namely $\|X\|=\|x\|$, which proves the 'only if'-part.

*Addendum.* After this paper had been prepared, we knew that the same subject was treated by A. W. Ingleton,[7] A. F. Monna[5] and I. S. Cohen[9]. We have not yet access to the papers of A. F. Monna and I. S. Cohen. We acknowledge that this paper has the essential part in common with the paper of Ingleton, but we submit this paper to publication, as there is some difference in the formulation of the final result in both papers, and we have a certain generalization of our result in view, which will be published on a later occasion.

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## Reference and Notes.

1) L. Nachbin, A theorem of the Hahn-Banach type for linear transformations. Trans. Amer. Math. Soc., 68 (1950).

2) E. Artin has introduced the normed space of finite dimension in his lecture 'Algebraic numbers and algebraic functions I.' (1950-1951). Princeton.

3) E. g. the algebraic extension field over a field $k$, or the totality of continuous mappings of a compact space into $k$ are non-archimedean.

4) We difine a circle of center $\xi$ and radius $r > 0$ to be a set $C = \{\eta\ ;\ \eta \in k, |\eta - \xi| < r\}$.

5) Nachbin has called such a property of the set of circles 'binary intersection property'. cf. l.c. (1).

6) l.c. (2), p. 39.

7) A. W. Ingleton, The Hahn-Banach theorem for non-Archimedean valued fields. Proc. Cambridge Phil. Soc., 48 (1952).

8) A F. Monna, Sur les espaces linéaires normés, III. Indag. Math., 8 (1946).

9) I. S. Cohen, On non-Archimedean normed spaces. Indag. Math., 10 (1948).