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On the Theory of Radicals in a Ring

Masayoshi NAGATA

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Introduction. In generalizing the classical notion of the radical in a ring, different kinds of radicals have been defined by many authors, including Azumaya [1],(0) Baer [2], Brown-McCoy [4], Jacobson [6], Köthe [9], Levitzki [10] and McCoy [12]. The main purpose of the present paper is to give a unified treatment to these theories, though we do not claim that we cover all of them, to introduce some new kinds of radicals, and further to study the properties of these radicals, both new and old. Our main concern is about p-radical, J-radical, e-radical and M-radical, each being a special case of C-radical; here p-radical and J-radical are the radicals in the sense of McCoy [12] and Jacobson [6] respectively.

In § 1, we first introduce a general concept of C-radical, and the notation $(1,R)$ which is a typical over-ring of a ring R containing identity. Further, we prove a fundamental characterization of prime ideals, and we define, by the way, the concept of irreducible ideals. In § 2, we introduce the notions of m-systems and m-families. A simmilar concept of m-systems has been already defined by McCoy [12], and ours is its modification (cf. foot-note 1); it plays, combined with the concept of prime ideals, an important rôle in this paper. In § 3, we treat semi-prime ideals, and define the concept of p-radical. Further, a fundamental characterization of semi-primeness (Proposition 8) is proved and we see that a radical ideal in the sense of Baer [2] is a semi-prime nil-ideal (and conversely) and therefore his lower radical coincides with our p-radical. § 4, is mostly devoted to J-radical, considered as a special case of C-radical, while in § 5, we introduce a concept of e-radical, and in § 6, we study some properties of J-radical and e-radical concerning idempotent elements. In § 7, we define M-radical and M-quasi-radical; the latter does not coincide with the classical notion of radical even when minimum condition is assumed for right ideals; for this reason we use the term quasi-radical. In § 8, we study the correspondences between radicals of R and those of $(1,R)$. In §9, we observe another typical over-ring $[1,R]$ of a ring $[R]$ which contains the identity. Further, we study

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(0) The numbers in brackets refer to the bibliography at the end.

in § 10, the correspondences between radicals of R and those of a ring of matrices over R, under certain conditions. In § 11. we observe rings satisfying maximum condition for semi-prime ideals, and we see that any semi-prime ideal in such a ring is an intersection of a finite number of prime ideals (this and other results in this section are evidently true for rings satisfying the minimum condition for semi-prime ideals). Finally, in Appendix some other radicals, namely, radicals in the sense of Baer [2], Fitting [5], Köthe [9] and Levitzki [10] are reviewed.

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1. PRELIMINARIES

C-radicals. *Definition* 0. Let C be a condition for rings; a ring satisfying C is called a C-ring, an ideal $\mathfrak{p}$ of a ring R such that $R/\mathfrak{p}$ is a C-ring is called a C-ideal, a ring which is isomorphic to a subdirect sum of C-rings is called a semi-C-ring, an ideal which is an intersection of C-ideals is called a semi-C-ideal, and the intersection of all C-ideals of a ring R is called the C-radical of R. Then evidently.

Proposition 0. 1) An ideal $\mathfrak{a}$ in a ring R is a semi-C-ideal if and only if $R/\mathfrak{a}$ is a semi-C-ring and 2) the C-radical of R is the smallest semi-C-ideal in R.

$(1,R)$. It is well known that we can define a ring $(1,R)$ for any ring R such that $(1,R)$ is the set of pairs (n,a) with rational integers n and elements a of R, and $(m,a)+(n,b)=(m+n,a+b)$, $(m,a)(n,b)=(mn,na+mb+ab)$. Further, every right (left) ideal in R is also a right (left) ideal in $(1,R)$. (Throughout this paper, $(1,R)$ has this meaning.)

Prime ideals. *Definition* 1. An ideal $\mathfrak{p}$ in a ring R is called prime if the following condition is satisfied: If $\mathfrak{a}$ and $\mathfrak{b}$ are ideals in R such as $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$ then $\mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{b} \subseteq \mathfrak{p}$. An ideal in R is called semi-simple if it is an intersection of prime ideals in R.

LEMMA 1. Let $\mathfrak{p}$ be a prime ideal in a ring R. Then $a \in \mathfrak{p}$ if Ra (or aR) $\subseteq \mathfrak{p}$, a being an element of R. (Cf. [12], § 2, Lemma 1.)

Proof. From the assumption that Ra (or aR) $\subseteq \mathfrak{p}$ we have $R(a)$ (or $(a)R$) $\subseteq \mathfrak{p}$, where (a) is the ideal generated by a. Therefore $a \in \mathfrak{p}$.

COROLLARY. Let $\mathfrak{a}$ be a semi-prime ideal in a ring R. Then $a \in \mathfrak{a}$ if Ra (or aR) $\subseteq \mathfrak{a}$.

Proposition 1. A necessary and sufficient condition for an ideal $\mathfrak{p}$ to be prime is that if a and b are two elements of R such as $R^{i_1}aR^{i_2}bR^{i_3} \subseteq$

$\mathfrak{p}$ with some (whence, any) combinations of positive integer i_2, and non-negative integers i_1 and i_3, then a or $b \in \mathfrak{p}$. (Cf. [12], Theorem 1.)

Proof. The necessity is evident in virtue of Lemma 1. Assume now that $\mathfrak{p}$ is not prime. Then we can find two ideals $\mathfrak{a}$ and $\mathfrak{b}$ such as $\mathfrak{p} \not\supseteq \mathfrak{a}$. $\mathfrak{p} \not\supseteq \mathfrak{b}$ and $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$. Let a and b be elements of $\mathfrak{a}$ and $\mathfrak{b}$ respectively which are not contained in $\mathfrak{p}$. Then $aRb \subseteq \mathfrak{p}$ therefore also $R^{i_1}aR^{i_2}bR^{i_3} \subseteq \mathfrak{p}$ if $i_2 > 0$.

Irreducible ideals. *Definition* 2. We say that an ideal $\mathfrak{p}$ in a ring R is irreducible (as an ideal) if $\mathfrak{p}$ satisfies the following condition: If $\mathfrak{p} = \mathfrak{a} \cap \mathfrak{b}$, then $\mathfrak{p} = \mathfrak{a}$ or $\mathfrak{p} = \mathfrak{b}$ for any two ideals $\mathfrak{a}$ and $\mathfrak{b}$ in R.

REMARK. It is clear that a prime ideal is irreducible. Further, if a prime ideal $\mathfrak{p}$ is an intersection of a right ideal $\mathfrak{r}$ and a left ideal $\mathfrak{l}$, $\mathfrak{p}$ is $\mathfrak{r}$ or $\mathfrak{l}$.

2. *m*-systems AND *m*-FAMILIES.

Definition 3. A subset M of a ring R is called an *m*-system if the following condition is satisfied: If $a, b \in M$, there exist x, y and z in R, or equivalently, in $(1,R)$ such as $xaybz \in M$[1].

Definition 4. A subfamily $\mathfrak{M}$ of ideals in a ring R is called an *m*-family if the following condition is satisfied: If $\mathfrak{a}, \mathfrak{b} \in \mathfrak{M}$, there exists an ideal $\mathfrak{c}$ of $\mathfrak{M}$ such as $\mathfrak{c} \subseteq \mathfrak{a}\mathfrak{b}$.

It is clear that the complementary set of a prime ideal $\mathfrak{p}$ is an *m*-system (which does not meet $\mathfrak{p}$), in virtue of Proposition 1, and that the family of ideals which are not contained in a prime ideal $\mathfrak{p}$ is an *m*-family (which has no ideal contained in $\mathfrak{p}$). Conversely,

Proposition 2. Let M be an *m*-system in a ring R which does not meet an ideal $\mathfrak{a}$. Then there exists an ideal $\mathfrak{p}$ such as 1) $\mathfrak{a} \subseteq \mathfrak{p}$, $\mathfrak{p} \cap M = \theta$[2] and 2) every ideal properly containing $\mathfrak{p}$ meets M; this $\mathfrak{p}$ is necessarily prime. (Cf. [12], § 2, Lemma 4.) And

Proposition 3. Let $\mathfrak{M}$ be an *m*-family in a ring R. Then an ideal $\mathfrak{p}$ is prime, if $\mathfrak{p}$ satisfies the following conditions: 1) No ideal of $\mathfrak{M}$ is contained in $\mathfrak{p}$ and 2) every ideal properly containing $\mathfrak{p}$ contains some ideals of $\mathfrak{M}$.

Proofs are immediate.

Definition 5. Such $\mathfrak{p}$ described in Proposition 2 (3) is called a maximal ideal with respect to *m*-system M (*m*-family $\mathfrak{M}$).

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(1) This is a generalization of m-system introduced by McCoy [12]. This is somewhat more covenient than his, as it seems to the writer; cf. for instance Proposition 26, § 10.

(2) θ denotes the empty set.

Proposition 4. Let $\mathfrak{M}$ be an m-family in a ring R. If every ideal of $\mathfrak{M}$ has a finite basis and if there exists an ideal $\mathfrak{a}$ which contains no ideal of $\mathfrak{M}$, there exists a maximal ideal with respect to $\mathfrak{M}$ which contains $\mathfrak{a}$.

Proof. This can be proved by Zorn's Lemma.

Proposition 5. Let S be a subring of a ring R, and $\mathfrak{p}_1$ a prime ideal in S. Then there exists a prime ideal $\mathfrak{p}$ in R such as $\mathfrak{p} \cap S = \mathfrak{p}_1$ if (and only if) there exists an ideal $\mathfrak{a}$ in R such as $\mathfrak{a} \cap S = \mathfrak{p}_1$. The last condition is satisfied if S is an ideal in R.

Proof. Let M be the complementary set of $\mathfrak{p}_1$ with respect to S. Then we can find such $\mathfrak{p}$ by Proposition 2, because M is an m-system in S, whence in R. As for the second assertion, let $\mathfrak{a}$ be the ideal generated by $\mathfrak{p}_1$ in R. Then $S\mathfrak{a}S \subseteq \mathfrak{p}_1$. Therefore $\mathfrak{a} \cap S = \mathfrak{p}_1$ since $\mathfrak{p}_1$ is a prime ideal in S.

REMARK 1. When S is an ideal in R and $\mathfrak{p}_1 \neq S$, no ideal properly containing $\mathfrak{p}$ has the property in our assertion.

REMARK 2. It follows that every semi-prime ideal in S is an ideal in R if S is an ideal in R. (Cf. Proposition 9a, below.)

On the other hand, it is known that

Proposition 6. If S is an ideal in R and $\mathfrak{p}$ is a prime ideal in R, then $\mathfrak{p} \cap S$ is a prime ideal in S (considered as a ring). [12, § 2, Lemma 2]

3. SEMI-PRIME IDEALS

Definition 6. A ring R is called prime if the zero ideal in R is prime. A ring R is called semi-prime if R is isomorphic to a subdirect sum of prime rings.

Definition 7. The p-radical of an ideal $\mathfrak{a}$ in a ring R is the intersection of all prime ideals containing $\mathfrak{a}$ which will be denoted by $\bar{\mathfrak{a}}$ throughout this paper. The p-radical of R is the p-radical of the zero ideal.

REMARK 1. It is easy to see that if a prime ideal $\mathfrak{p}$ contains an ideal $\mathfrak{a}$, there exists a minimal prime over-ideal of $\mathfrak{a}$ contained in $\mathfrak{p}$. Therefore the p-radical $\bar{\mathfrak{a}}$ of an ideal $\mathfrak{a}$ is the intersection of all minimal prime over-ideal of $\mathfrak{a}$.

REMARK 2. p-radical of an ideal $\mathfrak{a}$ is the radical of $\mathfrak{a}$ in the sense of McCoy [12].

REMARK 3. An ideal $\mathfrak{a}$ is semi-prime if and only if $\bar{\mathfrak{a}} = \mathfrak{a}$.

REMARK 4. Let $\mathfrak{a}$ be an ideal in a ring R. If $\mathfrak{a}$ is contained in the p-radical of R and if there exists an m-family $\mathfrak{M}$ consisting merely of powers of $\mathfrak{a}$ such that every ideal of $\mathfrak{M}$ has a finite basis, then $\mathfrak{a}$ is nilpotent. (This follows from Proposition 4.)

REMARK 5. Remark 4 shows also the fact that the p-radical in a ring which satisfies the maximum condition for ideals is nilpotent.

REMARK 6. The corollary to Lemma 1 (§ 1) shows also the fact that a minimal ideal in a semi-prime ring is a simple ring.

Proposition 7. If $\mathfrak{a}$ is an ideal in a ring R, then $\bar{\mathfrak{a}}$ is the set of elements r in R such as any m-system meeting r meets $\mathfrak{a}$. (Cf. [12], § 3, Definition 3.)

Proof is evident by virtue of Proposition 2.

Proposition 8. A ring R is semi-prime if (and only if) R contains no non-zero nilpotent ideal.

Proof. Assume that R is not semi-prime. Let $x_1 \neq 0$ be an element of the p-radical of R. Begining with x_1, we construct a sequence as follows: If $x_1, \cdots, x_n$ are already defined, let x_{n+1} be a non-zero element of the form $y_1 x_n y_2 x_n y_3 (y_i \in (1,R))$ if $(1,R)x_n(1,R)x_n(1,R) \neq 0$. Since x_1 is an element of the p-radical of R, this sequence must be finite. Let x_m be the last term. Then the square of the ideal generated by x_m is (0).

COROLLARY. An ideal $\mathfrak{a}$ is a radical ideal in the sense of Baer [2] if and only if $\mathfrak{a}$ is a semi-prime nil-ideal. His lower radical is the p-radical; for Baer's radical ideal cf. Appendix.

Proposition 9a. An ideal $\mathfrak{a}$ in a ring R, R being an ideal in a ring R', is semi-prime if and only if $\mathfrak{a}$ is an intersection of R and a semi-prime ideal in R'. Therefore the p-radical of R is the intersection of R and the p-radical of R'.

Proof. This follows from Propositions 5 and 6.

We prove, by way, a characterization of prime ideals:

Proposition 10. A semi-prime ideal $\mathfrak{p}$ is prime if and only if $\mathfrak{p}$ is irreducible, what amounts to the same that if an ideal $\mathfrak{a}$ is not only irreducible but also non-prime, then $\bar{\mathfrak{a}} \neq \mathfrak{a}$.

proof. We prove the last statement. Since $\mathfrak{a}$ is not prime, there exist two ideals $\mathfrak{b}$ and $\mathfrak{c}$ in the same ring such as $\mathfrak{a} \subset \mathfrak{b}$, $\mathfrak{a} \subset \mathfrak{c}$ and $\mathfrak{a} \supseteq \mathfrak{b}\mathfrak{c}$. Then $\mathfrak{b} \cap \mathfrak{c} \supset \mathfrak{a}$, for $\mathfrak{a}$ is irreducible. Therefore $\bar{\mathfrak{a}} \neq \mathfrak{a}$ because $(\mathfrak{b} \cap \mathfrak{c})^2 \subseteq \mathfrak{a}$.

4. *J*-RADICAL.

A ring which has a faithful, irreducible right module is called primitive[3].

(3) The term "primitive" is after Jacobson [6]. Primitive rings were called (right) irreducible in Nakayama [14] and Nakayama-Azumaya [15].

Definition 8. An ideal $\mathfrak{p}$ in a ring R is called primitive[3] if $R/\mathfrak{p}$ is primitive. An ideal $\mathfrak{a}$ in a ring R is called semi-primitive if $\mathfrak{a}$ is an intersection of primitive ideals. A ring R is called semi-primitive[4] if R is isomorphic to a subdirect sum of primitive rings.

REMARK. It is well known that a primitive ideal is a prime ideal.

Definition 9. The J-radical of a ring R is the intersection of all primitive ideals in R.

REMARK. The J-radical of a ring R is the radical of R in the sense of Jacobson [6]. It is well known that the J-radical of a ring R contains no idempotent element ; we will say an element e is idempotent if $e^2=e$ and if $e \neq 0$. Further, if we define the quasi-regularity[5] as follows, the J-radical of R is not only the largest quasi-regular ideal, but also the largest right-quasi-regular right ideal : An element a of R is called right-quasi-regular if there exist an element x, which is called a right-quasi-inverse, such that $a+x=ax$, left-quasi-regularity is defined by a same way, and if a is both right- and left-quasi-regular we say that a is quasi-regular[6] ; a (right) ideal $\mathfrak{a}$ in R is called (right-) quasi-regular if every element of $\mathfrak{a}$ is (right-) quasi-regular.

It is evident that a semi-primitive ideal is a semi-prime ideal, that if $\mathfrak{a}$ is an ideal in a ring R, then $R/\mathfrak{a}$ is semi-primitive if and only if $\mathfrak{a}$ is semi-primitive and that the J-radical is the smallest semi-primitive ideal in the same ring.

Proposition 11. Let R be a semi-primitive ring. If $(0)=\mathfrak{p}\cap\mathfrak{a}$ with an irreducible ideal $\mathfrak{p}$ and a non-zero ideal $\mathfrak{a}$, then $\mathfrak{p}$ is semi-primitive. Further, $\mathfrak{p}$ is prime.

Proof. We consider R as a subdirect sum of $R/\mathfrak{p}$ and $R/\mathfrak{a}$: Let N be the subset of R the residue classes of whose elements are contained in the J-radical of $R/\mathfrak{p}$. Then N is an ideal in R. Then $N\cap\mathfrak{a}$ is quasi-regular, whence $N\cap\mathfrak{a}$ is (0). Therefore $N\cap(\mathfrak{a}+\mathfrak{p})=\mathfrak{p}$, whence $N=\mathfrak{p}$ because $\mathfrak{p}$ is irreducible. This proves our first assertion. The remainder follows from Proposition 10.

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(4) A semi-primitive ring was called a semi-irreducible ring in Nakayama [14]. It was called semi-simple in Jacobson [6] and others. Primitive and semi-primitive rings were first effectively treated by Chevalley and applied by Nakayama [13].

(5) Cf. Jacobson [5] and Perlis [16].

(6) If a is quasi-regular and if x is a right-quasi-inverse of a, then x is the unique both right- and left-quasi-inverse of a, which is called the quasi-inverse of a.

5. e-RADICAL.

Definition 10. A ring R is called e-primitive if every non-zero ideal in R contains a definite idempotent element, An ideal $\mathfrak{p}$ in a ring R is called e-primitive if $R/\mathfrak{p}$ is e-primitive. A ring R is called semi-e-primitive if R is a subdirect sum of e-primitive rings. An ideal in a ring is called semi-e-primitive if it is an intersection of e-primitive ideals.

Proposition 12. If an e-primitive ring (ideal) is a subdirect sum (an intersection) of e-primitive rings (ideals), it coincides with one of them. Therefore an e-primitive ring (ideal) is primitive.

Proof. The first assertion is evident. The remainder follows from the fact that the J-radical of a ring contains no idempotent element and from the first assertion.

COROLLARY. A semi-e-primitive ring is a semi-primitive ring.

It is evident that if $\mathfrak{a}$ is an ideal in a ring R, then $R/\mathfrak{a}$ is semi-e-primitive if and only if $\mathfrak{a}$ is semi-e-primitive.

Definition 11. The e-radical of a ring R is the intersection of all e-primitive ideals in R.

It follows from this definition that the e-radical of a ring R is the smallest semi-e-primitive ideal in R.

Definition 12. An element a of a ring R is called semi-idempotent if the ideal generated by a^2-a in R does not contain a. An ideal in R is called quasi-nilpotent if it contains no semi-idempotent element.

Proposition 13. An element a of a ring R is semi-idempotent (if and) only if there exists an e-primitive ideal $\mathfrak{p}$ such as $\mathfrak{p} \ni a^2-a$, $\mathfrak{p} \not\ni a$.

Proof. If a is semi-idempotent, the residue class of a modulo the ideal (a^2-a) is an idempotent element in the residue ring $R/(a^2-a)$. This shows the existence of such $\mathfrak{p}$.

It follows that the e-radical is a quasi-nilpotent ideal. Furthermore,

Proposition 14. The e-radical $\mathfrak{e}$ is the largest quasi-nilpotent ideal. [4, Corollary 3 to Theorem 1][7]

Proof. If an ideal $\mathfrak{a}$ is not contained in $\mathfrak{e}$, then there exists an e-primitive ideal $\mathfrak{p}$ such as $\mathfrak{a} \not\subseteq \mathfrak{p}$. Then $\mathfrak{a}+\mathfrak{p}/\mathfrak{p}$ contains an idempotent element e^* of $R/\mathfrak{p}$. Let a be an element of $\mathfrak{a}$ which is a representative of e^*. Then $a^2-a \in \mathfrak{p}$, $a \notin \mathfrak{p}$.

Proposition 15. Let R be a semi-e-primitive ring. If $(0)=\mathfrak{p} \cap \mathfrak{a}$ with

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(7) This is the F-radical in the sense of Brown-McCoy [4] if we define $F(a)=(a^2-a)$.

an irreducible ideal $\mathfrak{p}$ and a non-zero ideal $\mathfrak{a}$, then $\mathfrak{p}$ is semi-e-primitive. Further, $\mathfrak{p}$ is prime.

Proof. This can be proved by the same way as Proposition 11.

REMARK. In this case, if $\mathfrak{p}$ is e-primitive then $\mathfrak{a}$ contains at least one idempotent element.

We mention, by the way, the evident

Proposition 16. If every non-zero ideal in a ring R contains some idempotent elements, then R is semi-e-primitive.

REMARK. It is evident that any right- or left-quasi-regular element is also non-semi-idempotent. But the converse is not true. In fact, let R be a matrix ring of a definite dimension, say 2, over a field. Then $a=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}$ is neither right- nor left-quasi-regular. On the other hand, a is non-semi-idempotent, because $a^2-a\neq 0$ and R is simple. Further, there are primitive rings which are quasi-nilpotent. This follows from the example of simple ring having no minimal right ideal that was given by Jacobson [7], p. 237 and the following.

LEMMA 2. Let R be a simple ring, and S a subring of R which can be expressed in the form ARB, A and B being subsets of R. If $SaS\neq 0$ for every non-zero element a of S, then S is a simple ring.

6. REMARKS ON J-RADICALS AND e-RADICALS CONCERNING IDEMPOTENT ELEMENTS.

It is evident that if a ring R posesses the radical $\mathfrak{m}_0$ in the sense of Azumaya [1], the e-radical $\mathfrak{e}$ of R is contained in $\mathfrak{m}_0$. In general case we have followings:

Definition 13. An element a of a ring R is called a root element of R if aR contains no idempotent element (8). An ideal (right ideal, left ideal, submodule) in R consisting merely of root elements is called a root ideal (right ideal, left ideal, submodule). The set C of all root element of R is if Ra called the root set of R.

REMARK. An element a of a ring R is a root-element of R if and only contains no idempotent element (8).

LEMMA 3. The subset $\mathfrak{M}=\{a\,;\ a+C\subseteq C\}$ of a ring R is a root submodule of R, where C is the root set of R.

Proof. If $a,b\in\mathfrak{M}$, $-b+c\in C$ for any $c\in C$, therefore $a-b+c\in C$ for any $c\in C$. Hence $a-b\in\mathfrak{M}$.

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(8) Cf. Azumaya [1].

Proposition 17. Let R be a ring. Then there exists an ideal in R such that: 1) Its sum with a root right (left) ideal is again a root right (left) ideal and 2) it is the largest among ideals which satisfy the condition 1). This ideal contains the J-radical $\mathfrak{m}$ of R.

Proof. Let $\mathfrak{M}$ be the module obtained by Lemma 3, and set $\mathfrak{m}_1=\{a \in \mathfrak{M}; xay \in \mathfrak{M}$ for any $x, y \in R\}$. Then $\mathfrak{m}_1$ is the largest ideal contained in M; therefore $\mathfrak{m}_1$ is the required one. As for the last assertion, let a be an element of $\mathfrak{m}$, and e an idempotent element. Set $c=e-a$. Then $ece=e-eae$ and eae is contained in the J-radical of eRe. Therefore $(ece)eRe=eRe$. This shows that ece, and therefore also c is not a root element of R. Therefore $\mathfrak{m}\subseteq\mathfrak{m}_1$.

COROLLARY. Let $\mathfrak{m}$ be the J-radical of a ring R and $\mathfrak{r}$ a root right (left) ideal of of R. The $\mathfrak{m}+\mathfrak{r}$ is again a root right (left) ideal.

Proposition 18. Let R be a ring. Then there exists an ideal in R such that: 1) its sum with a root ideal is again a root ideal and 2) it is the largest among ideals which satisfy the condition 1). This ideal is semi-e-primitive. Therefore any sum of the e-radical with a root ideal is again a root ideal.

Proof. The intersection of all maximal root ideals is the required ideal. Therefore the else is evident.

7. M-RADICAL.

Definition 14. The M-radical of a ring R is the intersection of all prime ideals $\mathfrak{p}$ such that $R/\mathfrak{p}$ is simple. The M-quasi-radical of R is the intersection of all ideals $\mathfrak{a}$ such that $R/\mathfrak{a}$ is simple.

Definition 15. If R is a subdirect sum of simple rings, R is called semi-simple [9].

Proposition 19. Let M and M' be the M-radical and M-quasi-radical of a ring R respectively. The $\bar{M}'=M$.

Proof. Let $\mathfrak{b}$ be the intersection of all ideals $\mathfrak{a}$ such as $R/\mathfrak{a}$ is simple and $R^2\subseteq\mathfrak{a}$. Then $M'=M\cap\mathfrak{b}$. Since $\mathfrak{b}\supseteq R^2$, $\bar{\mathfrak{b}}=R$. Therefore $M=\bar{M}=\bar{M}\cap\bar{\mathfrak{b}}=\bar{M}'$.

COROLLARY. The M-radical of a ring R is (0) if and only if R is semi-simple and semi-prime.

REMARK 1. If $R^2=R$, then $M=M'$.

REMARK 2. If R is a commutative ring, then the J-radical$=M$.

(9) Contrary to the usual usage of the term.

We want to offer, by the way, a problem to construct a prime, simple ring which is not primitive.

A special case (in some sense) of this problem is ; " Is there a prime, simple, non-zero nil-ring ? " This last problem is equivalent to the third problem of Levitzki [11].

8. CORRESPONDENCES BETWEEN RADICALS OF A RING R AND THOSE OF $(1,R)$.

First, we observe a general case ; let a ring R be an ideal in a ring R'. Then we have

Proposition 9b. Let $\mathfrak{p}_1$ be a primitive (e-primitive) ideal in R. If $\mathfrak{p}_1 \neq R$, every prime ideal $\mathfrak{p}$ in R' such that $\mathfrak{p} \cap R = \mathfrak{p}_1$ is a primitive (e-primitive) ideal So, an ideal $\mathfrak{a}$ in R is a (semi-) primitive ((semi-) e-primitive) ideal in R if and only if $\mathfrak{a}$ is an intersection of a (semi-) primitive) ((semi-) e-primitive) ideal in R' and R ; any p-radical of a semi-primitive (semi-e-primitive) ideal in R as an ideal in R' is a semi-primitive (semi-e-primitive) ideal.

Proof. This follows immediately from Proposition 5 and Remark 1 to it.

COROLLARY. If R is a semi-e-primitive ideal in R', p-, J- and e-radicals of R' coincides with those of R respectively. (As for p-radical, cf. Proposition 9a).

Now, we apply this to the case $R' = (1,R)$.

If we observe the fact that $(1,R)/R$ is isomorphic to the ring of rational integers, we have

Proposition 20. R is a prime, semi-e-primitive ideal in $(1,R)$.

This being said, we have, in virtue of Proposition 9a,

Proposition 21. An ideal in a ring R is semi-prime if and only if it is semi-prime in $(1,R)$.

COROLLARY 1. The p-radical of R is the p-radical of $(1,R)$.

COROLLARY 2. R is semi-prime if and only if $(1,R)$ is so.

Proposition 22. The J-radical of a ring R is the J-radical of $(1,R)$.

Proposition 23. The e-radical of a ring R is the e-radical of $(1,R)$.

Proofs are trivial by virtue of Corollary to Proposision 9b.

REMARK. The same does not holds for M-radical. For example, if R is a simple ring without identity, the M-radical of $(1,R)$ is R itself.

9. AN OVER-RING $[1,R]$ OF A RING R.

Definition 16. An element a of a ring R is called an n-fier[10] if $ba=ab=$

(10) The term "n-fier" is after Brown-McCoy [3].

nb for every $b \in R$, where n is an integer.

Proposition 24. Let $\mathfrak{a}$ be an ideal in $(1,R)$. Then $\mathfrak{a} \cap R=(0)$ if and only if $\mathfrak{a}$ is consists merely of pairs (n,a) such that $-a$ is an n-fier in R.

LEMMA 4. The totality of integers n such that there exists an n-fier is an ideal in the ring I of integers; the non-negative generator of this ideal is called the mode of the ring R.[11]

Proofs are trivial.

Now, let m be the mode of the ring R, and a an m-fier in R. We denote by $[1,R]$[12] the residue ring of $(1,R)$ modulo the ideal generated by $(m, -a)$. $[1,R]$ is an over-ring of the ring R and a right (left) ideal in R is also a right (left) ideal in $[1,R]$. Further, evidently

LEMMA 5. An intersection of a non-zero ideal in $[1,R]$ and R is not (0).

Proposition 25. $[1,R]$ is a prime (semi-prime, primitive, semi-primitive, e-primitive, semi-e-primitive) ring if and only if R is so.

Proof. This is an immediate consequence of Lemma 5.

REMARK. As for the correspondences between radicals of R and $[1,R]$, cf. Propositions 9a and 9b.

10. A RING OF MATRICES OVER A RING R.

Let R^* be a ring of matrices of definite pimension over a ring R such as R^* contains every matrix, of that dimension, which has only a finite number of non-zero components. Let $\mathfrak{n}$ and $\mathfrak{n}^*$ be the p-radicals of R and R^* respectively.

Proposition 26. Let A^* be a subset of R^* such that every matrix of A^* posesses at least one component which is not contained in a prime ideal $\mathfrak{p}$ in R. Then there exists an m-system M^* in R^* such as 1) $M^* \supseteq A^*$ and 2) every matrix of M^* posesses at least one component which is not contained in $\mathfrak{p}$.

Proof. For every $a^* \in A^*$, we select one component $a_{\lambda\mu}$ such that $a_{\lambda\mu} \notin \mathfrak{p}$. Then we can find elements b and c in R such as $ba_{\lambda\mu}c \notin \mathfrak{p}$. Let b^* be a matrix whose $(1,\lambda)$-component is b and whose other components are 0; let c^* be a matrix whose $(\mu,1)$-component is c and whose other components are 0. Then $b^*a^*c^*$ is a matrix whose (1.1)-comonent is not contained in $\mathfrak{p}$ and whose other components are 0. Therefore the union

(11) Cf. Brown-McCoy [3].

(12) $[1,R]$ is uniquely determined if and only if R contains no non-zero element x such that $xR=Rx=(0)$; $[1,R]$ contains the identity if $R \neq (0)$.

$M_1^* \cup A^*$ is a required m-system, where M_1^* denotes the set of all matrices whose (1,1)-components are not contained in $\mathfrak{p}$ and whose other components are 0.

This being proved, we have evidently

Proposition 27. Any component of an arbitrary element of $\mathfrak{n}^*$ is contained in $\mathfrak{n}$.

Proposition 28. R^* is prime if and only if R is so.

Proof. If R is prime, then R^* is so, by virtue of Proposition 26. Conversely, if R is not prime, we can find two non-zero elements a and b such as $aRb=0$. Let a^* and b^* be matrices each of which has only one non-zero component a or b respectively. Then clearly $a^*R^*b^*=0$, whence R^* is not prime.

COROLLARY. R^* is semi-prime if and only if R is so.

REMARK. It is easy to see that if R^* is the set of all matrices (of a definite dimension) each of which has only a finite number of non-zero components, then the p-radical of R^* is the set of all matrices contained in R^* whose components are in $\mathfrak{n}$. But in general case, such is not the case.

Similar assertions hold for J- and e-radicals. Let $\mathfrak{m}$ and $\mathfrak{m}^*$ be the J-radicals of R and R^* respectively, and let $\mathfrak{e}$ and $\mathfrak{e}^*$ be the e-radicals of R and R^* respectively.

Proposition 29. If an element a^* of R^* has a component which is not contained in $\mathfrak{m}$, then $a^* \notin \mathfrak{m}^*$.

Proof. Assume that $a^*=(a_{\lambda\mu})$, $a_{\lambda_1\mu_1} \notin \mathfrak{m}$. Then there exist two elements b and c in R such as $ba_{\lambda_1\mu_1}c \notin \mathfrak{m}$ because $\mathfrak{m}$ is semi-prime, by Corollary to Lemma 1 (§ 1). Therefore the ideal (a^*) generated by a^* in R^* contains an element d^* whose (1,1)-component is not contained in $\mathfrak{m}$ and whose other components are 0. Therefore (a^*) contains a non-quasi-regular element.

Proposition 30. R^* is semi-primitive if and only if R is so[13].

Proof. It is evident by virtue of Proposition 29 that if R is semi-primitive, then R^* is so. Converse follows from the fact that $a^* \in \mathfrak{m}^*$ if every component of a^* is in $\mathfrak{m}$ and if a^* has only a finite number of non-zero components.

Proposition 31. If an element a^* of R^* has a component which is not contained in $\mathfrak{e}$, then $a^* \notin \mathfrak{e}^*$.

(13) It is evident by virtue of the structure theorem of primitive rings (cf. Jacobson [8], Nakayama [13], [14] and Nakayama-Azumaya [15]) that if R is primitive then R^* is primitive.

Proof. This can be proved in the same way as Proposition 29.

Proposition 32. R^* is e-primitive if R is so[14].

Proof. Trivial.

COROLLARY. R^* is semi-e-primitive if R is so.

11. RINGS SATISFYING THE MAXIMUM CONDITION FOR SEMI-PRIME IDEALS.

Proposition 33. Let R be a ring satisfying the maximum condition for semi-prime ideals. Then any semi-prime ideal in R can be represented as an intersection of a finite number of prime ideals.

Proof. Our assertion follows easily from the fact that $\mathfrak{a}=\mathfrak{b}\cap\mathfrak{c}$ implies $\bar{\mathfrak{a}}=\bar{\mathfrak{b}}\cap\bar{\mathfrak{c}}$ and Proposition 10.

It is evident that

Proposition 34. Let $\mathfrak{a}$ be a semi-prime ideal in a ring R. If $\mathfrak{a}=\mathfrak{p}_1\cap\cdots\cap\mathfrak{p}_m=\mathfrak{q}_1\cap\cdots\cap\mathfrak{q}_n$ are two irredundant representations as intersections of prime ideals $\mathfrak{p}_i$ and $\mathfrak{q}_j$, then we have $m=n$, and in a suitable numbering $\mathfrak{p}_i=\mathfrak{q}_i$.

COROLLARY. There exists only a finite number of minimal prime over-ideals for an ideal in a ring R satisfying the maximum condition for semi-prime ideals.

Proposition 35. Let $\mathfrak{a}$ be a semi-primitive (semi-e-primitive) ideal in a ring R satisfying the maximum condition for semi-prime ideals. If $\mathfrak{p}$ is a minimal prime over-ideal of $\mathfrak{a}$, then $\mathfrak{p}$ is semi-primitive (semi-e-primitive).

Proof. This follows from Proposition 11 (15).

APPENDIX

Baer's radicals. Baer [2] defined radicals as follws : A radical ideal $\mathfrak{a}$ in a ring R is an ideal such that 1) $\mathfrak{a}$ is a nil-ideal and 2) $R/\mathfrak{a}$ contains no non-zero nilpotent ideal ; the lower radical of R is the intersection of all radical ideals in R, and the upper radical of R is the union of all radical ideals. It was already noticed (corollary to Proposition 8) that an ideal is a radical ideal if and only if it is a semi-prime nil-ideal and that the lower radical is the p-radical.

The upper radical is the radical in the sense of Fitting [5] (cf. below),

14) We want to offer a problem whether or not there exists a simple ring R without idempotent element such that the matrix ring of dimension 2 over R contains at least one idempotent element. If there is not such a ring, we have that R^* is e-primitive (if and) only if R is so.

because this last is the largest radical ideal.

Fitting's radical Fitting [5] defined radical as follows: An element a of a ring R is called properly nilpotent if the ideal generated by a is a nil-ideal; the radical N_1 of R is the set of all properly nilpotent elements. Whence N_1 is the largest nil-ideal in R.

It is evident that N_1 is contained in the J-radical of R, that there exists such an example of R that $N_1 \neq J$-radical, and furthermore that the radical in the same sense of $(1,R)$ is N_1.

Köthe's radical Köthe [9] defined radical as follows: If N_1 (in the same notation as above) of a ring R contains every one-sided nil-ideal, then N_1 is called the radical of R. Whether there is a ring which has no radical in this sense is an unsolved problem.

Levitzki's radical Levitzki [10] defined radical as follows: An ideal or a left or right ideal, in a ring R is called semi-nilpotent if every subring generated by a finite number of its elements is nilpotent; the radical N_2 of R is the sum of all semi-nilpotent ideals (and this coincides with the sum of all one-sided semi-nilpotent ideals).

It is known that $N_1 \supseteq N_2 \supseteq p$-radical and that there exists an example of a ring in which $N_2 \neq p$-radical (cf. Baer [2] But the problem whether there is a ring in which $N_1 \neq N_2$ is an open question: This problem is equivalent to the problem whether there is a ring such that 1) R is a non-zero prime ring, 2) R is a nil-ring, 3) R has a finite number of generators and 4) any non-zero ideal contains some power of R.

Nagoya University.

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