Journal of the Mathematical Society of Japan Vol. 4, No. 1, July, 1952.

On maximum modulus of integral functions.

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(Received Dec. 26, 1949)

Let D be a region on the z-plane, which lies in the disc $|z|<R$ $(0<R\leqq+\infty)$, and whose boundary Γ lying in $|z|<R$ consists of a finite or infinite number of analytic curves clustering nowhere in $|z|<R$. For any $0<r<R$, we denote by D_r the part of D lying in $|z|<r$. Let $A_k(r)$ $\big(k=1, \cdots, n(r)\big)$ be the arcs of $|z|=r<R$ contained in D, and $r\cdot\theta_k(r)$ be their lengths.

We define a function $\theta(r)$ in $0<r<R$ as follows: if $|z|=r$ is contained wholly in D, then $\theta(r)=+\infty$, and, otherwise, $\theta(r)=\max\limits_k \theta_k(r)$.

Using Carleman's method[1], we shall first prove

THEOREM 1. *Suppose that $\theta(r)>0$ for $0<r_0<r<R$, and let $u(z)$ be a harmonic function in D, which is >0 in D and $=0$ on Γ. We put*

$$m(r)=\frac{1}{2\pi}\sum_k\int_{A_k(r)}\Big[u(re^{i\varphi})\Big]^2 d\varphi \qquad (0<r<R)$$

and $$D(r)=\iint_{D_r}\left[\left(\frac{\partial u}{\partial \log r}\right)^2+\left(\frac{\partial u}{\partial\varphi}\right)^2\right]d\log r\, d\varphi\,.$$

Then, for any $0<r_0<r<R$,

$$D(r)\geqq D(r_0)\exp.\int_{r_0}^{r}\frac{2\pi}{r\theta(r)}\,dr$$

and $$m(r)-m(r_0)\geqq\frac{1}{\pi}D(r_0)\cdot\int_{r_0}^{r}\frac{dt}{t}\left[\exp.\int_{r_0}^{t}\frac{2\pi}{s\theta(s)}\,ds\right].$$

Let $f(z)$ be a regular analytic function in $|z|<R\leqq+\infty$. While applying Theorem 1 to $u(z)=\overset{+}{\log}|f(z)|$, we shall obtain some theorems on the modulus of $f(z)$.

PROOF OF THEOREM 1. Since $u=0$ on Γ, we have, by application of Green's formula,

$$(1)\qquad \frac{dm(r)}{d\log r}=\frac{1}{\pi}\sum_k\int_{A_k(r)}u\,\frac{\partial u}{\partial\log r}\,d\varphi=\frac{1}{\pi}\cdot D(r)>0\,,$$

$$(2)\qquad \frac{d^2m(r)}{(d\log r)^2}=\frac{1}{\pi}\,\frac{dD(r)}{d\log r}$$

$$=\frac{1}{\pi}\sum_k\int_{A_k(r)}\left[\left(\frac{\partial u}{\partial\log r}\right)^2+\left(\frac{\partial u}{\partial\varphi}\right)^2\right]d\varphi>0\,.$$

By Schwarz' inequality, we have from (1)

$$\left(\frac{dm(r)}{d\log r}\right)^2\leqq 2m(r)\cdot\frac{1}{\pi}\sum_k\int_{A_k(r)}\left(\frac{\partial u}{\partial\log r}\right)^2 d\varphi$$

$$(3)\qquad \text{or}\qquad \frac{1}{\pi}\sum_k\int_{A_k(r)}\left(\frac{\partial u}{\partial\log r}\right)^2 d\varphi\geqq\frac{1}{2m(r)}\left(\frac{dm(r)}{d\log r}\right)^2.$$

On the other hand, if $0<\theta(r)\leqq 2\pi$, we have, by Wirtinger's inequality,

$$\int_{A_k(r)}\left(\frac{\partial u}{\partial\varphi}\right)^2 d\varphi\geqq\frac{\pi^2}{\theta_k(r)^2}\int_{A_k(r)}u^2d\varphi\geqq\frac{\pi^2}{\theta(r)^2}\int_{A_k(r)}u^2d\varphi\,,$$

so that

$$(4)\qquad \frac{1}{\pi}\sum_k\int_{A_k(r)}\left(\frac{\partial u}{\partial\varphi}\right)^2 d\varphi\geqq\frac{2\pi^2}{\theta(r)^2}\,m(r)\,.$$

(4) holds also for the case $\theta(r)=+\infty$.

From (2), (3) and (4), we have

$$\frac{2}{m(r)}\cdot\frac{d^2m(r)}{(d\log r)^2}\geqq\frac{1}{m(r)^2}\left(\frac{dm(r)}{d\log r}\right)^2+\frac{4\pi^2}{\theta(r)^2}\,,$$

so that, putting $\log r=t$ and $\log m(r)=\lambda(t)$,

$$(5)\qquad \left(\frac{d\lambda}{dt}\right)^2+2\,\frac{d^2\lambda}{dt^2}\geqq\left(\frac{2\pi}{\theta(r)}\right)^2.$$

Since

$$\left[\frac{d}{dt}\left(\log\frac{d}{dt}\,e^\lambda\right)\right]^2=\left(\frac{d\lambda}{dt}+\frac{\dfrac{d^2\lambda}{dt^2}}{\dfrac{d\lambda}{dt}}\right)^2\geqq\left(\frac{d\lambda}{dt}\right)^2+2\,\frac{d^2\lambda}{dt^2}\,,$$

and since, by (1) and (2),

$$\frac{d}{dt}\left(\log \frac{d}{dt} e^{\lambda}\right) = \frac{d^2 m(r)}{(d \log r)^2} \Big/ \frac{dm(r)}{d \log r} > 0\,,$$

we have, from (5),

$$\frac{d}{dt}\left(\log \frac{d}{dt} e^{\lambda}\right) \geqq \frac{2\pi}{\theta(r)}\,.$$

Hence, by integration, we obtain the mentioned relations.

THEOREM 2. *Let $f(z)$ be an integral function, and D be the domain, where $|f(z)|>1$. Let $\theta(r)$ be defined as before for the domain D, and put $M(r)=\max_{|z|=r} |f(z)|$. Then, for any $0<\alpha<1$, we have*

$$\log_2 M(r) > \pi \int_{r_0}^{\alpha r} \frac{dr}{r\theta(r)} - c(\alpha, r_0)\,,$$

where $0<r_0<\alpha r$ and $c(\alpha, r_0)$ is independent of r.

PROOF. We apply Theorem 1 to $u(z)=\overset{+}{\log}|f(z)|$. Since $P(t) = \exp.\int_{r_0}^{t} \frac{2\pi}{s\theta(s)} ds$ is an increasing function of t, we have, for any $0<\alpha<1$,

$$m(r)-m(r_0) \geqq \frac{1}{\pi} D(r_0)\cdot \int_{r_0}^{r} \frac{P(t)}{t} dt \geqq \frac{1}{\pi} D(r_0)\cdot \int_{\alpha r}^{r} \frac{P(t)}{t} dt$$

$$\geqq \frac{1}{\pi} D(r_0)\cdot P(\alpha r) \int_{\alpha r}^{r} \frac{dt}{t} = \frac{1}{\pi} D(r_0)\cdot \log \frac{1}{\alpha} \cdot P(\alpha r)\,,$$

so that

$$\log m(r) \geqq \log P(\alpha r) - \text{const.} = 2\pi \int_{r_0}^{\alpha r} \frac{ds}{s\theta(s)} - \text{const.}$$

Hence and since

$$\log m(r) = \log\left[\frac{1}{2\pi}\int_0^{2\pi} \left(\overset{+}{\log}|f(re^{i\varphi})|\right)^2 d\varphi\right] \leqq 2\cdot \log_2 M(r)\,,$$

we have the mentioned result.

THEOREM 3. *Let $f(z)$ be an integral function of order ρ, then*

$$\rho \geqq \lim_{r\to\infty} \frac{1}{\log r} \int_{r_0}^{r} \frac{\pi}{r\theta(r)} dr\,.$$

PROOF. By Theorem 2, we have

$$\frac{\log_2 M(r)}{\log r} > \frac{\log \alpha r}{\log r} \cdot \frac{1}{\log \alpha r} \cdot \int_{r_0}^{\alpha r} \frac{\pi}{r\theta(r)} dr - O\left(\frac{1}{\log r}\right),$$

so that

$$\rho = \overline{\lim_{r\to\infty}} \frac{\log_2 M(r)}{\log r} \geqq \overline{\lim_{r\to\infty}} \frac{1}{\log \alpha r} \int_{r_0}^{\alpha r} \frac{\pi}{r\theta(r)} dr, \qquad \text{q. e. d.}$$

From Theorem 3, we obtain the following

THEOREM 4. *Let $f(z)$ be an integral function of finite order ρ, and, for any $K > 0$, let $\theta(r) = \theta(r, K)$ be defined as before for the domain where $|f(z)| > K$. Then,*

$$\overline{\lim_{r\to\infty}} \theta(r, K) \geqq \frac{\pi}{\rho}.$$

If $\rho < 1/2$, the above inequality means that there exists a sequence of circumferences $|z| = r_n$, on each of which $|f(z)| > K$.

PROOF. Suppose that, for a $0 < K < +\infty$ and for a $k\,(> \rho)$, $\theta(r, K) \leqq \pi/k$ would hold for any $r_0 < r < +\infty$. Then, by Theorem 3 applied to $f(z)/K$, we should have

$$\rho \geqq \overline{\lim_{r\to\infty}} \frac{1}{\log r} \int_{r_0}^{r} \frac{\pi}{r \cdot \frac{\pi}{k}} dr = k > \rho,$$

which is a contradiction.

By Theorem 4, we can state

THEOREM 5. *Let $f(z)$ be an integral function of finite order $\rho < k$, and $K_n \to \infty$ be a sequence of positive numbers. Then, there exists a sequence of circles C_n: $|z| = r_n \to \infty$, such that each C_n has an arc of length $> \frac{\pi}{k} \cdot r_n$, on which $|f(z)| > K_n$.*

Next, let $f(z)$ be an integral function of order $\rho < 1/2$. Then, the set of points z, where $|f(z)| < 1$, consists of an infinite number of bounded closed domains (islands) D_n $(n = 1, \cdots)$. Let λ_n, ρ_n be respectively the greatest and the least distance between D_n and the origin $z = 0$. Then,

THEOREM 6. (H. Milloux[2]).

$$\underline{\lim_{n\to\infty}} \frac{\log \lambda_n}{\log \rho_n} \leqq \frac{1}{1 - 2\rho}.$$

PROOF. By Theorem 3, we have

$$\rho \geqq \overline{\lim_{n\to\infty}} \frac{1}{\log r}\int_{r_0}^{r}\frac{\pi}{r\theta(r)}\,dr \geqq \overline{\lim_{n\to\infty}} \frac{1}{\log \lambda_n}\int_{\rho_n}^{\lambda_n}\frac{\pi}{r\theta(r)}\,dr$$

$$\geqq \lim_{n\to\infty} \frac{1}{\log \lambda_n}\,\frac{1}{2}\int_{\rho_n}^{\lambda_n}\frac{dr}{r} = \frac{1}{2} - \frac{1}{2}\lim_{n\to\infty}\frac{\log \rho_n}{\log \lambda_n}\,.$$

Hence the result.

Finally we shall prove

THEOREM 7. *Let $f(z)$ be regular in $|z|<1$, and let $\theta(r)$ be defined as before for the domain D, where $|f(z)|>1$. If $\lim_{r\to\infty}\frac{\theta(r)}{1-r}<2\pi$, then, either $|f(z)|<1$ in $|z|<1$ or*

$$\lim_{r\to\infty} \log_2 M(r)\Big/\log\frac{1}{1-r}>0\,.$$

PROOF. If $\theta(r)\equiv 0$, we have $|f(z)|<1$ in $|z|<1$. Otherwise, we have $\theta(r)>0$ for $r_0<r<1$. Then, by the assumption, there exists a positive number δ, such that

$$0<\theta(r)\leqq\frac{2\pi}{1+\delta}(1-r) \qquad \text{for } r_0<r_1<r<1\,.$$

Then, by a simple calculation, we have

$$\log\left[\int_{r_1}^{r}\frac{dt}{t}\exp.\int_{r_1}^{t}\frac{ds}{s\theta(s)}\right]\geqq\delta\cdot\log\frac{1}{1-r}-O(1)\,.$$

Hence, by Theorem 1 applied to $\overset{+}{\log}|f(z)|$, we obtain

$$2\cdot\log_2 M(r)\geqq\log m(r)\geqq\delta\cdot\log\frac{1}{1-r}-O(1)\,,$$

so that

$$\underline{\lim_{r\to\infty}}\log_2 M(r)\Big/\log\frac{1}{1-r}\geqq\frac{\delta}{2}>0\,, \qquad \text{q. e. d.}$$

References.

1) T. Carleman: Sur une inégalité différentielle dans la théorie des fonctions analytiques, C. r. Acad. Sci. Paris, 196 (1933).

2) H. Milloux: Sur les domaines de déterminations infinies des fonctions entières, Acta Math. 61 (1933).