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Fundamental theorems in potential theory.

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The potential theory plays an important rôle in function-theory, so that in this paper, I shall prove fundamental theorems in potential theory in the shortest lines. Almost all results are known and the proofs are not new, but are somewhat simpler than the usual ones.

Theorem 20 seems to be new and of some interest. In view of applications to function-theory, we confine ourselves to logarithmic potentials.

1. Maximum principle.

Let F be a bounded closed set on the z-plane and $\mu(e)\geqq 0$ be a positive mass distribution on F of finite total mass and consider the potential:

$$u(z)=\int_F \log\frac{1}{|z-a|}\,d\mu(a).$$

THEOREM 1. (*Maximum principle*).[1] *If* $u(z)\leqq K$ *on* F, *then* $u(z)\leqq K$ *in the whole* z-*plane.*

PROOF. Let D be the complement of F and $a_0\in F$ be its boundary point. It is sufficient to prove

$$\overline{\lim_{z\to a_0}}\, u(z)\leqq K \qquad (z\in D).$$

Let D_ρ be the part of D contained in $|z-a_0|<\rho$ and F_ρ be that of F

1) For Newtonian potentials: M. A. Maria: The potential of a positive mass and the weight function of Wiener. Proc. Nat. Acad. Sci. U S. A. 20 (1934). For general potentials: O. Frostman: Potentiel d'équilibre et capacité des ensembles, Lund (1935). Frostman's proof depends on Poincaré's sweeping-out process. A simple proof independent of the sweeping-out process was given by Y. Yosida: Sur le principe du maximum dans la théorie du potentiel. Proc. Imp. Acad. 17 (1941).

contained in $|z-a_0|\leqq\rho$. Since $u(z)\leqq K$ on F, a single point does not contain a positive mass, so that we take ρ so small that $\mu(F_\rho)<\varepsilon$. Let $z\in D_\rho$. We choose $z_1\in F_\rho$, such that $|z-z_1|\leqq|z-a|$ for any $a\in F_\rho$, then

$$|z_1-a|\leqq|z-z_1|+|z-a|\leqq 2|z-a| \quad \text{for any } a\in F_\rho,$$

so that

$$\int_{F_\rho}\log\frac{1}{|z-a|}\,d\mu(a)\leqq\mu(F_\rho)\log 2+\int_{F_\rho}\log\frac{1}{|z_1-a|}$$

$$d\mu(a)<\varepsilon\log 2+\int_F\log\frac{1}{|z_1-a|}\,d\mu(a)-\int_{F-F_\rho}\log\frac{1}{|z_1-a|}$$

$$d\mu(a)<\varepsilon\log 2+K-\int_{F-F_\rho}\log\frac{1}{|z_1-a|}\,d\mu(a),$$

$$\int_{F_\rho}\log\frac{1}{|z-a|}\,d\mu(a)+\int_{F-F_\rho}\log\frac{1}{|z_1-a|}\,d\mu(a)<\varepsilon\log 2+K.$$

Since $z_1\to a_0$, as $z\to a_0$, if z is sufficiently near to a_0,

$$\int_{F-F_\rho}\log\frac{1}{|z_1-a|}\,d\mu(a)>\int_{F-F_\rho}\log\frac{1}{|z-a|}\,d\mu(a)-\varepsilon,$$

hence

$$u(z)=\int_{F_\rho}\log\frac{1}{|z-a|}\,d\mu(a)+\int_{F-F_\rho}\log\frac{1}{|z-a|}\,d\mu(a)<\varepsilon\log 2+K+\varepsilon,$$

so that

$$\overline{\lim_{z\to a_0}}\,u(z)\leqq K.$$

THEOREM 2[2]. *Let $a_0\in F$ be a boundary point of D. If $u(z)$ is continuous at a_0 considered as a function on F, then $u(z)$ is continuous at a_0 considered as function in the full neighbourhood of a_0.*

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2) G. C. Evans: Application of Poincaré's sweeping-out process, Proc. Nat. Acad. Sci. U. S. A. 19 (1933). On potentials of positive mass, I, Trans. Amer. Math. Soc. 37 (1935). Vasilesco: Sur la continuité du potentiel à travers des masses et la démonstration d'un lemme de Kellogg, C. R. 200 (1935). For general potentials $u(P)=\int_F\Phi(\overline{PQ})d\mu(Q)$: T. Ugaheri: On the general potentials and capacity, Jap. Journ. Math. 20 (1950).

PROOF. In the above proof, if ρ is small, then $u(z_1) < u(a_0)+\varepsilon$, hence taking $K=u(a_0)+\varepsilon$, we have

$$\overline{\lim_{z\to a_0}}\, u(z) \leqq u(a_0) \quad (z \in D).$$

Since $u(z)$ is lower semi-continuous, we have $\lim\limits_{z\to a_0} u(z)=u(a_0)$, q. e. d.

THEOREM 3. *Let $\mu(e) \geqq 0$ be a positive mass distribution of finite total mass in a finite domain D. If*

$$u(z)= \int_D \log \frac{1}{|z-a|}\, d\mu(a)=\text{const.}=V$$

almost everywhere in a neighbourhood $U \subset D$, then $\mu(U)=0$.

PROOF. We suppose that $U: |z| < \rho$ and let D be contained in $|z| < R$. We put

$$\Omega(r)=\int_{|a|<r} d\mu(a) \qquad (0 < r < R).$$

Then since

$$\frac{1}{2\pi}\int_0^{2\pi} \log |re^{i\theta}-a|\, d\theta=\text{Max}(\log r,\ \log |a|),$$

$$\frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\, d\theta=-\int_0^R \text{Max}(\log r, \log t)\, d\Omega(t)$$

$$=\int_r^R \frac{\Omega(t)}{t}\, dt-\Omega(R)\log R.$$

By Fubini's theorem, $u(re^{i\theta})=V$ almost everywhere on $|z|=r$ $(0<r<\rho)$, except a null set of r in $(0, \rho)$, so that for a non-exceptional r,

$$\frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\, d\theta=V.$$

Hence for a non-exceptional r,

$$\int_r^R \frac{\Omega(t)}{t}\, dt=\text{const.} \quad (0 < r < \rho),$$

so that $\Omega(r)=0$ $(0 < r < \rho)$, or $\mu(U)=0$.

2. Capacity and conductor potential.

1. Let F be a bounded closed set on the z-plane and $\mu(e) \geqq 0$ be a positive mass distribution on F of total mass 1. We consider with Frostman the energy integral:

$$I(\mu)=\iint_F \log \frac{1}{|a-b|}\, d\mu(a)\, d\mu(b)\,, \quad \mu(F)=1\,, \tag{1}$$

and let

$$V=\inf_{\mu} I(\mu)\,. \tag{2}$$

We take $K \geqq 1$ so large that $|a-b| \leqq K$ for any $a \in F$, $b \in F$, then

$$I(\mu)=\iint_F \log \frac{K}{|a-b|}\, d\mu(a)\, d\mu(b) - \log K \geqq -\log K\,,$$

so that $V > -\infty$.

We define the capacity of F by

$$C(F)=e^{-V}\,. \tag{3}$$

If $C(F) > 0$, then there exists μ, such that $I(\mu)=V$.[3] We call μ the equilibrium distribution and

$$u(z)=\int_F \log \frac{1}{|z-a|}\, d\mu(a)\,, \quad \mu(F)=1 \tag{4}$$

the conductor potential of F. $u(z)$ is lower semi-continuous and super-harmonic.

The kernel F^* of μ is defined as the set of points a, such that any small neighbourhood of a contains a positive μ-mass. Evidently F^* is a closed sub-set of F.

The capacity of any Borel set E is defined by

$$C(E)=\sup_{F \subset E} C(F)\,, \tag{5}$$

where F are closed sub-sets of E.

It can be proved easily that if $C(E)=0$, then the measure of E is zero and if $C(E_n)=0$ $(n=1, 2, \cdots)$, then $C(\sum_{n=1}^{\infty} E_n)=0$.

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3) Frostman, l. c. 1).

In this paper "almost everywhere" means "except a set of capacity zero".

THEOREM 4. *Let $\mu(e)\geqq 0$ be a positive mass distribution on a bounded closed set F of total mass* 1 *and*

$$u(z)=\int_F \log\frac{1}{|z-a|}\,d\mu(a)\,,\quad \mu(F)=1\,.$$

If $\int_F u(z)\,d\mu(z)<\infty$, then any Borel set $e\subset F$ of capacity zero does not contain a positive μ-mass, i.e. $\mu(e)=0$.

PROOF. Suppose that $\mu(e)=m>0$. We may assume that e is closed. We take $K\geqq 1$, so large that $|a-b|\leqq K$ for any $a\in F$, $b\in F$, then

$$u(z)+\log K=\int_F \log\frac{K}{|z-b|}\,d\mu(b)\,,$$

hence

$$\int_F u(a)\,d\mu(a)+\log K\geqq\iint_e \log\frac{K}{|a-b|}\,d\mu(a)\,d\mu(b)$$

$$=\iint_e \log\frac{1}{|a-b|}\,d\mu(a)\,d\mu(b)+m^2\log K\,.$$

Hence $\iint_e \log\frac{1}{|a-b|}\,d\mu(a)\,d\mu(b)<\infty$, so that $C(e)>0$, which contradicts the hypothesis. Hence $\mu(e)=0$.

THEOREM 5. *Let E be a bounded F_σ-set of capacity zero, then we can distribute a positive mass $\mu(e)>0$ on E, such that*

$$u(z)=\int_E \log\frac{1}{|z-a|}\,d\mu(a)\,,\quad \mu(E)=1$$

tends to $+\infty$, when z tends to any point of E.

This follows from Evans's theorem[4], where E is closed.

We call $u(z)$ the Evans's function with respect to E. We use this function frequently in this paper.

2. Now we shall prove the following fundamental theorem in the potential theory.

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4) G. C. Evans: Potentials and positively infinite singularities of harmonic functions, Monatshefte f. Math. u. Phys. 43 (1936).

THEOREM 6.[5] *Let $u(z)$ be the conductor potential of a bounded closed set F, which is of positive capacity. Then $u(z)\leqq V$ in the whole z-plane and $u(z)=V$ "almost everywhere" on F, such that $u(z)=V$ on F, except at an F_σ-set of capacity zero.*

PROOF. First we shall prove

$$u(z)\geqq V \quad \text{"almost everywhere" on } F. \tag{1}$$

Suppose that $u(z)<V$ on a set $E\subset F$, such that $C(E)>0$, then we can find a suitable closed set $F_0\subset E$, such that $C(F_0)>0$ and

$$u(z)<V-2\varepsilon \quad \text{on } F_0 \quad \text{for some } \varepsilon>0\,. \tag{2}$$

Let F^* be the kernel of μ, then since $\int_{F^*} u(a)\,d\mu(a)=V$, there exists $a_0\in F^*$, such that $u(a_0)>V-\varepsilon$, then a_0 lies outside F_0. By the lower semi-continuity of $u(z)$,

$$u(z)>V-\varepsilon \qquad \text{in } U(a_0)\,, \tag{3}$$

where $U(a_0)$ is a neighbourhood of a_0, which we take so small that $U(a_0)$ and F_0 have a positive distance.

Since $C(F_0)>0$, there exists $\sigma\geqq 0$ on F_0, such that

$$\sigma(F_0)=\mu\big(U(a_0)\big)=m>0\,,\quad I(\sigma)=\iint_{F_0}\log\frac{1}{|a-b|}\,d\sigma(a)\,d\sigma(b)<\infty\,. \tag{4}$$

We put

$$\sigma_1=-\mu \text{ in } U(a_0)\,,\quad \sigma_1=\sigma \text{ in } F_0\,,\quad \sigma_1=0 \text{ elsewhere,} \tag{5}$$

then

$$I(\sigma_1)=\iint_F\log\frac{1}{|a-b|}\,d\sigma_1(a)\,d\sigma_1(b)\neq\infty\,,\quad \sigma_1(F)=0\,.$$

Hence for $0<\eta<1$, we have $\mu+\eta\sigma_1>0$, $\int_F d(\mu+\eta\sigma_1)=1$ and

$$\begin{aligned}\delta I&=I(\mu+\eta\sigma_1)-I(\mu)=2\eta\int_F u\,d\sigma_1+\eta^2 I(\sigma_1)\\ &<2\eta\Big[m(V-2\varepsilon)-m(V-\varepsilon)\Big]+\eta^2 I(\sigma_1)=-\eta\Big[\,2m\varepsilon-\eta I(\sigma_1)\,\Big]<0\end{aligned}$$

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5) Frostman, l.c. 1).

for a small $\eta > 0$, which contradicts the definition of $I(\mu)$. Hence $u(z) \geqq V$ "almost everywhere" on F.

Next we shall prove

$$u(z) \leqq V \text{ on } F^* .$$

Suppose that $u(a_0) > V+\varepsilon$ $(a_0 \in F^*, \varepsilon > 0)$, then by the lower semi-continuity of $u(z)$,

$$u(z) > V+\varepsilon \text{ in } U_0 = U(a_0) . \tag{7}$$

Hence by (1) and Theorem 4,

$$V = \int_{F^* U_0} u(a)\, d\mu(a) + \int_{F^* - F^* U_0} u(a)\, d\mu(a) \geqq (V+\varepsilon)\, \mu(U_0)$$
$$+ V\big(1-\mu(U_0)\big) = V + \varepsilon\mu(U_0) > V ,$$

which is absurd, so that $u(z) \leqq V$ on F^*. Hence by the maximum principle, $u(z) \leqq V$ in the whole z-plane, so that by (1), $u(z)=V$ "almost everywhere" on F.

Since $u(z)$ is lower semi-continuous and $u(z) \leqq V$, the set of points of F, such that $u(z) < V$ is an F_σ-set. Hence our theorem is proved.

We have easily

THEOREM 7. *If* $u(a_0)=V$ $(a_0 \in F)$, *then* $u(z)$ *is continuous at* a_0.

The complement of F consists of at most a countable number of connected domains $D_\infty + \{D_\nu\}$, where D_∞ contains $z=\infty$. Let Γ be the boundary of D_∞. We call Γ the outer boundary of F.

THEORFM 8.[6] *μ-mass lies on the outer boundary Γ of F and $\Gamma - F^*$ is of capacity zero.*

PROOF. $u(z)$ is a bounded harmonic function in D_ν and by Theorem 6, its boundary value is V "almost everywhere", so that $u(z) \equiv V$ in D_ν. Let z_0 be an inner point of F and $U: |z-z_0| < \rho$ be contained in F, then $u(z)=V$ almost everywhere in U, so that by Theorem 3, $\mu(U)=0$, hence $u(z_0)$ is harmonic in U and $u(z_0)=V$. Hence $u(z)=V$ at inner points of F. Now the complement of Γ consists of at most a countable number of connected domains $D_\infty + \{\Delta_\nu\}$. Since $u(z)=V$ in D_ν and at inner points of F and $u(z)=V$ "almost everywhere" on the boundary of D_ν, $u(z)=V$ almost everywhere in Δ_ν, so that by Theorem

6) Frostman, l. c. 1).

3, $\mu(\Delta_\nu)=0$. Hence the mass lies on Γ. We put $E=\Gamma-F^*$ and suppose that $C(E)>0$. Then E contains a closed sub-set F_0 of positive capacity. Since F_0 and F^* have a positive distance, $u(z)<V$ on F_0, but by Theorem 6, F contains a point, such that $u(z)=V$, which is absurd. Hence $C(E)=0$.

THEOREM 9.[7] *μ is unique.*

PROOF. Suppose that $I(\mu_1)=I(\mu_2)=V$ and let

$$u_1(z)=\int_\Gamma \log\frac{1}{|z-a|}\,d\mu_1(a)\,,\qquad u_2(z)=\int_\Gamma \log\frac{1}{|z-a|}\,d\mu_2(a)\,,$$

$$\mu_1(\Gamma)=\mu_2(\Gamma)=1\,,$$

then $u_1(z)=u_2(z)=V$ "almost everywhere" on Γ. Hence

$$u(z)=u_1(z)-u_2(z)=\int_\Gamma \log\frac{1}{|z-a|}\,d\mu(a)\,,\quad (\mu=\mu_1-\mu_2)$$

is a bounded harmonic function in D_∞ and its boundary value vanishes "almost everywhere", so that $u(z)\equiv 0$ in D_∞. Since $u(z)=0$ in Δ_ν and "almost everywhere" on Γ, $u(z)=0$ almost everywhere in the whole z-plane, so that by Theorem 3, $\mu\equiv 0$, or $\mu_1=\mu_2$.

3. Green's functions.

1. Let D be a finite or an infinite domain, but we assume that its boundary Γ is a bounded closed set of positive capacity. Let $\mu\geqq 0$ be a positive mass-distribution on Γ of total mass 1 and z_0 be a fixed point of D.

We consider with Frostman

$$G(\mu)=\int_\Gamma\left(\int_\Gamma \log\frac{1}{|a-b|}\,d\mu(a)-2\log\frac{1}{|b-z_0|}\right)d\mu(b)\,, \tag{1}$$

$$G=\inf_\mu G(\mu)\,. \tag{2}$$

Then there exists μ_{z_0}, such that $G(\mu_{z_0})=G$[8]. μ_{z_0} is called the mass of balayage.

7) Frostman, l. c. 1).
8) Frostman, l. c. 1).

Since $G=G(\mu_{z_0})\neq\infty$,

$$\iint_{\Gamma}\log\frac{1}{|a-b|}\,d\mu_{z_0}(a)\;d\mu_{z_0}(b)<\infty\,,$$

hence by Theorem 4, a set of capacity zero on Γ does not contain a positive μ_{z_0}-mass.

We put

$$h(z)=\int_{\Gamma}\log\frac{1}{|z-a|}\,d\mu_{z_0}(a)-\log\frac{1}{|z-z_0|}\qquad\left(\mu_{z_0}(\Gamma)=1\right)\tag{3}$$

and let

$$\gamma(z_0)=\sup_{\Gamma^*}\;h(z)\,,\tag{4}$$

where Γ^* is the kernel of μ_{z_0}. Then we can prove similarly as Theorem 6,

$$h(z)=\gamma(z_0)\ \text{“almost everywhere” on }\Gamma\,.\tag{5}$$

First we shall prove

$$h(z)\geqq\gamma(z_0)\ \text{“almost everywhere” on }\Gamma\,.\tag{6}$$

If $h(z)<\gamma(z_0)$ on a set $E\subset\Gamma$ of positive capacity, then by the same notation as in the proof of Theorem 6,

$$h(z)<\gamma(z_0)-2\varepsilon\ \text{ on }F_0\,,\quad C(F_0)>0\,.$$

By the definition of $\gamma(z_0)$, there exists $a_0\in\Gamma^*$, such that $h(a_0)>\gamma(z_0)-\varepsilon$, so that

$$h(z)>\gamma(z_0)-\varepsilon\quad\text{in}\quad U_0=U(a_0)\,.$$

We define σ,σ_1 as before, then for $0<\eta<1$,

$$\begin{aligned}\delta G=G(\mu_{z_0}+\eta\sigma_1)-G(\mu_{z_0})&=2\eta\int_{\Gamma}h(a)\,d\sigma_1(a)+\eta^2I(\sigma_1)\\&<2\eta\Big[m\big(\gamma(z_0)-2\varepsilon\big)-m\big(\gamma(z_0)-\varepsilon\big)\Big]+\eta^2I(\sigma_1)\\&=-\eta\Big[2m\varepsilon-\eta I(\sigma_1)\Big]<0\end{aligned}$$

for a small $\eta>0$, which contradicts the definition of $G(\mu_{z_0})$. Hence $h(z)\geqq\gamma(z_0)$ “almost everywhere” on Γ.

Since $h(z) \leqq \gamma(z_0)$ on Γ and $h(z)$ is harmonic at $z=\infty$, by the maximum principle,

$$h(z) \leqq \gamma(z_0) \text{ in the whole } z\text{-plane}, \tag{7}$$

so that by (6),

$$h(z)=\gamma(z_0) \quad \text{"almost everywhere" on } \Gamma, \qquad \text{q. e. d.}$$

If D is a finite domain, then $\gamma(z_0)=0$. For let

$$u(z)=\int_\Gamma \log \frac{1}{|z-a|}\, d\mu(a) \quad \left(\mu(\Gamma)=1\right)$$

be the conductor potential of Γ, then since $u(z_0)=V$ and $u(z)=V$ "almost everywhere" on Γ and a set of capacity zero does not contain a positive μ and μ_{z_0} mass, we have by (5),

$$\gamma(z_0)=\int_\Gamma h(z)\, d\mu(z)$$

$$=\int_\Gamma d\mu_{z_0}(a)\int_\Gamma \log \frac{1}{|z-a|}\, d\mu(z)-\int_\Gamma \log \frac{1}{|z-z_0|}\, d\mu(z)=V-V=0. \tag{8}$$

If D is an infinite domain, then $\gamma(z_0)>0$. For

$$\gamma(z_0)=V-\int_\Gamma \log \frac{1}{|z-z_0|}\, d\mu(z)>0. \tag{9}$$

2. We define the Green's function $g(z, z_0)$ of D by

$$\left.\begin{aligned} g(z, z_0)&=\gamma(z_0)-h(z) \quad (z_0 \neq \infty), \\ g(z, \infty)&=V-\int_\Gamma \log \frac{1}{|z-a|}\, d\mu(a), \end{aligned}\right\} \tag{10}$$

where μ is the equilibrium distribution on Γ.

Hence:

(i) If D is a finite domain,

$$g(z, z_0)=\log \frac{1}{|z-z_0|}-\int_\Gamma \log \frac{1}{|z-a|}\, d\mu_{z_0}(a); \tag{11}$$

(ii) If D is an infinite domain,

$$\left.\begin{aligned} g(z, z_0) &= \log \frac{1}{|z-z_0|} + \gamma(z_0) - \int_\Gamma \log \frac{1}{|z-a|}\, d\mu_{z_0}(a) \quad (z_0 \neq \infty), \\ g(z, \infty) &= V - \int_\Gamma \log \frac{1}{|z-a|}\, d\mu(a). \end{aligned}\right\} \qquad (12)$$

Hence from (9) and (12),

$$\gamma(z_0) = g(z_0, \infty) = g(\infty, z_0). \qquad (13)$$

$g(z, z_0)$ is upper semi-continuous and subharmonic. Hence the set of points on Γ, such that $g(z, z_0) > 0$ is an F_σ-set.

We can prove easily that $g(z, z_0) = 0$ in the complement of $D + \Gamma$. By this and the upper semi-continuity and subharmonicity of $g(z, z_0)$, we can prove that $g(a_0, z_0) = 0$ $(a_0 \in \Gamma)$, when and only when $\lim_{z \to a_0} g(z, z_0) = 0$ $(z \in D)$. Hence we have proved:

Theorem 10.[9] *$g(z, z_0) > 0$ in D and $g(z, z_0) = 0$ "almost everywhere" on Γ. The set of points z on Γ such that $g(z, z_0) > 0$ is an F_σ-set of capacity zero.*

Let $a_0 \in \Gamma$, then $g(a_0, z_0) = 0$, when and only when $\lim_{z \to a_0} g(z, z_0) = 0$ $(z \in D)$.

Since $h(z) \leqq \gamma(z_0)$ in the whole z-plane and $h(z) = \gamma(z_0)$ "almost everywhere" on Γ, we can prove similarly as Theorem 8,

Theorem 11. *Let $\Gamma^*_{z_0}$ be the kernel of μ_{z_0}, then $\Gamma - \Gamma^*_{z_0}$ is of capacity zero.*

Theorem 12.[10] *μ_{z_0} is unique.*

Proof. Suppose that "almost everywhere" on Γ,

$$\log \frac{1}{|z-z_0|} + \gamma_1(z_0) - \int_\Gamma \log \frac{1}{|z-a|}\, d\mu^1_{z_0}(a) = 0,$$

$$\log \frac{1}{|z-z_0|} + \gamma_2(z_0) - \int_\Gamma \log \frac{1}{|z-a|}\, d\mu^2_{z_0}(a) = 0,$$

then "almost everywhere" on Γ,

$$u(z) = \int_\Gamma \log \frac{1}{|z-a|}\, d\mu(a) = \gamma_1(z_0) - \gamma_2(z_0) \quad (\mu = \mu^1_{z_0} - \mu^2_{z_0}).$$

9) Frostman, l. c. 1).

10) Frostman, l. c. 1).

From this we can prove easily that $u(z)=\gamma_1(z_0)-\gamma_2(z_0)$ almost everywhere in the whole z-plane, so that by Theorem 3, $\mu\equiv 0$ or $\mu^1_{z_0}=\mu^2_{z_0}$.

THEOREM 13.[11] $g(z_2, z_1)=g(z_1, z_2)$.

PROOF. Since $g(z, \infty)=g(\infty, z)$, we assume that $z_1\neq\infty$, $z_2\neq\infty$. It suffices to prove

$$\gamma(z_1)-\int_\Gamma \log\frac{1}{|z_2-a|}\,d\mu_{z_1}(a)=\gamma(z_2)-\int_\Gamma \log\frac{1}{|z_1-a|}\,d\mu_{z_2}(a)\,. \qquad (1)$$

Since $g(z, z_2)=0$ "almost everywhere" on Γ,

$$\log\frac{1}{|z_2-a|}=-\gamma(z_2)+\int_\Gamma \log\frac{1}{|a-b|}\,d\mu_{z_2}(b)$$

for "almost all" a on Γ. Since the exceptional set does not contain a positive μ_{z_1}-mass,

$$\int_\Gamma \log\frac{1}{|z_2-a|}\,d\mu_{z_1}(a)=-\gamma(z_2)+\iint_\Gamma \log\frac{1}{|a-b|}\,d\mu_{z_1}(a)\,d\mu_{z_2}(b)\,,$$

$$\gamma(z_1)-\int_\Gamma \log\frac{1}{|z_2-a|}\,d\mu_{z_1}(a)=\gamma(z_1)+\gamma(z_2)-\iint_\Gamma \log\frac{1}{|a-b|}\,d\mu_{z_1}(a)d\mu_{z_2}(b)\,.$$

Hence by the symmetry, we have (1).

THEOREM 14. *We approximate D by a sequence of domains $D_1\subset D_2\subset\cdots\subset D_n\to D$, where the boundary Γ_n of D_n consists of a finite number of analytic Jordan curves and $z_0\in D_1$. Let $g_n(z, z_0)$ be the Green's function of D_n and*

$$d\mu^n_{z_0}(a)=\frac{1}{2\pi}\frac{\partial g_n(a, z_0)}{\partial\nu}\,ds \qquad (a\in\Gamma_n)\,,$$

where ν is the inner normal and ds the arc element of Γ_n at a. Then

$$g_n(z, z_0)\to g(z, z_0)\,, \qquad \mu^n_{z_0}\to\mu_{z_0}\,(n\to\infty)\,.$$

Since $d\mu^n_{z_0}(a)$ is a bounded harmonic function of z_0, $d\mu_{z_0}(a)$ is a bounded harmonic function of z_0.

Hence if $f(a)$ is a bounded B-measurable function on Γ, then

$$u(z)=\int_\Gamma f(a)\,d\mu_z(a)$$

11) Frostman, l. c. 1).

is a bounded harmonic function in D.

PROOF. Since $g_n(z, z_0) \leqq g_{n+1}(z, z_0)$, let

$$\lim_{n\to\infty} g_n(z, z_0)=G(z, z_0)\,. \tag{1}$$

We shall prove $G(z, z_0)=g(z, z_0)$.

Since $g(z, z_0)>0$ on Γ_n, $g(z, z_0)>g_n(z, z_0)$ in D_n, so that

$$g(z, z_0)\geqq G(z, z_0) \quad \text{in } D\,. \tag{2}$$

Let e be the F_σ-set of capacity zero on Γ, where $g(z, z_0)>0$ and $v(z)$ be the Evans's function with respect to e, then since $\underline{\lim}\, G(z, z_0)\geqq 0$ on Γ, we have for any $\varepsilon>0$,

$$G(z, z_0)>g(z, z_0)-\varepsilon-\varepsilon v(z) \text{ in } D\,,$$

so that for $\varepsilon\to 0$,

$$G(z, z_0)\geqq g(z, z_0) \text{ in } D\,. \tag{3}$$

Hence from (2), (3), we have

$$G(z, z_0)=g(z, z_0) \text{ in } D\,. \tag{4}$$

Next we shall prove the second part of the theorem. If D is a finite domain, then

$$g_n(z, z_0)=g_n(z_0, z)=\log\frac{1}{|z-z_0|}-\frac{1}{2\pi}\int_{\Gamma_n}\log\frac{1}{|z-a|}\frac{\partial g_n(a, z_0)}{\partial\nu}ds$$

$$=\log\frac{1}{|z-z_0|}-\int_{\Gamma_n}\log\frac{1}{|z-a|}d\mu^n_{z_0}(a)\,.$$

Since $\int_{\Gamma_n} d\mu^n_{z_0}(a)=1$, we select a partial sequence, such that $\mu^{n_\nu}_{z_0}\to\nu_{z_0}$, so that by (4),

$$g(z, z_0)=\log\frac{1}{|z-z_0|}-\int_{\Gamma}\log\frac{1}{|z-a|}d\nu_{z_0}(a)\,.$$

Hence by the uniqueness of μ_{z_0}, we have $\nu_{z_0}=\mu_{z_0}$. Since μ_{z_0} is independent of the choice of n_ν, $\lim_n \mu^n_{z_0}$ exists and $=\mu_{z_0}$. If D is an infinite domain, then let $\gamma_0:|\zeta-z_0|=\rho$, $\gamma:|\zeta-z|=\rho$, $C:|\zeta|=R$ and Δ_n be the domain bounded by Γ_n, γ_0, γ, C. If we apply the Green's formula $\int\left(u\frac{\partial v}{\partial\nu}-v\frac{\partial u}{\partial\nu}\right)ds=0$ for $u=g_n(\zeta, z_0)$, $v=\log\dfrac{1}{|\zeta-z|}$ in Δ_n

and make $\rho \to 0$, $R \to \infty$, we have

$$g_n(z, z_0) = \log \frac{1}{|z-z_0|} + g_n(\infty, z_0) - \frac{1}{2\pi}\int_{\Gamma_n} \log \frac{1}{|z-a|} \frac{\partial g_n(a, z_0)}{\partial \nu} ds$$

$$= \log \frac{1}{|z-z_0|} + g_n(\infty, z_0) - \int_{\Gamma_n} \log \frac{1}{|z-a|} d\mu_{z_0}^n(a).$$

Hence

$$g(z, z_0) = \log \frac{1}{|z-z_0|} + g(\infty, z_0) - \int_{\Gamma} \log \frac{1}{|z-a|} d\nu_{z_0}(a),$$

so that $\nu_{z_0} = \mu_{z_0}$ and $\lim_n \mu_{z_0}^n$ exists and $= \mu_{z_0}$.

4. Dirichlet problem.

1. Let D be a finite or an infinite domain. We assume that its boundary Γ is a bounded closed set of positive capacity. Let $f(a)$ be a given continuous function on Γ. We extend it to a continuous function $F(z)$ in the whole z-plane, such that $f = F$ on Γ. We approximate D by a sequence of domains $D_1 \subset D_2 \subset \cdots \subset D_n \to D$, where the boundary Γ_n of D_n consists of a finite number of analytic Jordan curves. Let $u_n(z)$ be the solution of the Dirichlet problem for D_n with F as its boundary value. Then Wiener[12] proved that $\lim_{n\to\infty} u_n(z) = u(z)$ exists, where $u(z)$ is independent of the choice of D_n and F. By means of Theorem 14, we can prove

THEOREM 15. $\lim_{n\to\infty} u_n(z) = u(z) = \int_{\Gamma} f(a) d\mu_z(a)$,
where $d\mu_z(a)$ is the mass of balayage.

PROOF. Let $g_n(\zeta, z)$ be the Green's function of D_n, then

$$u_n(z) = \frac{1}{2\pi}\int_{\Gamma_n} F(a) \frac{\partial g_n(a, z)}{\partial \nu} ds.$$

Hence by Theorem 14,

$$u_n(z) \to \int_{\Gamma} f(a)\, d\mu_z(a), \qquad \text{q. e. d.}$$

2. Let a_0 be a point of Γ. If $\lim_{z\to a_0} u(z) = f(a_0)$ for any f, a_0 is called

12) N. Wiener: Certain notions in potential theory, Journ. Math. Massachusetts Inst. Technology, 1924.

a regular point and, otherwise, a_0 is called an irregular point. By Theorem 15, we have

THEOREM 16.[13] *$a_0 \in \Gamma$ is a regular point, when and only when $d\mu_z(a) \to 1$ as $z \to a_0$.*

$d\mu_z(a) \to 1$ $(z \to a_0)$ means that for any $\delta > 0$ and for any neighbourhood $U(a_0)$ of a_0, if $|z-a_0| < \eta = \eta(\delta)$, then the mass $d\mu_z(a)$ contained in $U(a_0)$ is $> 1-\delta$.

THEOREM 17.[14] *$a_0 \in \Gamma$ is a regular point, when and only when $\lim_{z \to a_0} g(z, z_0) = 0$ $(z \in D)$.*

PROOF. If D is a finite domain, then

$$g(z, z_0) = \log \frac{1}{|z-z_0|} - \int_\Gamma \log \frac{1}{|z_0-a|} d\mu_z(a). \tag{1}$$

Hence if $d\mu_z(a) \to 1$ $(z \to a_0)$, then $g(z, z_0) \to 0$.

Next suppose that $g(z, z_0) \to 0$ $(z \to a_0)$, then we choose a sequence $z_k \to a_0$, such that $\mu_{z_k} \to \nu$, so that putting $z = z_k$ in (1), we have

$$\log \frac{1}{|a_0-z_0|} = \int_\Gamma \log \frac{1}{|z_0-a|} d\nu(a). \tag{2}$$

Since if $g(z, z_0) \to 0$, then $g(z, z_1) \to 0$ for any $z_1 \in D$, (2) holds for any $z_0 \in D$. Since $\log \frac{1}{|a_0-z_0|}$ is a harmonic function of z_0 $(\neq a_0)$, we have by Theorem 3, $d\nu(a)=0$, if $a \neq a_0$ and $d\nu(a)=1$, if $a=a_0$. Hence $d\mu_z(a) \to 1$ as $z \to a_0$.

If D is an infinite domain and $d\mu_z(a) \to 1$ $(z \to a_0)$, then

$$w(z) = \int_\Gamma |a-a_0| \, d\mu_z(a) \to 0 \text{ as } z \to a_0.$$

We enclose Γ in a Jordan curve C, such that z_0 lies outside of C. Let D_0 be the domain bounded by Γ and C. We take $K > 0$, so large that

$$Kw(z) > g(z, z_0) \text{ on } C,$$

then for any $\varepsilon > 0$,

$$Kw(z) > g(z, z_0) - \varepsilon - \varepsilon\, v(z) \text{ in } D_0,$$

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13) de la Vallée-Poussin: Les nouvelles méthodes de la théorie du potentiel et le problème généralisé de Dirichlet, Actualités scientifiques et industrielles, 1937.

14) Bouligand: Sur le problème de Dirichlet, Ann. de la Soc. Polonaise de Math. 1925.

where $v(z)$ is the Evans's function with respect to the set of points of Γ such that $g(z, z_0)>0$. Hence for $\varepsilon \to 0$,

$$Kw(z) \geqq g(z, z_0) \text{ in } D_0.$$

Since $w(z) \to 0$, we have $g(z, z_0) \to 0$ as $z \to a_0$.

Next suppose that $g(z, z_0) \to 0$, $(z \to a_0)$. Since

$$g(z, z_0)=\log \frac{1}{|z-z_0|} + g(z, \infty) - \int_\Gamma \log \frac{1}{|z_0-a|} d\mu_z(a),$$

and $g(z, \infty) \to 0$ with $g(z, z_0) \to 0$ as $z \to a_0$, we have

$$\log \frac{1}{|z-z_0|} - \int_\Gamma \log \frac{1}{|z_0-a|} d\mu_z(a) \to 0 \qquad (z \to a_0),$$

so that $d\mu_z(a) \to 1$ as $z \to a_0$, q. e. d.

From Theorem 10 and 17, we have

THEOREM 18.[15] *The set of irregular points is an F_σ-set of capacity zero.*

From the expression of $g(z, \infty)$ in (12) of § 3. 2 and Theorem 10 and 17, we have

THEOREM 19. *Let $u(z)$ be the conductor potential of a bounded closed set F and Γ be its outer boundary and a_0 be a point of Γ. Then $u(a_0)=V$, when and only when a_0 is a regular point of D_∞.*

3. By Theorem 16, if a_0 is a regular point, then $d\mu_z(a) \to 1$ as $z \to a_0$. If a_0 is an irregular point, then $d\mu_z(a)$ is dispersed on Γ in such a way as the following theorem.

THEOREM 20. *Let D be a finite domain and a_0 be an irregular point on its boundary Γ, so that $\lim_k g(z_k, z_0)>0$ for some $z_k \to a_0$. We select a partial sequence, which we denote again k, such that $\mu_{z_k} \to \nu$. Let Γ^* be the kernel of ν, then $\Gamma-\Gamma^*$ is of capacity zero.*

PROOF. From

$$g(z_k, z_0)=\log \frac{1}{|z_k-z_0|} - \int_\Gamma \log \frac{1}{|z_0-a|} d\mu_{z_k}(a) \geqq \eta > 0 \quad (k=1, 2, \cdots),$$

we have

15) O. D. Kellogg: Unicité des fonctions harmoniques, C. R. 187 (1928). G. C. Evans: Application of Poincaré's sweeping-out process, Proc. Nat. Acad. Sci. U. S. A. (1933).

$$\log \frac{1}{|a_0-z_0|} - \int_{\Gamma^*} \log \frac{1}{|z_0-a|}\, d\nu(a) \geqq \eta > 0. \tag{1}$$

Since $g(z_k, z) = \log \frac{1}{|z_k-z|} - \int_\Gamma \log \frac{1}{|z-a|}\, d\mu_{z_k}(a) > 0$ for any $z \in D$,

$$u(z) = \log \frac{1}{|a_0-z|} - \int_{\Gamma^*} \log \frac{1}{|z-a|}\, d\nu(a) \geqq 0 \text{ in } D, \tag{2}$$

so that $\lim u(z) \geqq 0$ on Γ^*, hence by the upper semi-continuity of $u(z)$, $u(z) \geqq 0$ on Γ^*. Since $u(z)$ is harmonic at $z=\infty$, by the maximum principle, $u(z) \geqq 0$ in the complement of Γ^*. Since by (1), $u(z_0) > 0$, we have

$$u(z) > 0 \text{ in the complement of } \Gamma^*. \tag{3}$$

From

$$\log \frac{1}{|z_k-z|} - \int_\Gamma \log \frac{1}{|z-a|}\, d\mu_{z_k}(a)$$

$$= \log \frac{1}{|z_k-z|} - \int_\Gamma \log \frac{1}{|z_k-a|}\, d\mu_z(a),$$

we have

$$\int_\Gamma \log \frac{1}{|z-a|}\, d\mu_{z_k}(a) = \int_\Gamma \log \frac{1}{|z_k-a|}\, d\mu_z(a),$$

so that by Fatou's theorem,

$$\int_{\Gamma^*} \log \frac{1}{|z-a|}\, d\nu(a) \geqq \int_\Gamma \log \frac{1}{|a_0-a|}\, d\mu_z(a).$$

Hence by (2),

$$0 < u(z) \leqq \log \frac{1}{|a_0-z|} - \int_\Gamma \log \frac{1}{|a_0-a|}\, d\mu_z(a) \text{ in } D. \tag{4}$$

We put $E = \Gamma - \Gamma^*$ and suppose that $C(E) > 0$. Then E contains a regular point α and if z tends to α, $d\mu_z(a) \to 1$, so that the right hand side of (4) tends to zero, hence $u(z) \to 0$ as $z \to \alpha$. But since α lies outside Γ^*, $u(z)$ is harmonic at α and $u(\alpha) > 0$, which is absurd. Hence $C(E) = 0$.

4. Let D be a domain and Γ be its boundary and $a_0 \in \Gamma$ and D_ρ be the part of D, which is contained in $|z-a_0| < \rho$. If $w(z)$ satisfies the following condition, then $w(z)$ is called a barrier at a_0.

(i) $w(z) > 0$ and is continuous and superharmonic in D_{ρ_0},

(ii) $\lim_{z \to a_0} w(z) = 0 \quad (z \in D)$,

(iii) $w(z) \geqq d(\rho) > 0$ on $|z - a_0| = \rho \quad (0 < \rho \leqq \rho_0)$.

THEOREM 21.[16] *$a_0 \in \Gamma$ is a regular point, when and only when a barrier exists at a_0.*

Hence the regularity and irregularity is a local property.

PROOF. (i) Suppose that a barrier $w(z)$ exists at a_0. We take ρ so small than $\rho < |z_0 - a_0|$. Then if we take $K > 0$ sufficiently large, we have for any $\varepsilon > 0$,

$$Kw(z) > g(z, z_0) - \varepsilon - \varepsilon\, v(z) \text{ in } D_\rho,$$

where $v(z)$ is the Evans's function with respect to the set of irregular points. Hence for $\varepsilon \to 0$,

$$Kw(z) \geqq g(z, z_0) \text{ in } D_\rho,$$

so that $g(z, z_0) \to 0$. Hence a_0 is a regular point.

(ii) Next suppose that a_0 is a regular point and put

$$w(z) = \int_\Gamma |a - a_0|\, d\mu_z(a),$$

then by Theorem 16, $w(z) \to 0$ as $z \to a_0$ and the boundary value of $w(z)$ coincides with that of $|z - a_0|$ "almost everywhere" on Γ. If D is a finite domain, then $|z - a_0|$ is subharmonic in D, so that

$$w(z) \geqq |z - a_0| \text{ in } D,$$

hence $w(z)$ is a barrier at a_0. If D is an infinite domain, we enclose Γ in a Jordan curve C and D_0 be the domain bounded by Γ and C. We take $K \geqq 1$, so large that

$$Kw(z) \geqq |z - a_0| \text{ on } C,$$

then $Kw(z) \geqq |z - a_0|$ in D_0, so that $w(z)$ is a barrier at a_0.

5. Let D be a domain and Γ be its boundary, which is a bounded closed set of positive capacity. Let $f(z)$ be a bounded B-measurable function on Γ. We define with Brelot an upper function $\psi(z)$ as follows:

(i) $\psi(z)$ is continuous and superharmonic in D,

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16) O. D. Kellogg: Foundations of potential theory, Berlin (1929) p. 326.

(ii) $\underline{\lim}_{z\to a} \psi(z) \geqq f(a)$ $(z \in D)$ on Γ.

We put

$$\overline{H}_f(z)=\inf_{\psi} \psi(z)\,. \tag{1}$$

A lower function $\varphi(z)$ is defined as follows :

(i) $\varphi(z)$ is continuous and subharmonic in D,

(ii) $\overline{\lim}_{z\to a} \varphi(z) \leqq f(a)$ $(z \in D)$ on Γ.

We put

$$\overline{H}_f(z)=\sup_{\varphi} \varphi(z)\,. \tag{2}$$

Brelot called $\overline{H}_f(z)$ the hyperfonction, $\underline{H}_f(z)$ the hypofonction.

THEOREM 22.[17] $\underline{H}_f(z)=\overline{H}_f(z)=H_f(z)=\int_\Gamma f(a)d\mu_z(a)$, *where $d\mu_z(a)$ is the mass of balayage. Hence*

$$\varphi(z) \leqq \int_\Gamma f(a)\, d\mu_z(a) \leqq \psi(z) \quad \textit{in } D\,.$$

PROOF. Let

$$u(z)=\int_\Gamma f(a)\, d\mu_z(a)\,, \tag{1}$$

then $u(z)$ is a bounded harmonic function in D (§ 3). If $f(a)$ is lower semi-continuous on Γ, then at a regular point a,

$$\underline{\lim}_{z\to a} u(z) \geqq f(a)\,.$$

Let $v(z)$ be the Evans's function with respect to the set of irregular points, then for any $\varepsilon > 0$, $u(z)+\varepsilon\, v(z)$ is an upper function, so that $\overline{H}_f(z) \leqq u(z)+\varepsilon v(z)$, hence for $\varepsilon \to 0$,

$$\overline{H}_f(z) \leqq u(z)\,. \tag{2}$$

If $f(a)$ is upper semi-continuous on Γ, then similarly

$$\underline{H}_f(z) \geqq u(z)\,. \tag{3}$$

Let $f(a)$ be a bounded B-measurable function on Γ, then by Vitali-Carathéodory's theorem, there exist upper semi-continuous functions $U_n(a)$ and lower semi-continuous functions $L_n(a)$, such that

17) M. Brelot: Familles de Perron et problême de Dirichlet, Acta de Szeged **19** (1938).

$$U_1(a) \leqq U_2(a) \leqq \cdots \leqq U_n(a) \leqq f(a) \leqq L_n(a) \leqq \cdots \leqq L_2(a) \leqq L_1(a)\,,$$

$$\psi_n(z) = \int_\Gamma L_n(a) d\mu_z(a) \to \int_\Gamma f(a) d\mu_z(a) = u(z)\,,$$

$$\varphi_n(z) = \int_\Gamma U_n(a) d\mu_z(a) \to \int_\Gamma f(a) d\mu_z(a) = u(z)\,.$$

By (2), (3),

$$\overline{H}_{L_n}(z) \leqq \psi_n(z)\,, \qquad \underline{H}_{U_n}(z) \geqq \varphi_n(z)\,.$$

Since $f \leqq L_n$ on Γ, $\overline{H}_f(z) \leqq \overline{H}_{L_n}(z)$, so that $\overline{H}_f(z) \leqq \psi_n(z) \to u(z)$. Hence $\overline{H}_f(z) \leqq u(z)$. Similarly $\underline{H}_f(z) \geqq u(z)$, so that $\underline{H}_f(z) = \overline{H}_f(z) = H_f(z) = u(z)$.

6. By means of Theorem 20 and 22, we can prove easily the following theorem.

THEOREM 23.[18] *Let D be a domain and a_0 be a point on its boundary Γ. If there exists a continuous superharmonic function $w(z) > 0$ in D, such that $\lim_{z \to a_0} w(z) = 0$ $(z \in D)$, then a_0 is a regular point.*

PROOF. We may assume that $w(z)$ is bounded, since otherwise we consider $\mathrm{Min}(w(z), 1)$ instead of $w(z)$. Let D_ρ be the part of D, which is contained in $|z - a_0| < \rho$ and Λ_ρ be its boundary. We take ρ so small that $C: |z - a_0| = \rho$ contains an inner point of D. Let

$$0 < m \leqq w(z) \leqq M \quad \text{on an arc } \widehat{\alpha\beta} \text{ of } C\,. \tag{1}$$

For any $a \in \Lambda_\rho$, we put

$$\varliminf_{z \to a} w(z) = \underline{w}(a) \quad (z \in D_\rho)\,,$$

then $\underline{w}(a)$ is a bounded lower semi-continuous function on Λ_ρ. Let

$$u(z) = \int_{\Lambda_\rho} \underline{w}(a)\, d\mu_z(a) \quad (z \in D_\rho)\,, \tag{2}$$

where $d\mu_z(a)$ is the mass of balayage with resepct to D_ρ. Since $w(z)$ is an upper function of $\underline{w}(a)$, we have by Theorem 22,

$$0 < u(z) \leqq w(z) \text{ in } D_\rho\,.$$

Hence

$$\lim_{z \to a_0} u(z) = 0\,. \tag{3}$$

18) Brelot, l. c. 17).

Suppose that a_0 is an irregular point of D, then a_0 is an irregular point of D_ρ, so that by Theorem 20,

$$\int_{\widehat{\alpha\beta}} d\mu_{z_k}(a) \geqq \eta > 0 \quad \text{for some } z_k \to a_0 .$$

Since $\underline{w}=w \geqq m > 0$ on $\widehat{\alpha\beta}$,

$$u(z_k) \geqq \int_{\widehat{\alpha\beta}} m \, d\mu_{z_k}(a) \geqq m\eta > 0 \ (z_k \to a_0),$$

which contradicts (3). Hence a_0 is a regular point.

5. Elliptic capacity and elliptic conductor potential.

Let K be the Riemann sphere of diameter 1, which touches the z-plane at $z=0$ and $[a, b]=\dfrac{|a-b|}{\sqrt{(1+|a|^2)(1+|b|^2)}}$ be the spherical distance of a, b.

Let F be a closed set on K and $\mu(e)>0$ be a positive mass distribution on F of total mass 1 and let

$$I(\mu)=\iint_F \log \frac{1}{[a, b]} d\mu(a) d\mu(b), \tag{1}$$

$$V_+ = \inf_\mu I(\mu). \tag{2}$$

We define the elliptic capacity of F by

$$C_+(F)=e^{-V_+} \tag{3}$$

and of any Borel set E by

$$C_+(E)=\sup_{F \subset E} C_+(F), \tag{4}$$

where F are closed sub-sets of E. The capacity defined in §2 may be called the parabolic capacity.[19] We can prove easily that $C_+(E)=0$, when and only when $C(E)=0$.

If $C_+(F)>0$, then there exists μ, such that $I(\mu)=V_+$ and

19) Similarly we can define the hyperbolic capacity and the elliptic and hyperbolic transfinite diameter. I have proved the identity of the elliptic (hyperbolic) capacity with the elliptic (hyperbolic) transfinite diameter in another paper, M. Tsuji: Some metrical theorems on Fuchsian groups, Jap. Journ. Math. 19 (1947).

$$u(z)=\int_F \log \frac{1}{[z,a]}\, d\mu(a)\,, \quad \mu(F)=1 \tag{5}$$

is called the elliptic conductor potential of F. $u(z)$ is lower semi-continuous on K and is subharmonic outside the mass. We can prove similarly as Theorem 6,

THEOREM 24. *$u(z)\leqq V_+$ on the whole sphere K, and $u(z)=V_+$ on F except at an F_σ-set of capacity zero.*

We shall prove

THEOREM 25. *$u(z)=V_+$ at inner points of F and the density of μ at inner points is $1/\pi$.*

PROOF. We consider at $z=0$ and put

$$\Omega(r)=\int_{|a|<r} d\mu(a)\,.$$

Since

$$u(re^{i\theta})=\int_F \log \frac{\sqrt{(1+r^2)(1+|a|^2)}}{|re^{i\theta}-a|}\, d\mu(a)$$

$$=\log\sqrt{1+r^2}+\int_F \log \frac{\sqrt{1+|a|^2}}{|re^{i\theta}-a|}\, d\mu(a)\,,$$

we have

$$\frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})d\theta=\log\sqrt{1+r^2}+\int_0^\infty \left(\log\sqrt{1+t^2}-\mathrm{Max}(\log r, \log t)\right) d\Omega(t)$$

$$=\log\sqrt{1+r^2}+\int_0^\infty \frac{\Omega(t)}{t(1+t^2)}\,dt-\int_0^r \frac{\Omega(t)}{t}\,dt\,.$$

Since

$$u(0)=\int_F \log \frac{\sqrt{1+|a|^2}}{|a|}\, d\mu(a)=\int_0^\infty \frac{\Omega(t)}{t(1+t^2)}\,dt\,,$$

we have

$$\frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\,d\theta=\log\sqrt{1+r^2}+u(0)-\int_0^r \frac{\Omega(t)}{t}\,dt\,. \tag{1}$$

Hence

$$\lim_{r\to 0}\frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\,d\theta=u(0)\,. \tag{2}$$

If $z=0$ is an inner point of F, then $u(z)=V_+$ almost everywhere in a neighbourhood of $z=0$, so that from (2), we have $u(0)=V_+$. If $|z|<\rho$ belongs to F, then $u(re^{i\theta})=V_+$ $(0<r<\rho)$, $u(0)=V_+$, so that from (1),

$$\int_0^r \frac{\Omega(t)}{t}\,dt=\log\sqrt{1+r^2}\,.$$

Hence $\Omega(r)=r^2/1+r^2$, so that $\lim_{r\to 0}\Omega(r)/\pi r^2=1/\pi$. Hence the density of μ at $z=0$ is $1/\pi$.

REMARK. If $|z|<\rho$ lies outside F, then $\Omega(r)=0$ $(0<r<\rho)$, so that from (1),

$$\frac{1}{2\pi}\int_0^{2\pi} u(re^{i\theta})\,d\theta=\log\sqrt{1+r^2}+u(0)>u(0)\,, \tag{3}$$

which expresses that $u(z)$ is subharmonic outside the mass. If F consists of a finite number of Jordan domains and A be the area of F, then the mass contained inside of F is A/π, so that the boundary of F contains a mass $(\pi-A)/\pi$, where $\pi-A$ is the area of the complement of F.

THEOREM 26. *μ is unique.*

PROOF. Let

$$u_1(z)=\int_F \log\frac{1}{[z,a]}\,d\mu_1(a)\,,\qquad u_2(z)=\int_F \log\frac{1}{[z,a]}\,d\mu_2(a)\,,$$

$$\mu_1(F)=\mu_2(F)=1\,,$$

such that $u_1(z)=u_2(z)=V_+$ "almost everywhere" on F. We may assume, by a suitable rotation of K, that F is projected on a finite distance on the z-plane. Then

$$u(z)=u_1(z)-u_2(z)=\int_F \log\frac{1}{|z-a|}\,d\mu_1(a)-\int_F \log\frac{1}{|z-a|}\,d\mu_2(a)+$$

$$+\text{const.}=\int_F \log\frac{1}{|z-a|}\,d\mu(a)+\text{const.}\quad(\mu=\mu_1-\mu_1)$$

is harmonic outside F. Since $u(z)=0$ "almost everywhere" on the boundary of the complement of F and $u=0$ at inner points of F, $u(z)=0$ almost everywhere in the whole z-plane, so that by Theorem 3, $\mu\equiv 0$, or $\mu_1=\mu_2$.

THEOREM 27. *Let F be a closed set on the Riemann sphere K and the complement of F consists of only one domain D. Let a_0 be a boundary point of D, then $u(a_0)=V_+$, when and only when a_0 is a regular point of D.*

PROOF. (i) Suppose that $u(a_0)=V_+$. Then by the lower semi-continuity of $u(z)$ and $u(z)\leqq V_+$, we have $\lim_{z\to a_0} u(z)=V_+$ $(z\in D)$. Since $w(z)=V_+-u(z)>0$ is superharmonic in D and tends to zero, by Theorem 23, a_0 is a regular point of D.

(ii) Next suppose that $a_0=0$ is a regular point of D. Let D_ρ be the part of D contained in $|z|<\rho$ and F_ρ be that of F contained in $|z|\leqq\rho$. Then if we put

$$\begin{aligned}u(z)&=\int_{F_\rho}\log\frac{\sqrt{(1+|z|^2)(1+|a|^2)}}{|z-a|}\,d\mu(a)\\&\quad+\int_{F-F_\rho}\log\frac{\sqrt{(1+|z|^2)(1+|a|^2)}}{|z-a|}\,d\mu(a)\\&=\int_{F_\rho}\log\frac{\sqrt{1+|a|^2}}{|z-a|}\,d\mu(a)\\&\quad+\int_{F-F_\rho}\log\frac{\sqrt{1+|a|^2}}{|a|}\,d\mu(a)+\varphi(z),\end{aligned}$$

then $\lim_{z\to 0}\varphi(z)=0$, so that if $|z|<\rho_1<\rho$ then $|\varphi(z)|<\delta$. We put

$$u_1(z)=\int_{F_\rho}\log\frac{\sqrt{1+|a|^2}}{|z-a|}\,d\mu(a)+\int_{F-F_\rho}\log\frac{\sqrt{1+|a|^2}}{|a|}\,d\mu(a),\tag{1}$$

then

$$u(z)=u_1(z)+\varphi(z).\tag{2}$$

Since $u(z)=V_+$ on F, except at an F_σ-set E of capacity zero,

$$V_+-u_1(z)=\varphi(z)<\delta \text{ on } F_{\rho_1}, \text{ except at } E.\tag{3}$$

Let $w(z)$ be a barrier at $z=0$, then for a large $K\geqq 1$,

$$Kw(z)>V_+-u_1(z)-\delta-\varepsilon v(z) \text{ in } D_{\rho_1}$$

for any $\varepsilon>0$, where $v(z)$ is the Evans's function with respect to E, so that making $\varepsilon\to 0$,

$$Kw(z)\geqq V_+-u_1(z)-\delta \text{ in } D_{\rho_1},$$

or

$$u(z)-\varphi(z)+\delta+Kw(z)\geqq V_+ \text{ in } D_{\rho_1}.$$

Since $w(z)\to 0$, $\varphi(z)\to 0$ as $z\to 0$ and δ is arbitrary, $\lim\limits_{z\to 0} u(z)\geqq V_+$.
Since $u(z)\leqq V_+$, we have $\lim\limits_{z\to 0} u(z)=V_+$ $(z\in D)$.
Since $u(z)=V_+$ "almost everywhere" on F and $u(z)\to V_+$, $(z\to 0)$, we have

$$u(z)>V_+-\delta$$

almost everywhere in a full neighbourhood $|z|<\rho$ of $z=0$, where $\delta\to 0$ with $\rho\to 0$. Hence by (2) of the proof of Theorem 25, we have $u(0)=V_+$.

6. Functions of U^*-class.

Let $w(z)$ be regular and $|w(z)|<1$ in $|z|<1$ and $\lim\limits_{r\to 1}|w(re^{i\theta})|=1$ almost everywhere on $|z|=1$, then $w(z)$ is called a function of U-class. Frostman[20] proved that a function of U-class takes any value a $(|a|<1)$, except a set of capacity zero.

We generalize the definition of U-class as follows. Let F be a closet set of positive capacity on the Riemann sphere K and $w(z)$ be meromorphic in $|z|<1$ and does not take values on F and $\lim\limits_{r\to 1} w(re^{i\theta}) =w(e^{i\theta})$ belongs to F almost everywhere on $|z|=1$. We call $w(z)$ a function of U^*-class and F its lacunary set. The complement of F consists of at most a countable number of connected domains $\{D_\nu\}$. Let D be one of D_ν, which contains $w(0)$, then $w(z)$ belongs to D.

Similarly as Frostman, we shall prove

THEOREM 28. *Let $w(z)$ be a function of U^*-class in $|z|<1$. Then $w(z)$ takes any value in D, except a set of capacity zero.*

PROOF. Let $g(w,a)$ $(a\in D)$ be the Green's function of D, then $g(w,a)=0$ on F, except at a set E of capacity zero, where E is independent of a. Let c be a bounded closed set of positive capacity contained in D and μ be the equilibrium distribution of e, then we can prove easily

20) Frostman, l. c. 1).

$$u(w)=\int_e g(w, a)\, d\mu(a)$$

is bounded in D. If $w(z)$ does not take values on e, then

$$\int_0^{2\pi} u\big(w(re^{i\theta})\big)d\theta=\int_e d\mu(a)\int_0^{2\pi} g\big(w(re^{i\theta}), a\big)\, d\theta=2\pi\int_e g\big(w(0), a\big)\, d\mu(a)$$
$$=2\pi\, u\big(w(0)\big)>0 \quad (0<r<1).$$

Since $u(w)$ is bounded, by Lebesgue's theorem, if we make $r\to 1$,

$$2\pi\, u\big(w(0)\big)=\int_0^{2\pi} u\big(w(e^{i\theta})\big)\, d\theta=\int_e d\mu(a)\int_0^{2\pi} g\big(w(e^{i\theta}), a\big)\, d\theta.$$

Since E is of capacity zero, its image on $|z|=1$ is of measure zero, so that $g\big(w(e^{i\theta}), a\big)=0$ almost everywhere on $|z|=1$, hence $u\big(w(0)\big)=0$, which is absurd. Hence $w(z)$ takes any value in D, except a set of capacity zero.

THEOREM 29. *Let D be a domain on the w-plane and its boundary Γ is of positive capacity. We map D on $|z|<1$ conformally. Let E be a closed sub-set of Γ, such that E and $\Gamma-E$ have a positive distance and let e be the image of E on $|z|=1$. Then the measure of e is positive, when and only when $C(E)>0$.*

PROOF. If $me>0$, then by Frostman's theorem,[21] $C(E)>0$. Next suppose that $C(E)>0$ and we shall prove $me>0$. If $C(\Gamma-E)=0$, then $\Gamma-E$ is mapped on a null set on $|z|=1$, so that $me=2\pi>0$. If $C(\Gamma-E)>0$ and $me=0$, then the mapping function $w=f(z)$ belongs to U^*-class, whose lacunary set is $\Gamma-E$. Since $f(z)$ does not take values on E $\big(C(E)>0\big)$, this contradicts Theorem 28. Hence $me>0$.

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21) Frostman, l. c. 1).