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The class number of embedding of the space with projective connection.

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In this paper we shall concern with the embedding of projectively connected spaces. This is a generalization of the embedding problem of Riemann spaces.

The first five sections are of an introductory character, that is, we get the Gauss, Codazzi and Ricci equations of n-dimensional variety in the space with projective connection of m-dimensions and define the class number of the projectively connected space, and then deduce the fundamental theorem of embedding. We then give a necessary and sufficient condition for a space with projective connection to be of class one.

1. Frame on a sub-variety.

Let P_m be an m-dimensional space with projective connection and (A_0, A_α) $(\alpha=1, \cdots, m)$ be a frame attaching to a current point A_0; where A_0 is determined by a system of coordinates $(y^1, \cdots, y^m)$. Then the connection of P_m is given by the following equations:

$$(1.1) \qquad \begin{cases} dA_0 = A_\alpha dy^\alpha , \\ dA_\lambda = (\Pi^0_{\lambda\alpha} A_0 + \Pi^\mu_{\lambda\alpha} A_\mu) dy^\alpha . \end{cases}$$

Next, consider an n-dimensional variety V_n in P_m defined by a system of equations

$$(1.2) \qquad y^\alpha = \varphi^\alpha(x^1, \cdots, x^n) ,$$

where φ's are analytic functions in a certain domain of x's and that the functional matrix

$$\left\| \begin{matrix} B_{1}^{\cdot 1} \cdots\cdots B_{1}^{\cdot m} \\ \vdots \qquad \vdots \\ B_{n}^{\cdot 1} \cdots\cdots B_{n}^{\cdot m} \end{matrix} \right\| \qquad \left(B_{i}^{\cdot \alpha} = \frac{\partial y^{\alpha}}{\partial x^{i}} \right)$$

is everywhere of rank n. When a current point A_0 displaces along a curve on V_n, we have

$$dA_0 = A_\alpha dy^\alpha = A_\alpha B_i^{\cdot\alpha} dx^i .$$

Now n linearly independent points $A_i (i=1, \cdots, n)$ defined by

$$A_i = A_\alpha B_i^{\cdot\alpha} \tag{1.3}$$

are on the n-dimensional tangent plane of V_n which lies in the osculating projective space of P_m at A_0. Moreover we take in the space $(m-n)$ points A_P defined by

$$A_P = A_\alpha B_P^{\cdot\alpha} \qquad (P=n+1, \cdots, m), \tag{1.4}$$

where the quantities $B_P^{\cdot\alpha}$ are arbitrarily chosen under a condition that the determinant $| B_i^{\cdot\alpha} B_P^{\cdot\alpha} |$ does not vanish. Those $(m+1)$ linearly independent points A_0, A_i and A_P determine the frame, which is called *the frame on V_n.*

2. The fundamental equations of sub-variety.

We have immediately

$$dA_0 = A_i dx^i , \tag{2.1}$$

and further we put, along V_n,

$$dA_i = (\Pi^0_{ij} A_0 + \Pi^k_{ij} A_k + H^Q_{ij} A_Q)\, dx^j , \tag{2.2}$$

$$dA_P = (H^0_{Pj} A_0 + H^k_{Pj} A_k + H^Q_{Pj} A_Q)\, dx^j . \tag{2.3}$$

Differentiating (1.3) and (1.4), and making use of (1.1) we get

$$\begin{cases} dA_i = \left\{ \Pi^0_{\alpha\beta} B_i^{\cdot\alpha} B_j^{\cdot\beta} A_0 + \left(\dfrac{\partial B_i^{\cdot\alpha}}{\partial x^j} + \Pi^\alpha_{\beta\gamma} B_i^{\cdot\beta} B_j^{\cdot\gamma} \right) A_\alpha \right\} dx^j , \\ dA_P = \left\{ \Pi^0_{\alpha\beta} B_P^{\cdot\alpha} B_j^{\cdot\beta} A_0 + \left(\dfrac{\partial B_P^{\cdot\alpha}}{\partial x^j} + \Pi^\alpha_{\beta\gamma} B_P^{\cdot\beta} B_j^{\cdot\gamma} \right) A_\alpha \right\} dx^j . \end{cases}$$

On the other hand, substitution of (1.3) and (1.4) in (2.2) and (2.3) gives

$$\begin{cases} dA_i = \{\Pi^0_{ij}A_0 + (\Pi^k_{ij}B^{\cdot\alpha}_k + H^P_{ij}B^{\cdot\alpha}_P)A_\alpha\}dx^j , \\ dA_P = \{H^0_{Pj}A_0 + (H^k_{Pj}B^{\cdot\alpha}_k + H^Q_{Pj}B^{\cdot\alpha}_Q)A_\alpha\}dx^j . \end{cases}$$

Therefore, we have the following equations:

$$\begin{cases} \Pi^0_{ij} = \Pi^0_{\alpha\beta}B^{\cdot\alpha}_i B^{\cdot\beta}_j , \\ H^0_{Pj} = \Pi^0_{\alpha\beta}B^{\cdot\alpha}_P B^{\cdot\beta}_j , \end{cases} \tag{2.4}$$

$$\begin{cases} \Pi^k_{ij}B^{\cdot\alpha}_k + H^P_{ij}B^{\cdot\alpha}_P = \dfrac{\partial B^{\cdot\alpha}_i}{\partial x^j} + \Pi^\alpha_{\beta\gamma}B^{\cdot\beta}_i B^{\cdot\gamma}_j , \\ H^k_{Pj}B^{\cdot\alpha}_k + H^Q_{Pj}B^{\cdot\alpha}_Q = \dfrac{\partial B^{\cdot\alpha}_P}{\partial x^j} + \Pi^\alpha_{\beta\gamma}B^{\cdot\beta}_P B^{\cdot\gamma}_j . \end{cases} \tag{2.5}$$

By these equations the coefficients of the equations (2.2) and (2.3) are determined. We shall call the equations (2.1), (2.2) and (2.3) *the fundamental equations of V_n in P_m.*

When we replace the system of coordinates y's in P_m by $\bar{y}$'s, the equations of transformation of the components of the connection $\Pi^0_{\alpha\beta}$ and $\Pi^\gamma_{\alpha\beta}$ of P_m are given by

$$\begin{cases} \bar{\Pi}^0_{\lambda\mu} = \Pi^0_{\alpha\beta}\, \underline{P}^\alpha_\lambda\, \underline{P}^\beta_\mu , \\ \bar{\Pi}^\lambda_{\mu\nu} = \bar{P}^\lambda_\alpha \left(\dfrac{\partial \underline{P}^\alpha_\mu}{\partial \bar{y}^\nu} + \Pi^\alpha_{\beta\gamma}\, \underline{P}^\beta_\mu\, \underline{P}^\gamma_\nu \right), \end{cases}$$

where $\bar{P}^\lambda_\alpha$ and $\underline{P}^\alpha_\lambda$ are

$$\bar{P}^\lambda_\alpha = \frac{\partial \bar{y}^\lambda}{\partial y^\alpha} , \qquad \underline{P}^\alpha_\lambda = \frac{\partial y^\alpha}{\partial \bar{y}^\lambda} .$$

The quantities $B^{\cdot\alpha}_i$ enjoy the transformation $\bar{B}^{\cdot\lambda}_i = B^{\cdot\lambda}_i \bar{P}^\lambda_\alpha$, and further we put $\bar{B}^{\cdot\lambda}_P = B^{\cdot\alpha}_P \bar{P}^\lambda_\alpha$, so that A_i and A_P are invariant. Then, making use of (2.4) and (2.5) we see easily that the quantities Π^0_{ij}, Π^k_{ij}, H^P_{ij}, H^0_{Pi}, H^j_{Pi} and H^Q_{Pi} are all invariant.

Next, when we replace the coordinates x's in V_n by $\bar{x}$'s, we have $\bar{B}^{\cdot\alpha}_a = B^{\cdot\alpha}_i \underline{P}^i_a$ and $\bar{A}_a = A_i \underline{P}^i_a$, where $\underline{P}^i_a = \dfrac{\partial x^i}{\partial \bar{x}^a}$, $\bar{P}^a_i = \dfrac{\partial \bar{x}^a}{\partial x^i}$. Further we put $\bar{B}^{\cdot\alpha}_P = B^{\cdot\alpha}_P$, and then we see easily from (2.4) and (2.5) that the quantities Π^0_{ij}, Π^k_{ij}, H^P_{ij}, H^j_{Pi} and H^Q_{Pi} enjoy the transformation

$$\begin{cases} \bar{\Pi}^0_{ab} = \Pi^0_{ij}\, \underline{P}^i_a\, \underline{P}^j_b , \\ \bar{\Pi}^c_{ab} = \bar{P}^c_k \left(\dfrac{\partial \underline{P}^k_a}{\partial x^b} + \Pi^k_{ij}\, \underline{P}^i_a\, \underline{P}^j_b \right), \end{cases} \tag{2.6}$$

$$(2.7)\qquad \begin{cases} \overline{H}^{P}_{ab}=H^{P}_{ij}\,\underline{P}^{i}_{a}\,\underline{P}^{j}_{b}\,, & \overline{H}^{0}_{Pa}=H^{0}_{Pi}\,\underline{P}^{i}_{a}\,,\\ \overline{H}^{b}_{Pa}=H^{j}_{Pi}\,\overline{P}^{b}_{j}\,\underline{P}^{i}_{a}\,, & \overline{H}^{Q}_{Pa}=H^{Q}_{Pi}\,\underline{P}^{i}_{a}\,. \end{cases}$$

From (2.6) we see that the quantities Π^0_{ij} and Π^k_{ij} are transformed analogously to the components of the projective connection, so that we shall call Π^0_{ij} and Π^k_{ij} *the components of the projective connection of V_n induced from P_m.*

3. The torsion and curvature tensors of sub-variety.

Let a point A_0 on V_n have coordinates $y(x)$ and consider the infinitesimal circuit on V_n consisting of $A_0(y)$, $A_0(y+dy)$, $A_0(y+dy+\delta y)$, $A_0(y+\delta y)$ and $A_0(y)$. The projective transformation of the frame (A_0, A_i, A_P) for this circuit, that is,

$$\begin{cases} \Delta A_0=(\delta d-d\delta)A_0\,,\\ \Delta A_i=(\delta d-d\delta)A_i\,,\\ \Delta A_P=(\delta d-d\delta)A_P\,, \end{cases}$$

are given by the following equations:

$$(3.1)\qquad \begin{cases} \Delta A_0=(R_{0\cdot\alpha\beta}^{\;\;0}\,A_0+R_{0\cdot\alpha\beta}^{\;\;\lambda}\,A_\lambda)B_{i}^{\cdot\alpha}\,B_{j}^{\cdot\beta}\,dx^i\,\delta x^j\,,\\ \Delta A_i=(R_{\mu\cdot\alpha\beta}^{\;\;0}\,A_0+R_{\mu\cdot\alpha\beta}^{\;\;\lambda}\,A_\lambda)B_{i}^{\cdot\mu}\,B_{j}^{\cdot\alpha}\,B_{k}^{\cdot\beta}\,dx^j\,\delta x^k\,,\\ \Delta A_P=(R_{\mu\cdot\alpha\beta}^{\;\;0}\,A_0+R_{\mu\cdot\alpha\beta}^{\;\;\lambda}\,A_\lambda)B_{P}^{\cdot\mu}\,B_{j}^{\cdot\alpha}\,B_{k}^{\cdot\beta}\,dx^j\,\delta x^k\,, \end{cases}$$

by virtue of (1.1), where the quantities $R_{0\cdot\alpha\beta}^{\;\;0}$, $R_{0\cdot\alpha\beta}^{\;\;\lambda}$, $R_{\mu\cdot\alpha\beta}^{\;\;0}$ and $R_{\mu\cdot\alpha\beta}^{\;\;\lambda}$ are the torsion and curvature tensors of P_m defined by

$$\begin{cases} R_{0\cdot\alpha\beta}^{\;\;0}=\Pi^0_{\alpha\beta}-\Pi^0_{\beta\alpha}\,, \qquad R_{0\cdot\alpha\beta}^{\;\;\lambda}=\Pi^\lambda_{\alpha\beta}-\Pi^\lambda_{\beta\alpha}\,,\\ R_{\mu\cdot\alpha\beta}^{\;\;0}=\dfrac{\partial\Pi^0_{\mu\alpha}}{\partial y^\beta}-\dfrac{\partial\Pi^0_{\mu\beta}}{\partial y^\alpha}+\Pi^\sigma_{\mu\alpha}\,\Pi^0_{\sigma\beta}-\Pi^\sigma_{\mu\beta}\,\Pi^0_{\sigma\alpha}\,,\\ R_{\mu\cdot\alpha\beta}^{\;\;\lambda}=\dfrac{\partial\Pi^\lambda_{\mu\alpha}}{\partial y^\beta}-\dfrac{\partial\Pi^\lambda_{\mu\beta}}{\partial y^\alpha}+\Pi^\sigma_{\mu\alpha}\,\Pi^\lambda_{\sigma\beta}-\Pi^\sigma_{\mu\beta}\,\Pi^\lambda_{\sigma\alpha}+\Pi^0_{\mu\alpha}\,\delta^\lambda_\beta-\Pi^0_{\mu\beta}\,\delta^\lambda_\alpha\,. \end{cases}$$

On the other hand, making use of the fundamental equations (2.1), (2.2) and (2.3) we have

$$(3.2)\qquad \begin{cases} \Delta A_0=(R_{0\cdot ij}^{\;\;0}\,A_0+R_{0\cdot ij}^{\;\;k}\,A_k+H^P_{[ij]}\,A_P)dx^i\,\delta x^j\,,\\ \Delta A_i=(\Pi_{i\cdot jk}^{\;\;0}\,A_0+\Pi_{i\cdot jk}^{\;\;l}\,A_l+\Pi_{i\cdot jk}^{\;\;P}\,A_P)\,dx^j\,\delta x^k\,,\\ \Delta A_P=(\Pi_{P\cdot ij}^{\;\;0}\,A_0+\Pi_{P\cdot ij}^{\;\;k}\,A_k+\Pi_{P\cdot ij}^{\;\;Q}\,A_Q)\,dx^i\,\delta x^j\,, \end{cases}$$

where we put

$$(3.3)\qquad \begin{cases} \Pi_{i\cdot jk}^{\ 0}=R_{i\cdot jk}^{\ 0}+H_{ij}^{P}\,H_{Pk}^{0}-H_{ik}^{P}\,H_{Pj}^{0}\,, \\ \Pi_{i\cdot jk}^{\ l}=R_{i\cdot jk}^{\ l}+H_{ij}^{P}\,H_{Pk}^{l}-H_{ik}^{P}\,H_{Pj}^{l}\,, \\ \Pi_{i\cdot jk}^{\ P}=H_{ij,k}^{P}-H_{ik,j}^{P}+H_{ij}^{Q}\,H_{Qk}^{P}-H_{ik}^{Q}\,H_{Qj}^{P}\,, \end{cases}$$

$$(3.4)\qquad \begin{cases} \Pi_{P\cdot ij}^{\ 0}=H_{Pi,j}^{0}-H_{Pj,i}^{0}+H_{Pi}^{k}\,\Pi_{kj}^{0}-H_{Pj}^{k}\,H_{ki}^{0}+H_{Pi}^{Q}\,H_{Qj}^{0}-H_{Pj}^{Q}\,H_{Qi}^{0}\,, \\ \Pi_{P\cdot ij}^{\ k}=H_{Pi,j}^{k}-H_{Pj,i}^{k}+H_{Pi}^{Q}\,H_{Qj}^{k}-H_{Pj}^{Q}\,H_{Qi}^{k}+H_{Pi}^{0}\,\delta_{j}^{k}-H_{Pj}^{0}\,\delta_{i}^{k}\,, \\ \Pi_{P\cdot ij}^{\ Q}=H_{Pi,j}^{Q}-H_{Pj,i}^{Q}+H_{Pi}^{k}\,H_{kj}^{Q}-H_{Pj}^{k}\,H_{ki}^{Q}+H_{Pi}^{R}\,H_{Rj}^{Q}-H_{Pj}^{R}\,H_{Pi}^{Q}\,. \end{cases}$$

The quantities $R_{0\cdot ij}^{\ 0}$, $R_{0\cdot ij}^{\ k}$, $R_{i\cdot jk}^{\ 0}$ and $R_{i\cdot jk}^{\ l}$ are *the torsion and curvature tensors of V_n constructed from the components of the projective connection of V_n induced from P_m*, and the comma means the covariant differentiation with respect to Π_{ij}^{k}. Hence we obtain from (3.1) and (3.2) the following equations:

$$(3.5)\qquad R_{0\cdot\alpha\beta}^{\ 0}\,B_{i}^{\cdot\alpha}\,B_{j}^{\cdot\beta}=R_{0\cdot ij}^{\ 0}\,,$$

$$(3.6)\qquad R_{0\cdot\alpha\beta}^{\ \lambda}\,B_{i}^{\cdot\alpha}\,B_{j}^{\cdot\beta}=R_{0\cdot ij}^{\ k}\,B_{k}^{\cdot\lambda}+(H_{ij}^{P}-H_{ji}^{P})B_{P}^{\cdot\lambda}\,,$$

$$(3.7)\qquad R_{\mu\cdot\alpha\beta}^{\ 0}\,B_{i}^{\cdot\mu}\,B_{j}^{\cdot\alpha}\,B_{k}^{\cdot\beta}=\Pi_{i\cdot jk}^{\ 0}\,,$$

$$(3.8)\qquad R_{\mu\cdot\alpha\beta}^{\ \lambda}\,B_{i}^{\cdot\mu}\,B_{j}^{\cdot\alpha}\,B_{k}^{\cdot\beta}=\Pi_{i\cdot jk}^{\ l}\,B_{l}^{\cdot\lambda}+\Pi_{i\cdot jk}^{\ P}\,B_{P}^{\cdot\lambda}\,,$$

$$(3.9)\qquad R_{\mu\cdot\alpha\beta}^{\ 0}\,B_{P}^{\cdot\mu}\,B_{i}^{\cdot\alpha}\,B_{j}^{\cdot\beta}=\Pi_{P\cdot ij}^{\ 0}\,,$$

$$(3.10)\qquad R_{\mu\cdot\alpha\beta}^{\ \lambda}\,B_{P}^{\cdot\mu}\,B_{i}^{\cdot\alpha}\,B_{j}^{\cdot\beta}=\Pi_{P\cdot ij}^{\ k}\,B_{k}^{\cdot\lambda}+\Pi_{P\cdot ij}^{\ Q}\,B_{Q}^{\cdot\lambda}\,.$$

4. The fundamental theorem of embedding.

Let Π_{ij}^{0} and Π_{ij}^{k} be components of the connection of the given n-dimensional space P_n with projective connection. We say that P_n can be *embedded* in a projective space S_m of m-dimensions, if there exists an n-dimensional sub-space S_n, whose components of projective connection induced from S_m are equal to the given Π_{ij}^{0} and Π_{ij}^{k}. The space P_n is called of *class p*, if P_n can be embedded in a projective space of $(n+p)$-dimensions but not of $(n+q)$ $(p>q\geqq 0)$-dimensions.

It is clear that P_n is of class zero if, and only if, the torsion and curvature tensors vanish. Also it has been proved by S. S. Chern, see [4], that the class number is at most $n(n-1)/2+(n-1)/2$ (n: odd), or $n(n-1)/2+n/2$ (n: even).

Since P_m in preceding sections is flat hereafter and the determinant $|B_i^{;\alpha} B_P^{;\alpha}|$ does not vanish, we have from (3.5), (3.6), (3.7), (3.8), (3.9) and (3.10) the following equations:

$$R_0{}^0{}_{\cdot ij}=0\,, \qquad R_0{}^k{}_{\cdot ij}=0\,, \tag{4.1}$$

$$H^P_{ij}-H^P_{ji}=0\,, \tag{4.2}$$

$$R_h{}^i{}_{\cdot jk}=-H^P_{hj}\,H^i_{Pk}+H^P_{hk}\,H^i_{Pj}\,, \tag{4.3}$$

$$R_i{}^0{}_{\cdot jk}=-H^P_{ij}\,H^0_{Pk}+H^P_{ik}\,H^0_{Pj}\,, \tag{4.4}$$

$$H_{ij,k}^{P}-H_{ik,j}^{P}+H^Q_{ij}\,H^P_{Qk}-H^Q_{ik}\,H^P_{Qj}=0\,, \tag{4.5}$$

$$H_{Pj,k}^{i}-H_{Pk,j}^{i}+H^Q_{Pj}\,H^i_{Qk}-H^Q_{Pk}\,H^i_{Qj}+H^0_{Pj}\,\delta^i_k-H^0_{Pk}\,\delta^i_j=0\,, \tag{4.6}$$

$$H_{Pj,k}^{0}-H_{Pk,j}^{0}+H^l_{Pj}\,\Pi^0_{kl}-H^l_{Pk}\Pi^0_{jl}+H^Q_{Pj}\,H^0_{Qk}-H^Q_{Pk}\,H^0_{Qj}=0\,, \tag{4.7}$$

$$H^Q_{Pj,k}-H^Q_{Pk,j}+H^l_{Pj}\,H^Q_{kl}-H^l_{Pk}\,H^Q_{jl}+H^R_{Pj}\,H^Q_{Rk}-H^R_{Pk}\,H^Q_{Rj}=0\,. \tag{4.8}$$

The equations (4.3) and (4.4) are called *the Gauss equations;* (4.5), (4.6) and (4.7) *the Codazzi equations;* (4.8) *the Ricci equation.* From (4.1) we see that *the connection is necessarily symmetric, if the space can be embedded in a projective space.* Also from (4.2) we see that the functions H^P_{ij} are symmetric with respect to i and j.

Conversely, suppose that there exist in P_n, whose connection is symmetric, four systems of functions H^P_{ij} $(=H^P_{ji})$, H^0_{Pj}, H^i_{Pj} and H^Q_{Pj} $(i, j=1, \cdots, n;\ P, Q=n+1, \cdots, m)$ satisfying the Gauss, Codazzi and Ricci equations. Consider the fixed frame (E_0, E_α) $(\alpha=1, \cdots, m)$ in an m-dimensional projective space S_m and let A^0_0 and A^α_0 be coordinates of a current point A_0 with reference to the frame. Then a system of partial differential equations, namely the fundamental equations of sub-variety

$$\frac{\partial A_0}{\partial x^i}=A_i\,, \tag{4.9}$$

$$\frac{\partial A_i}{\partial x^j}=\Pi^0_{ij}\,A_0+\Pi^k_{ij}\,A_k+H^P_{ij}\,A_P\,, \tag{4.10}$$

$$\frac{\partial A_P}{\partial x^i}=H^0_{Pi}\,A_0+H^k_{Pi}\,A_k+H^Q_{Pi}\,A_Q\,, \tag{4.11}$$

is integrable. In fact, the integrability condition of (4.9), that is to say,

$$\frac{\partial}{\partial x^j}\left(\frac{\partial A_0}{\partial x^i}\right)-\frac{\partial}{\partial x^i}\left(\frac{\partial A_0}{\partial x^j}\right)=0$$

is equal to

$$\frac{\partial A_i}{\partial x^j}-\frac{\partial A_j}{\partial x^i}=0,$$

which is satisfied by virture of symmetry of the connection of P_n and that of H^P_{ij}. Next the integrability condition of (4.10)

$$\frac{\partial}{\partial x^k}\left(\frac{\partial A_i}{\partial x^j}\right)-\frac{\partial}{\partial x^j}\left(\frac{\partial A_i}{\partial x^k}\right)=0$$

is satisfied by means of (4.3), (4.4) and (4.5), substituting (4.9), (4.10) and (4.11). Finally (4.11) is also integrable by means of (4.6), (4.7) and (4.8) Thus the n-dimensional surface, that is the locus of the point A_0 so defined as the solution of the above system, has the same connection as given P_n. Hence we obtain the fundamental theorem of embedding as follows:

THEOREM I: *A space with projective connection of n-dimensions can be embedded in an $(n+p)$-dimensional projective space if, and only if, the connection is symmetric and there exist four systems of functions H^P_{ij} $(=H^P_{ji})$, H^0_{Pi}, H^i_{Pj} and H^Q_{Pi}. $(i,j=1,\cdots,n;\ P,Q=n+1,\cdots,n+p)$ satisfying the Gauss, Codazzi and Ricci equations.*

5. Projectively connected space of class one.

Consider the case $p=1$. We replace A_{n+1} by

(5.1) $$\bar{A}_{n+1}=\rho\, A_{n+1} \qquad (\rho\neq 0),$$

where $\bar{A}_{n+1}$ coincides geometrically with A_{n+1}, and we put, instead of (2.2) and (2.3)

(5.2) $$dA_i=(\bar{\Pi}^0_{ij}A_0+\bar{\Pi}^k_{ij}A_k+\bar{H}^{n+1}_{ij}\bar{A}_{n+1})\,dx^j\,,$$

(5.3) $$d\bar{A}_{n+1}=(\bar{H}_{n+1j}^{\ 0}A_0+\bar{H}_{n+1j}^{\ k}A_k+\bar{H}^{n+1}_{n+1j}\bar{A}_{n+1})\,dx^j\,.$$

Substitution of (5.1) in (5.2) and comparison with (2.2) give $\bar{H}^{n+1}_{ij}=\dfrac{1}{\rho}H^{n+1}_{ij}$, $\bar{\Pi}^0_{ij}=\Pi^0_{ij}$ and $\bar{\Pi}^k_{ij}=\Pi^k_{ij}$. Next, differentiating (5.1) and making use of (2.3) and (5.3) we obtain $\bar{H}_{n+1j}^{\ 0}=\rho H_{n+1j}^{\ 0}$, $\bar{H}_{n+1j}^{\ i}=\rho H_{n+1j}^{\ i}$ and

$$\overline{H}_{n+1}^{n+1}{}_{j}=\rho H_{n+1}^{n+1}{}_{j}+\frac{\partial\rho}{\partial x^{j}}. \tag{5.4}$$

It is well known that $R_{l}{}^{l}{}_{ij}=0$ $(i,j=1,\cdots,n)$ refering to the natural frame of P_n, see [3], and hence the second member of (4.8) vanishes. And also the third member vanishes identically. Therefore we have

$$\frac{\partial H_{n+1}^{n+1}{}_{j}}{\partial x^{k}}-\frac{\partial H_{n+1}^{n+1}{}_{k}}{\partial x^{j}}=0,$$

and so we can take such a function ρ that $\overline{H}_{n+1}^{n+1}{}_{j}=0$ identically.

When we refer to such a frame $(A_0,\ A_i, A_{n+1})$ on P_n, the Gauss and Codazzi equations are expressible as follows:

$$R_{l}{}^{i}{}_{jk}=-H_{lj}H_{k}^{i}+H_{lk}H_{j}^{i}, \tag{5.5}$$

$$R_{i}{}^{0}{}_{jk}=-H_{ij}H_{k}+H_{ik}H_{j}, \tag{5.6}$$

$$H_{ij,\,k}-H_{ik,\,j}=0, \tag{5.7}$$

$$H_{j,\,k}^{i}-H_{k,\,j}^{i}+H_{j}\delta_{k}^{i}-H_{k}\delta_{j}^{i}=0, \tag{5.8}$$

$$H_{j,\,k}-H_{k,\,j}+H_{j}^{i}\Pi_{ki}^{0}-H_{k}^{i}\Pi_{ji}^{0}=0; \tag{5.9}$$

where we put $H_{ij}=H_{ij}^{n+1}$, $H_{j}^{i}=H_{n+1}{}^{i}{}_{j}$ and $H_{j}=H_{n+1}{}^{0}{}_{j}$; and finally the Ricci equation is satisfied identically.

6. The second and third Codazzi equations as consequences of the Gauss and the first Codazzi equations.

In the case of Riemann space V_n of class one, the Codazzi equation is automatically satisfied, if V_n is of type $\geqq 4$ and the Gauss equation is satisfied, see [5]. And also in the case of class greater than one, three systems of conditions are not independent in general, see [1] and [6]. Now, in our case, we shall prove similarly that the conditions (5.8) and (5.9) are consequences of the remaining (5.5), (5.6) and (5.7).

(**A**) *The equation* (5.8).

Differentiating (5.5) covariantly with respect to x^h and summing three equations obtained from the first by cyclic permutation of i, j and h, we have

$$H_{lij}H_{h}^{k}+H_{ljh}H_{i}^{k}+H_{lhi}H_{j}^{k}=H_{li}H_{jh}^{k}+H_{lj}H_{hi}^{k}+H_{lh}H_{ij}^{k} \tag{6.1}$$

on account of (5.6) and the Bianchi identity

$$R_{b\cdot(jk,l)}^{\ a}+R_{b\cdot(jk}^{\ 0}\delta_{l)}^{a}=0\,,$$

where we put

$$H_{lij}=H_{li,\,j}-H_{lj,\,i}\,,$$

$$H_{ij}^{k}=H_{i,\,j}^{k}-H_{j,\,i}^{k}+H_{i}\,\delta_{j}^{k}-H_{j}\,\delta_{i}^{k}\,.$$

Then, if (5.7) is satisfied, i. e. $H_{lij}=0\,(l,i,j=1,\cdots,n)$, we obtain from (6.1)

$$H_{lj}\,H_{hi}^{k}+H_{lh}\,H_{ij}^{k}+H_{li}\,H_{jh}^{k}=0\,. \tag{6.2}$$

[1] Suppose the determinant $|H_{ij}|\neq 0$. Let $\|H^{ij}\|$ be the inverse matrix of the $\|H_{ij}\|$; then contracting (6.2) by H^{li} we have

$$(n-2)\,H_{jh}^{k}=0 \qquad (j,k,h=1,\cdots,n)\,,$$

and hence $H_{jh}^{k}=0$ for $n\geqq 3$.

[2] Suppose the matrix $\|H_{ij}\|$ be of rank $\sigma\,(n>\sigma\geqq 3)$. Transform the coordinates x^i of P_n so that at the origin the matrix $\|H_{ij}\|$ has the form

$$\|H_{ij}\|=\left\|\begin{array}{c:c} H_\sigma & 0 \\ \hdashline 0 & 0 \end{array}\right\|, \qquad |H_\sigma|=\begin{vmatrix} H_{11} & \cdots & H_{1\sigma} \\ \vdots & & \vdots \\ H_{\sigma 1} & \cdots & H_{\sigma\sigma} \end{vmatrix}\neq 0\,,$$

and let $\|H^{ij}\|$ $(i,j=1,\cdots,\sigma)$ be the inverse of the $\|H_\sigma\|$. First, taking l,i,j and h for $1,\cdots,\sigma$ in (6.2) and in the same way as [1] we obtain $H_{jh}^{k}=0$ $(j,h=1,\cdots,\sigma;\,k=1,\cdots,n)$. Next, taking $h>\sigma$ and $l,i,j=1,\cdots,\sigma$ in (6.2) we have

$$H_{li}\,H_{jh}^{k}+H_{lj}\,H_{hi}^{k}=0\,.$$

Contraction of the above equation by H^{li} gives $H_{jh}^{k}=0$ $(j=1,\cdots,\sigma;\,h>\sigma;\,k=1,\cdots,n)$. Finally, taking $j,h>\sigma$ and $i,l=1,\cdots,\sigma$ in (6.2) we have $H_{jh}^{k}=0$ $(j,h>\sigma;\,k=1,\cdots,n)$. Since H_{ij}^{k} is skew-symmetric with respect to i and j, all of H_{ij}^{k} vanish.

Hence, *if the matrix* $\|H_{ij}\|$ *has the rank* $\sigma\,(\geqq 3)$, (5.8) *is a consequence of* (5.5), (5.6) *and* (5.7).

(**B**) *The equation* (5.9).

Similarly, differentiating (5.6) covariantly and making use of (5.5) and (5.7) and also the Bianchi identity

$$R_{l\cdot(ij,k)}^{\ 0} - \Pi_{a(i}^{0} R_{[l]\cdot jk)}^{\ a} = 0 ,$$

we have

(6.3) $$H_{li} A_{jk} + H_{lj} A_{ki} + H_{lk} A_{ij} = 0 ,$$

where we put

$$A_{jk} = H_{j,k} - H_{k,j} + H_j^a \Pi_{ak}^0 - H_k^a \Pi_{aj}^0 .$$

In the same way as in (A) we can prove that (5.9) *is a consequence of* (5.5), (5.6) *and* (5.7) *if the matrix* $\| H_{ij} \|$ *has the rank* $\sigma (\geq 3)$.

In this way the rank of the matrix $\| H_{ij} \|$ plays an important part in our problem, but it seems to us that this integer σ can not be determined by intrinsic properties of P_n. This circumstance is like to a property of type number of Riemann spaces of class greater than one, which was defined by C. B. Allendoerfer, see [1].

7. Type number of hypersurfaces in projective space.

We put

(7.1) $$K_{h\cdot ij}^{\ k} = H_{hi} H_j^k ,$$

and by contraction

(7.2) $$K_{ij} = K_{a\cdot ij}^{\ a} = H_{ai} H_j^a .$$

The above tensor K_{ij} is symmetric on account of $R_{a\cdot ij}^{\ a} = 0$. We introduce the equation which determines the intrinsic form of K_{ij}. Interchanging indices j and k in the following expression

$$K_{a\cdot ij}^{\ b} K_{kl} = H_{ai} H_l^c H_{ck} H_j^b ,$$

and subtracting, we have in accordance with (5.5)

$$K_{a\cdot i[j}^{\ b} K_{k]l} = H_{ai} H_l^c R_{c\cdot jk}^{\ b} .$$

Moreover, interchanging i and l, and subtracting, we obtain

(7.3) $$R_{a\cdot li}^{\ c} R_{c\cdot jk}^{\ b} = K_{a\cdot i[j}^{\ b} K_{k]l} - K_{a\cdot l[j}^{\ b} K_{k]i} .$$

Contraction of (7.3) with respect to a and b gives

(7.4) $$M_{ijkl} = K_{ik} K_{jl} - K_{il} K_{jk} ,$$

making use of (7.2), where we put

$$M_{ijkl} = -\frac{1}{2} R_{a\cdot ij}^{\ b} R_{b\cdot kl}^{\ a}. \tag{7.5}$$

It is easily seen that the intrinsic tensor M_{ijkl} satisfies the following identities

$$M_{ijkl} = -M_{jikl} = M_{klij}, \tag{7.6}$$

by means of the properties of the tensor $R_{a\cdot ij}^{\ b}$.

On the other hand, multiplying (5.5) by H_{ah} and summing three equations obtained from the first by cyclic permutation of i, j and h, we have from (5.5)

$$H_{a(h} R_{|l|\cdot ij)}^{\ k} + H_{l(h} R_{|a|\cdot ij)}^{\ k} = 0.$$

And further, multiplying the above equation by H_m^b and subtracting three equations obtained by interchanging i and m, j and m, h and m, we obtain from (5.5)

$$\text{(I)} \quad R_{a\cdot m(h}^{\ b} R_{|l|\cdot ij)}^{\ k} + R_{a\cdot (ij}^{\ b} R_{|l\cdot m|h)}^{\ k} + R_{l\cdot m(h}^{\ b} R_{|a|\cdot ij)}^{\ k} + R_{l\cdot (ij}^{\ b} R_{|a\cdot m|h)}^{\ k} = 0.$$

This equation is a necessary condition that there may exist two systems of functions H_{ij} and H_j^i satisfying (5.5). Contracting (I) with respect to a and k, and moreover b and l, we get

$$M_{i(jkl)} = 0. \tag{7.7}$$

Thus the tensor M_{ijkl} has the same properties (7.6) and (7.7) as the curvature tensor of Riemann spaces and (7.4) has the same form as the Gauss equation of hypersurfaces in euclidean space. Therefore, by means of (7.4), we can obtain similar facts and theorems which have been already proved by T. Y. Thomas in his excellent theory of Riemann spaces of class one, see [5]. Let us enumerate those facts and theorems without proofs.

(**A**) *Definition of type number.*

A hypersurface S in a projective space will be said to be of *type one* if the rank of the matrix $\| K_{ij} \|$ is zero or one. It will be said to be of *type* τ where τ is an integer of the set $2, \cdots, n$ if the rank of the above matrix is τ. It is easily seen that *S is of type one if, and only if, the tensor M_{ijkl} is identically equal to zero.* And we can prove that *the type number τ ($\geqq 3$) of a hypersurface is equal to the rank of the matrix*

$$\begin{Vmatrix} M_{1abc} & M_{2abc} \cdots\cdots M_{nabc} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ M_{1pqr} & M_{2pqr} \cdots\cdots M_{npqr} \end{Vmatrix}.$$

Hence the type number is determined by intrinsic properties of S.

(**B**) *Uniqueness of solution K's of* (7.4).

We can prove that it holds the following relation between the type number and the number of solutions of (7.4). That is, *if a hypersurface S is of type* $\tau\,(\geqq 3)$, *the system of functions* K_{ij} *satisfying* (7.4) *is uniquely determined to within algebraic sign.*

(**C**) *Reality condition.*

When the hypersurface S is real, the tensor K_{ij} is naturally real. *The solution K's of* (7.4) *will be real if, and only if, the matrix condition*

$$\text{(II)} \qquad M_{(a,b,c)(i,j,k)} \equiv \begin{vmatrix} M_{abij} & M_{abjk} & M_{abki} \\ M_{bcij} & M_{bcjk} & M_{bcki} \\ M_{caij} & M_{cajk} & M_{caki} \end{vmatrix} \geqq 0$$

is satisfied, when S is of type $\tau\,(\geqq 3)$.

(**D**) *Resultant system.*

Let us write (7.4) in the homogeneous form, namely

$$\text{(7.4')} \qquad t^2 \cdot M_{ijkl} = K_{ik} K_{jl} - K_{il} K_{jk}\,,$$

and we obtain easily from (7.4)

$$\text{(7.8)} \qquad K_{h[i} M_{m]ljk} - K_{l[j} M_{k]him} = 0\,.$$

Represent the resultant system of (7.4′) and (7.8), a set of polynomials in the components M's such that the vanishing of these polynomials is a necessary and sufficient condition for the existence of a non-trivial solution, by $R_n(M)$. We can prove that (7.4) *will have a real solution if, and only if, the inequalities* (II) *and*

$$\text{(III)} \qquad \sum_{a,b,c,i,j,k} M^2_{(a,b,c)(i,j,k)} > 0\,,$$

and the equation

$$\text{(IV)} \qquad R_n(M) = 0$$

are satisfied.

Now there is a relation between the type number τ and the rank σ of the matrix $\| H_{ij} \|$. That is, from (7.2) we have

$$\| K_{ij} \| = \| H_{ij} \| \cdot \| H^i_j \| . \tag{7.9}$$

On account of the well-known theorem for the rank of the product of two matrices, see [2], we get $\tau \leqq \sigma$; this is similar to a relation between two kinds of type numbers of Riemann space of class two, the one defined by C. B. Allendoerfer and the other by the writer, see [6]. Hence, if P_n of class one is of type $\tau\,(\geqq 3)$, the rank σ of the matrix $\| H_{ij} \|$ is not less than three. Consequently from the results in the last section we have the

THEOREM II: *If a space with symmetric projective connection P_n of dimensions $n\,(\geqq 3)$, satisfying the equation* (I), *is of type $\tau\,(\geqq 3)$, then P_n will be of class one if, and only if, there exist three system of functions $H_{ij}\,(=H_{ji})$, H^i_j and H_i $(i, j=1, \cdots, n)$ satisfying the equations* (5.5), (5.6) *and* (5.7).

8. The first Gauss equation.

In this section we shall find the tensor $K_{h\cdot ij}^{\;\;k}$ defined by (6.1) making use of the intrinsic tensor K_{ij} determined in the last section, and get a necessary and sufficient condition for the existence of two systems of functions H_{ij} and H^i_j satisfying (5.5), as P_n is of type $\tau\,(\geqq 3)$.

First we prove that, if (7.3) has a solution $K_{h\cdot ij}^{\;\;k}$ $(h, i, j, k=1, \cdots, n)$, it is uniquely determined. In fact let $K_{h\cdot ij}^{\;\;k}$ and $\overline{K}_{h\cdot ij}^{\;\;k}$ be two solutions and we put

$$\overline{K}_{h\cdot ij}^{\;\;k} = K_{h\cdot ij}^{\;\;k} + D_{h\cdot ij}^{\;\;k} ,$$

and then we see that $D_{h\cdot ij}^{\;\;k}$ satisfies the equation

$$D_{a\cdot ij}^{\;\;b} K_{kl} - D_{a\cdot ik}^{\;\;b} K_{jl} - D_{a\cdot lj}^{\;\;b} K_{ki} + D_{a\cdot lk}^{\;\;b} K_{ji} = 0 . \tag{8.1}$$

[1] Suppose the type $\tau = n\,(\geqq 3)$. Let $\| K^{ij} \|$ be the inverse matrix of $\| K_{ij} \|$; then contraction of (8.1) by K^{kl} gives

$$(n-2)\; D_{a\cdot ij}^{\;\;b} + K^{kl}\, D_{a\cdot lk}^{\;\;b} K_{ij} = 0 . \tag{8.2}$$

Moreover, contracting (8.2) by K^{ji} we have $K^{kl}\, D_{a\cdot lk}^{\;\;b} = 0$. Hence, from (8.2) we have $D_{a\cdot ij}^{\;\;b} = 0$ $(a, b, i, j=1, \cdots, n)$.

[2] Suppose the type $\tau\,(n>\tau\geqq 3)$. Transform the coordinates x^i of P_n such that in the origin the matrix $\|K_{ij}\|$ has the form

$$\|K_{ij}\| = \left\| \begin{array}{c:c} K_\tau & 0 \\ \hdashline 0 & 0 \end{array} \right\|, \quad |K_\tau| = \begin{vmatrix} K_{11} \cdots\cdots K_{1\tau} \\ \vdots \\ K_{\tau 1} \cdots\cdots K_{\tau\tau} \end{vmatrix} \neq 0 .$$

Let $\|K^{ij}\|$ $(i, j=1, \cdots, \tau)$ be the inverse matrix of $\|K_\tau\|$; taking i, j, k and l for $1, \cdots, \tau$ in (8.1), we obtain, in the same way as in [1], $D_{h\cdot ij}^{\ k}=0\,(h, k=1, \cdots, n;\ i, j=1, \cdots, \tau)$. Next, taking $l>\tau$ and $i, j, k=1, \cdots, \tau$ in (8.1) we have

$$-D_{a\cdot lj}^{\ b}\,K_{ki}+D_{a\cdot lk}^{\ b}K_{ij}=0\,.$$

Contraction by K^{ij} gives $D_{h\cdot ij}^{\ k}=0\,(i>\tau;\ j=1, \cdots, \tau;\ h, k=1, \cdots, n)$. Finally taking $k, l>\tau$; $i, j=1, \cdots, \tau$ in (8.1) we obtain $K_{ij}D_{a\cdot kl}^{\ b}=0$ and so we have $D_{a\cdot kl}^{\ b}=0\,(k, l>\tau;\ a, b=1, \cdots, n)$. Hence the above statement is proved.

Now we get a necessary and sufficient condition that (7.3) has a solution. Let us write (7.3) in the homogeneous form

$$t\cdot R_{a\cdot li}^{\ c}\,R_{c\cdot jk}^{\ b}=K_{a\cdot i[j}^{\ b}K_{k]l}-K_{a\cdot l[j}^{\ b}K_{k]i}\,. \tag{7.3$'$}$$

Represent the resultant system of (7.3′) a set of polynomials in the components of the curvature tensor and K_{ij} such that the vanishing of these polynomials is necessary and sufficient for the existence of a non-trivial solution, by $R_n(K)$. We see that (7.3) has a solution if, and only if, the equation

$$R_n(K)=0 \tag{V}$$

is satisfied. In fact, putting $t=0$ in (7.3′) we have all $K_{a\cdot ij}^{\ b}=0$, in the similar way, by which we proved the uniqueness of solution. Thus we have the intrinsic tensor $K_{a\cdot ij}^{\ b}$ as the unique solution of (7.3), which is clearly real.

Next, it is easily seen that this tensor so determined must satisfy the following equations:

$$R_{h\cdot ij}^{\ k}=K_{h\cdot ji}^{\ k}-K_{h\cdot ij}^{\ k}\,, \tag{VI}$$

$$K_{i\cdot jb}^{\ a}=K_{j\cdot ib}^{\ a}\,, \tag{VII}$$

$$\text{(VIII)} \qquad \begin{vmatrix} K_{i\cdot jb}^{\ a} & K_{i\cdot jd}^{\ c} \\ K_{k\cdot lb}^{\ a} & K_{k\cdot ld}^{\ c} \end{vmatrix} = 0 .$$

The first is given by (5.5) and (7.1), the second by symmetry of H_{ij}, the third by (7.1).

But, it can be proved that $K_{a\cdot ij}^{\ a} = K_{ij}$, i. e. (7.2). In fact, contracting (7.3) with respect to a and b, and making use of (7.4) and (7.5), we obtain

$$K_{a\cdot i[j}^{\ a} K_{k]l} - K_{a\cdot l[j}^{\ a} K_{k]i} = -2K_{l[j} K_{k]i} . \tag{8.3}$$

[1] Suppose the type $\tau = n\ (\geqq 3)$. Contraction of (8.3) by K^{lj} gives

$$(n-2)\, K_{a\cdot ik}^{\ a} + K^{lj}\, K_{a\cdot lj}^{\ a}\, K_{ik} = 2\,(n-1)\, K_{ik} . \tag{8.4}$$

Moreover, contracting by K^{ik} we have $K^{ik} K_{a\cdot ik}^{\ a} = n$, so that we get (7.2) from (8.4).

[2] Suppose the type $\tau\ (n > \tau \geqq 3)$. Taking $i, j, k, l = 1, \cdots, \tau$ in (8.3) we have similarly (7.2) $(i, j = 1, \cdots, \tau)$. Next, taking $l > \tau;\ i, j, k = 1, \cdots, \tau$ in (8.3) we obtain

$$-K_{a\cdot lj}^{\ a}\, K_{ki} + K_{a\cdot lk}^{\ a}\, K_{ij} = 0 .$$

Contraction by K^{ki} gives $K_{a\cdot lj}^{\ a} = 0\ (= K_{lj})\ (l > \tau;\ j = 1, \cdots, \tau)$. Finally taking $l, k > \tau;\ i, j = 1, \cdots, \tau$ in (8.3) we have $K_{a\cdot lk}^{\ a} K_{ij} = 0$, and hence $K_{a\cdot lk}^{\ a} = 0\ (= K_{lk})\ (l, k > \tau)$. Therefore the above statement is proved.

Now, by means of (VIII), it is seen that the matrix $\| K_{i\cdot jb}^{\ a} \|$ (i, j: rows; a, b: columns) has a rank not greater than one. But it can not happen that the rank be equal to zero; when it does, from (7.3) the tensor M_{ijkl} vanishes contrarily to the supposition of the type $\tau\ (\geqq 3)$, so that the rank of $\| K_{i\cdot jb}^{\ a} \|$ is equal to one. Consequently there are two systems of functions H_{ij} and H_j^i satisfying (7.1). From (VII) H_{ij} is symmetric and from (7.2) the rank of the matrix $\| H_{ij} \|$ is $\sigma\ (\geqq \tau)$. Finally, from (VI) and (7.1) those H_{ij} and H_j^i satisfy (5.5). Therefore we get the

THEOREM III: *If an* $n\ (\geqq 3)$*-dimensional space with symmetric projective connection is of type* $\tau\ (\geqq 3)$, *there exist two systems of real functions* $H_{ij}\ (= H_{ji})$ *and* $H_j^i\ (i, j = 1, \cdots, n)$ *satisfying* (5.5) *if, and only if, the inequalities* (II) *and* (III), *and the equations* (I), (IV), (V), (VI), (VII) *and* (VIII) *are satisfied.*

If we take the functions $\overline{H}_{ij}$ and $\overline{H}^i_j$, instead of H_{ij} and H^i_j, satisfying (7.1), we have

$$H_{ij}H^a_b=\overline{H}_{ij}\overline{H}^a_b,$$

from which we obtain

$$\frac{\overline{H}_{ij}}{H_{ij}}=\frac{H^a_b}{\overline{H}^a_b} \qquad (a,b,i,j=1,\cdots,n),$$

so that we have

$$\overline{H}_{ij}=\kappa H_{ij}, \qquad \overline{H}^i_j=\frac{1}{\kappa}H^i_j, \tag{8.5}$$

hence the general solution of (5.5) is given by (8.5).

It is to be noted here that the equation (7.4) does not determine the algebraic sign of the solution K_{ij}, but the sign can be chosen by the condition (VI), because that of $K_h{}^k{}_{ij}$, the solution of (7.3) depends on that of K_{ij}.

9. The second Gauss equation.

At the end of the last section we take H_{ij} and H^i_j arbitrarily, satisfying (7.1), and hence it remains to find such a condition under which we can take H_{ij} satisfying (5.7). Be that as it may, in this section, we shall find a condition for the algebraic equation (5.6) having a solution H_i $(i=1,\cdots,n)$.

Contracting (5.6), i. e.

$$R_a{}^0{}_{jc}=H_{ac}H_j-H_{aj}H_c \tag{9.1}$$

by $H_{ik}H^a_b$ and making use of (7.1) and (7.2), we obtain

$$K_i{}^a{}_{kb}R_a{}^0{}_{jc}=K_{bc}H_{ik}H_j-K_{bj}H_{ik}H_c. \tag{9.2}$$

[1] Suppose the type $\tau=n$ $(\geqq 3)$. Contraction of (9.2) by K^{bc} gives

$$(n-1)H_{ik}H_j=K^{bc}K_i{}^a{}_{kb}R_a{}^0{}_{jc},$$

hence by substitution in (5.6) we have

$$(n-1)R_i{}^0{}_{jk}=K^{bc}(K_i{}^a{}_{kb}R_a{}^0{}_{jc}-K_i{}^a{}_{jb}R_a{}^0{}_{kc}), \tag{IX}$$

which is necessary for (5.6) to be satisfied. Conversely, if (IX) be satisfied, we have

$$(n-1)\, R_{i\,\cdot\,jk}^{\;0} = H_{ik}\,(K^{bc}\, H_b^a\, R_{a\,\cdot\,jc}^{\;0}) - H_{ij}\,(K^{bc}\, H_b^a\, R_{a\,\cdot\,kc}^{\;0}),$$

substituting (7.1) in (IX), and hence if we put

$$H_j = \frac{1}{n-1}\, K^{bc}\, H_b^a\, R_{a\,\cdot\,jc}^{\;0}, \tag{9.3}$$

we obtain (5.6).

[2] Suppose the type $\tau\,(n > \tau \geqq 3)$. Similarly we obtain

$$(\tau-1)\, R_{i\,\cdot\,jk}^{\;0} = K^{ab}\,(K_{i\,\cdot\,kb}^{\;a}\, R_{a\,\cdot\,jc}^{\;0} - K_{i\,\cdot\,jb}^{\;a}\, R_{a\,\cdot\,kc}^{\;0}) \tag{IX$_1$}$$

$$(b, c, j, k = 1, \cdots, \tau;\; a, i = 1, \cdots, n).$$

Conversely, if (IX$_1$) be satisfied, we put

$$H_j = \frac{1}{\tau-1}\, K^{bc}\, H_b^a\, R_{a\,\cdot\,jc}^{\;0} \qquad (b, c, j = 1, \cdots, \tau;\; a = 1, \cdots, n), \tag{9.4}$$

which satisfies the equation

$$R_{i\,\cdot\,jk}^{\;0} = H_{ik}\, H_j - H_{ij}\, H_k \quad (i = 1, \cdots, n;\; j, k = 1, \cdots, \tau).$$

Next, taking $j > \tau$ and $b, c = 1, \cdots, \tau$ in (9.2) we have

$$K_{cb}\, H_{ik}\, H_j = K_{i\,\cdot\,kb}^{\;a}\, R_{a\,\cdot\,jc}^{\;0},$$

and so, substituting this in (5.6), we have

$$K_{cb}\, R_{i\,\cdot\,jk}^{\;0} = K_{i\,\cdot\,kb}^{\;a}\, R_{a\,\cdot\,jc}^{\;0} - K_{i\,\cdot\,jb}^{\;a}\, R_{a\,\cdot\,kc}^{\;0} \tag{IX$_2$}$$

$$(j, k > \tau;\; b, c = 1, \cdots, \tau;\; a = 1, \cdots, n).$$

Conversely, if (IX$_2$) be satisfied, we have, contracting (IX$_2$) by K^{cb},

$$\tau R_{i\,\cdot\,jk}^{\;0} = H_{ik}\,(K^{cb}\, H_b^a\, R_{a\,\cdot\,jc}^{\;0}) - H_{ij}\,(K^{cb}\, H_b^a\, R_{a\,\cdot\,kc}^{\;0}).$$

Hence we put

$$H_j = \frac{1}{\tau}\, K^{cb}\, H_b^a\, R_{a\,\cdot\,jc}^{\;0} \qquad (j > \tau;\; b, c = 1, \cdots, \tau;\; a = 1, \cdots, n), \tag{9.5}$$

which satisfies the equation

$$R_{i\,\cdot\,jk}^{\;0} = H_{ik}\, H_j - H_{ij}\, H_k \qquad (j, k > \tau;\; i = 1, \cdots, n).$$

Finally, from (9.4) and (9.5) we have

$$R_{i\,\cdot\,jk}^{\;0} = H_{ik}\, H_j - H_{ij}\, H_k$$

$$= H_{ik}\left(\frac{1}{\tau-1}\, K^{bc}\, H_b^a\, R_{a\,\cdot\,jc}^{\;0}\right) - H_{ij}\left(\frac{1}{\tau}\, K^{bc}\, H_b^a\, R_{a\,\cdot\,kc}^{\;0}\right)$$

$$(k > \tau;\ b, c, j=1, \cdots, \tau;\ a, i=1, \cdots, n),$$

and hence we obtain by (7.1)

$$\text{(IX}_3\text{)} \qquad R_{i\cdot jk}^{\ 0} = K^{bc}\left(\frac{1}{\tau-1} K_{i\cdot kb}^{\ a} R_{a\cdot jc}^{\ 0} - \frac{1}{\tau} K_{i\cdot jb}^{\ a} R_{a\cdot kc}^{\ 0}\right)$$

$$(k > \tau;\ b, c, j=1, \cdots, \tau;\ a, i=1, \cdots, n).$$

Conversely, if (IX_3) be satisfied, from (9.4) and (9.5) we get

$$R_{i\cdot jk}^{\ 0} = H_{ik} H_j - H_{ij} H_k \qquad (j=1, \cdots, \tau;\ k > \tau;\ i=1, \cdots, n).$$

Thus, if (IX) or ($IX_{1,2,3}$) is satisfied, there exists a system of functions H_i $(i=1, \cdots, n)$ satisfying (5.6). Therefore we have the

THEOREM IV: *If an* $n(\geqq 3)$*-dimensional space with symmetric projective connection* P_n *is of type* $\tau(\geqq 3)$, *there exist three systems of real functions* $H_{ij}(=H_{ji})$, H_j^i *and* H_i $(i, j=1, \cdots, n)$ *satisfying* (5.5) *and* (5.6) *if, and only if, the inequalities* (II) *and* (III), *and the equations* (I), (IV), (V), (VI), (VII), (VIII) *and* (IX′) *are satisfied,* where we mean by (IX′) either (IX) or ($IX_{1,2,3}$).

We get, in accordance with the type number of τ, the condition that (5.6) may have a solution H_i $(i=1, \cdots, n)$. For our purpose is to find a concrete expression of H_i. But we see that the condition can be written in only one system of equations.

If P_n is of type $(n-1)$, of course, (IX_2) is unnecessary.

It is clear that the general solution of (5.5) and (5.6) is given by

$$\text{(9.6)} \qquad \overline{H}_{ij} = \kappa H_{ij}, \qquad \overline{H}_j^i = \frac{1}{\kappa} H_j^i, \qquad \overline{H}_i = \frac{1}{\kappa} H_i.$$

10. The first Codazzi equation.

Finally, let us find a system of functions H_{ij} satisfying (7.1) and (5.7). From (7.1) we get

$$\text{(10.1)} \qquad L_{ij} = H_{ij} \cdot H \qquad (H \equiv H_a^a),$$

where we put

$$\text{(10.2)} \qquad L_{ij} = K_{i\cdot ja}^{\ a}.$$

The known tensor L_{ij} is clearly symmetric by (VII). If the quantity H is equal to zero, the tensor L_{ij} vanishes. Conversely if L_{ij} does not vanish identically, the matrix $\| L_{ij} \|$ has the rank σ equal to one of the matrix $\| H_{ij} \|$. In this section we shall confine our considerations to such a domain in P_n that L_{ij} does not vanish.

Then, the rank of $\| L_{ij} \|$ is not less than that of $\| K_{ij} \|$, that is, the type number. From (10.1) we have $H \neq 0$ and put

$$H_{ij} = e^{\rho} L_{ij}, \tag{10.3}$$

where $H = e^{-\rho}$. Let us find such a function ρ that H_{ij} defined by (10.3) satisfies (5.7). Substituting this expression in (5.7) we obtain

$$\rho_k L_{ij} - \rho_j L_{ik} + L_{ijk} = 0; \tag{10.4}$$

where we put

$$L_{ijk} = L_{ij,k} - L_{ik,j}, \tag{10.5}$$

$$\rho_i = \frac{\partial \log \rho}{\partial x^i}. \tag{10.6}$$

Let us write (10.4) in the homogeneous form, namely

$$\rho_k L_{ij} - \rho_j L_{ik} + t L_{ijk} = 0. \tag{10.4'}$$

We represent by $R_n(L)$ the resultant system of (10.4′), which is a set of polynomials in the components L_{ij} and L_{ijk} such that the vanishing of these polynomials is necessary and sufficient for the existence of a non-trivial solution. Then it follows that

$$R_n(L) = 0 \tag{X}$$

is a necessary condition for P_n to be of class one.

Assume (X) be satisfied and let (ρ_i, t) be a non-trivial solution of (10.4′). Suppose $t=0$ in this solution. Then we have

$$\rho_k L_{ij} - \rho_j L_{ik} = 0, \tag{10.7}$$

from which we can prove that all of ρ_i are equal to zero contrary to the hypothesis of the non-trivial solution as follows.

[1] Suppose the determinant $| L_{ij} | \neq 0$. Let $\| L^{ij} \|$ be the inverse matrix of $\| L_{ij} \|$; contracting (10.7) by L^{ij} we have $(n-1)\rho_k = 0$, which shows that all of ρ_k $(k=1, \cdots, n)$ vanish.

[2] Suppose the matrix $\| L_{ij} \|$ has the rank σ $(n > \sigma \geqq \tau \geqq 3)$. Transform the coordinates x^i in P_n such that in the origin the matrix $\| L_{ij} \|$ has the form

$$\| L_{ij} \| = \left\| \begin{array}{c:c} L_\sigma & 0 \\ \hdashline 0 & 0 \end{array} \right\| , \qquad | L_\sigma | = \left| \begin{array}{ccc} L_{11} & \cdots & L_{1\sigma} \\ \vdots & & \vdots \\ L_{\sigma 1} & \cdots & L_{\sigma\sigma} \end{array} \right| \neq 0 ,$$

and let $\| L^{ij} \|$ $(i, j = 1, \cdots, \sigma)$ be the inverse matrix of $\| L_\sigma \|$. Taking $i, j, k = 1, \cdots, \sigma$ in (10.7) we obtain similarly $\rho_k = 0\,(k = 1, \cdots, \sigma)$. Next taking $k > \sigma$; $i, j = 1, \cdots, \sigma$ in (10.7) we get $\rho_k L_{ij} = 0$, from which it follows $\rho_k = 0\,(k > \sigma)$.

Therefore, our proposition is proved. Consequently we must have $t \neq 0$ so that the quantities ρ_i / t can be defined and these constitute a solution of (10.4). Further it is easily seen that the solution of (10.4) is uniquely determined. In fact, assume both ρ_i and $\bar{\rho}_i$ are solutions and put

$$\bar{\rho}_i = \rho_i + \gamma_i ,$$

and then by substitution in (10.4), we have

$$\gamma_k L_{ij} - \gamma_j L_{ik} = 0 ,$$

from which we obtain $\gamma_k = 0\,(k = 1, \cdots, n)$ in the same way as we have proved above $\rho_k = 0$ for $t = 0$.

Now we consider the differential equation (10.6), by which the quantity ρ will be determined, where in this equation ρ_i is the solution of (10.4) above found. The integrability condition of (10.6), that is,

$$\frac{\partial}{\partial x^j} \left(\frac{\partial \log \rho}{\partial x^i} \right) - \frac{\partial}{\partial x^i} \left(\frac{\partial \log \rho}{\partial x^j} \right) = 0$$

is written in a covariant form

$$\text{(XI)} \qquad \rho_{i,j} - \rho_{j,i} = 0 .$$

This equation, which is necessary for P_n to be of class one, is constituted from L_{ij} and its covariant derivatives of first and second orders. Conversely, if (XI) is satisfied and we give an initial value ρ_0 in the point (x_0^i), then we obtain a solution ρ of (10.6). Now, define a system of functions $H_{ij}\,(i, j = 1, \cdots, n)$ by (10.3); then it is clear from (10.4) that H_{ij} so defined satisfy (5.7).

Since all of L_{ij} are not equal to zero, say $L_{11} \neq 0$, we obtain $H_{11} \neq 0$ from (10.3). From (7.1) we get

$$(10.8) \qquad K_1{}^a{}_{1b} = H_{11} H_b^a \qquad (a, b = 1, \cdots, n) .$$

This gives $H^a_b\,(a, b=1, \cdots, n)$, and from (VIII) we see

$$\begin{vmatrix} K_i{}^{a}_{\cdot\, jb} & K_i{}^{k}_{\cdot\, jl} \\ K_1{}^{a}_{\cdot\, 1b} & K_1{}^{k}_{\cdot\, 1l} \end{vmatrix} = 0 .$$

Contraction with respect to k and l gives $L_{11} K_i{}^{a}_{\cdot\, jb} = L_{ij} K_1{}^{a}_{\cdot\, 1b}$ by means of (10.2) and substitution of (10.8) and (10.3) gives $e^{-\rho} H_{11} K_i{}^{a}_{\cdot\, jb} = e^{-\rho} H_{ij} H_{11} H^a_b$, and hence we have (7.1). Finally, making use of those H_{ij} and H^a_b, we obtain H_i by the method in § 4.

Consequently, just now, we attain to the following main theorem of this paper.

THEOREM V: *If an* $n\,(\geqq 3)$*-dimensional space with symmetric projective connection* P_n *is of type* $\tau\,(\geqq 3)$ *and the tensor* L_{ij} *does not vanish,* P_n *is of class one if, and only if, the inequalities* (II) *and* (III), *and the equations* (I), (IV), (V), (VI), (VII), (VIII), (IX′), (X) *and* (XI) *are satisfied.*

It is easy to write polynomials (V) and (X) concretely, but is not necessary for our discussions.

Though we excluded in our discussion such a space that L_{ij} is identically equal to zero, we easily see that this condition imposes no restriction in space with symmetric *normal* projective connection of type $\tau\,(\geqq 2)$. In fact, we have for the normal connection $R_i{}^{a}_{\cdot\, ja} = 0$ and from (VI)

$$R_i{}^{a}_{\cdot\, ja} = K_i{}^{a}_{\cdot\, aj} - K_i{}^{a}_{\cdot\, ja} = K_a{}^{a}_{\cdot\, ij} - K_i{}^{a}_{\cdot\, ja} ,$$

and hence, if $L_{ij} = K_i{}^{a}_{\cdot\, ja} = 0$, $K_a{}^{a}_{\cdot\, ij} = K_{ij} = 0$ in contradiction to the assumption concerning the type number. Consequently, a space with symmetric normal projective connection of type $\tau\,(\geqq 3)$ is of class one if, and only if, the conditions of Theorem V are satisfied.

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