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Some Remarks on the Theory of Picard Varieties

Jun-ichi IGUSA

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Let V^d be a non-singular projective model in the algebraic geometry with the universal domain of all complex numbers and let

$$(\gamma)=(\gamma_1, \gamma_2, \dots, \gamma_{2q})$$

be a base of the first integral Betti group of V, then we can find the "invariant cycles" on the generic curve[1] $C(M)$ in V

$$(\beta)=(\beta_1, \beta_2, \dots, \beta_{2q}),$$

which is homologous to (γ) modulo Q. Moreover let ω be the period matrix of the Picard integrals of the first kind in V along (γ) and let

$$E={}^tI_\beta^{-1}$$

be the transposed inverse of the intersection matrix of the invariant cycles on $C(M)$ then E is one of the principal matrices of the Riemann matrix ω.

We have also attached the *Albanese variety* A^q and the *Picard variety* P^q to the Riemann matrices ω and

$$\hat{\omega}=\omega\epsilon^{-1}E,$$

where ϵ means the Pfaffian of E. More precisely if we denote by $[\omega]$ the discrete subgroup of rank $2q$ in the complex vector space S^q, then A^q is isomorphic with the complex toroid $S^q/[\omega]$; and similary for P^q and $\hat{\omega}$.

On the other hand let

$$(\Gamma)=(\Gamma_1\ \Gamma_2, \dots, \Gamma_{2q})$$

be a base of the integral homology group of $(2d-1)$ dimension in V, which is dual to (γ) in the sense

$$I(\gamma_i, \Gamma_j; V)=\delta_{ij} \qquad (1\leq i, j\leq 2q).$$

Then if we put

$$\hat{\gamma}_i=\Gamma_i\cdot C(M) \qquad (1\leq i\leq 2q),$$

We shall use freely the results and terminology of Weil's book: Foundations of algebraic geometry, Am. Math. Soc. Colloq., Vol. 29 (1946).

1) See my paper, On the Picard varieties attached to algebraic varieties, Amer. J. of Math. Vol. 74 (1952). We cite this paper as (P).

we see readily the relation

$$(\hat{\gamma})\sim(\gamma)'E \qquad (\text{mod. } Q)\ ;$$

which I have missed in my previous papers.[2]

Now let $\boldsymbol{Y}$ be a $\boldsymbol{V}$-divisor, which is continuously equivalent to 0, then $\boldsymbol{Y}$ induces a character $\chi_{\boldsymbol{Y}}(\gamma)$ of the first integral Betti group of $\boldsymbol{V}$. We shall denote by $\hat{\varphi}(\boldsymbol{Y})$ the point of $\boldsymbol{P}$ with "coordinates" (v) modulo $[\hat{\omega}]$ such that

$$'(v)=\frac{1}{2\pi i}(\log\chi_{\boldsymbol{Y}}(\gamma_1),\dots,\ \log\chi_{\boldsymbol{Y}}(\gamma_{2q}))'\hat{\omega}.$$

As in (P) we shall denote by $\mathfrak{G}_c(\boldsymbol{V})$ and $\mathfrak{G}_l(\boldsymbol{V})$ the groups of $\boldsymbol{V}$-divisors which are defined respectively by continuous and linear equivalences. Then we have the following *supplement* to our theorem 5 in (P).

(P1) *The mapping $\hat{\varphi}$ induces an isomorphism between the factor group $\mathfrak{G}_c(\boldsymbol{V})/\mathfrak{G}_l(\boldsymbol{V})$ and the Picard variety $\boldsymbol{P}$.*

Moreover φ is "continuous" in the following sense.

(P2) *There exists a common field of definition K of $\boldsymbol{V}$ and $\boldsymbol{P}$ such that if a $\boldsymbol{V}$-divisor $\boldsymbol{Y}$ in $\mathfrak{G}_c(\boldsymbol{V})$ has the specialization $\overline{\boldsymbol{Y}}$ over a field L containing K, the image $\hat{\varphi}(\overline{\boldsymbol{Y}})$ is the uniquely determined specialization of $\hat{\varphi}(\boldsymbol{Y})$ over that specialization.*

In fact $\boldsymbol{A}$ is the parameter variety of the *Poincaré family* $\boldsymbol{X}$ with the property that if z-z' has the coordinates (v) modulo $[\omega]$, the point $\hat{\varphi}(\boldsymbol{X}(z)$-$\boldsymbol{X}(z'))$[3] has the same coordinates (v) modulo $[\hat{\omega}]$. Therefore if z' is the neutral element in $\boldsymbol{A}$ with coordinates in $[\omega]$, the mapping

$$\lambda(z)=\hat{\varphi}(\boldsymbol{X}(z)\text{-}\boldsymbol{X}(z'))$$

gives an analytic and hence[4] algebraic homomorphism from $\boldsymbol{A}$ onto $\boldsymbol{P}$. Let K be a common field of definition of $\boldsymbol{X}$ and λ, hence of $\boldsymbol{V}$, $\boldsymbol{A}$, $\boldsymbol{P}$,

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2) Cf. (P) and also my paper, Algebraic correspondences between algebraic varieties, Jap. J. of Math., Vol. 3 (1951).

3) The $\boldsymbol{V}$-divisor $\boldsymbol{X}(z)$ was defined in the following way. Let k be a field of definition of the continuous family $\boldsymbol{X}$ and let w be a generic point of $\boldsymbol{A}$ over k, then $\boldsymbol{X}(z)$ is one of the specializations of

$$\boldsymbol{X}(w)=pr_{\boldsymbol{V}}[\boldsymbol{X}\cdot(w\times\boldsymbol{V})]$$

over the specialization $w\to z$ with reference to k.

4) Cf. W. L. Chow, On compact complex analytic varieties, Amer. J. of Math., Vol. 71 (1949), theorem 7.

and of the composition functions in $\boldsymbol{A}$ and $\boldsymbol{P}$ over which $\boldsymbol{X}(z')$ is rational.

Now let $\boldsymbol{X}(z)$-$\boldsymbol{X}(z')$ be a $\boldsymbol{V}$-divisor which is linearly equivalent to $\boldsymbol{Y}$ and let π be any specialization of the point

$$\hat{\varphi}(\boldsymbol{Y})=\lambda(z)$$

over $\boldsymbol{Y}\rightarrow\overline{\boldsymbol{Y}}$ with reference to L. Let $\boldsymbol{X}(\bar{z})$ be any specialization of $\boldsymbol{X}(z)$ over this specialization, then since the linear equivalence is preserved by specializations,[5] we see that $\boldsymbol{X}(\bar{z})$-$\boldsymbol{X}(z')$ is linearly equivalent to $\overline{\boldsymbol{Y}}$. Moreover since the homomorphism λ is everywhere defined, the value of λ at $\bar{z}$ is the uniquely determined specialization of $\lambda(z)$ over $z\rightarrow\bar{z}$ with reference to L. Therefore we must have

$$\hat{\varphi}(\overline{\boldsymbol{Y}})=\lambda(\bar{z})=\pi,$$

which proves our assertion.

It follows from the above complement that *if $\boldsymbol{Y}$ is rational over L, the image $\hat{\varphi}(\boldsymbol{Y})$ is a rational point of $\boldsymbol{P}$ with reference to L.* Moreover we see readily that *the Picard variety $\boldsymbol{P}$ is uniquely determined by* (P1) *and* (P2).[6] Therefore we can take them together as a convenient definition of the Picard variety. In the same way Albanese variety $\boldsymbol{A}$ is uniquely determined by the following two postulates.[7]

(A1) *There exists a function φ from $\boldsymbol{V}$ into $\boldsymbol{A}$ whose image is a set of generators of $\boldsymbol{A}$.*

Moreover φ is "universal" in the following sense.

(A2) *Let f be any function from $\boldsymbol{V}$ into an Abelian variety $\boldsymbol{B}$, then f can be put in the form*

$$f=\mu\circ\varphi+\text{const.}$$

with some homomorphism μ from $\boldsymbol{A}$ into $\boldsymbol{B}$.

Now let $\boldsymbol{V}_1^{d_1}$ and $\boldsymbol{V}_2^{d_2}$ be two non-singular projective models; we shall attach 1 and 2 to (γ), (Γ) etc., which we have defined for $\boldsymbol{V}$. Let $\boldsymbol{X}$

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5) This fact can be reduced easily to the case of curves. In this case the assertion is proved in Weil's book: Variétés Abéliennes et courbes algébriques, Act. Sc. et Ind. n° 1064 (1948), lemme 10.

6) Indeed this idea had been applied in my first (unpublished) proof of the birational invariance of the Picard variety.

7) I have borrowed this formulation (with a slight modification) from a letter of A. Weil to W. L. Chow in February 12, 1951. We note also that the functions $\hat{\varphi}$ and φ are essentially unique, which correspond to the "canonical function" (Cf. loc. cit. 5) in the case of curves.

be a $(V_1\times V_2)$-divisor, then it can be written uniquely in the form

$$X\sim\Gamma_1\times V_2+V_1\times\Gamma_2+\sum_{i,j}s_{ij}(\Gamma_{1i}\times\Gamma_{2j})$$

over integers and the mapping

$$X\longrightarrow S=(s_{ij})$$

induces an isomorphism between the module of correspondences $C(V_1, V_2)$ and the module of integral matrices S satisfying the linear relation

$$\hat{\omega}_1 S{}^t\hat{\omega}_2=0.$$

Therefore ${}^tS$ and S are the complex multiplicalions from ω_1 to $\hat{\omega}_2$ and from ω_2 to $\hat{\omega}_1$.[8] Since the converse is also true, *we have the natural isomorphisms between the module* $C(V_1, V_2)$ *and the modules of complex multiplications from* ω_1 *to* $\hat{\omega}_2$ *and from* ω_2 *to* $\hat{\omega}_1$. Although this is a matter of interpretation in the case of curves, where the Albanese variety is isomorphic with the Picard variety, it is of some importance in the general theory of Picard varieties.

The above isomorphisms have the following geometrical meaning. Let K be a field containing K_1 and K_2 over which X is rational and let M_0 and M be two independent generic points of V_1 over K, then if we put $L=K(M_0)$, the V_2-divisor $X(M)$-$X(M_0)$ is rational over $L(M)$. Therefore the point $\hat{\varphi}_2(X(M)\text{-}X(M_0))$ in P_2 is rational over $L(M)$, hence there exists a function f from V_1 into P_2, which is defined over L, such that

$$f(M)=\hat{\varphi}_2(X(M)\text{-}X(M_0)).$$

As we have remarked before f can be written in the form

$$f=\mu\circ\varphi_1+\text{const}.$$

with some homomorphism μ from A_1 into P_2, which is also defined over L. It can be readily verified that μ has the representation ${}^tS$ in the coordinate systems with respect to ω_1 and $\hat{\omega}_2$. In the same way X determines a homomorphism μ' from A_2 into P_1 with the representation S in the coordinate systems with respect to ω_2 and $\hat{\omega}_1$. Therefore

we can identify the three modules

$$C(V_1, V_2),\ H(A_1, P_2)\ \textit{and}\ H(A_2, P_1)$$

by these natural isomorphisms.

Kyôto University.

8) An integral matrix M is called a complex multiplication from ω_1 to $\hat{\omega}_2$ if there exists a complex matrix μ such that $\mu\omega_1=\hat{\omega}_2 M$.