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Theory of the Spherically Symmetric Space-Times, I
Characteristic System

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§ 1. Definition

A spherically symmetric space-time is a 4-dimensional Riemannian space whose fundamental form is reducible to

$$ds^2=-A(r,\ t)dr^2-B(r,\ t)(d\theta^2+\sin^2\theta d\phi^2)+C(r,\ t)dt^2 \tag{1·1}$$

where A, B and C are any positive valued functions of r and t. Historically (1·1) was obtained by generalizing the metric of the Minkowski space-time. Eiesland defined this space-time from the standpoint of the group of motions using the group of ordinary 3-dimensional rotations.(1) In this paper, (1) we shall give a new definition of the s. s. (spherically symmetric) space-time S_0 using some tensor equations to be satified by g_{ij}. (2) At the same time we shall define a set of vectors and scalars characterizing this space-time. (3) Then we shall show that this new definition coincides with Eiesland's one. (4) Finally we shall obtain some properties of the s. s. space-time.

Definition : *Spherically symmetric space-time* is a 4-dimensional Riemannian space with the following properties :

(I) Its curvature tensor satisfies the equation

$$K_{ijlm}=-\overset{1}{\rho}\alpha_{[i}\alpha_{[l}\beta_{j]}\beta_{m]}-\overset{2}{\rho}g_{[i[l}\alpha_{j]}\alpha_{m]}+\overset{3}{\rho}g_{[i[l}\beta_{j]}\beta_{m]}+\overset{4}{\rho}g_{[i[l}g_{j]m]} \tag{F_1}$$

where α_i and β_i are mutually orthogonal unit vectors (real or complex) satisfying

$$\nabla_i\alpha_j=\sigma\alpha_i\beta_j+\varkappa(g_{ij}+\alpha_i\alpha_j-\beta_i\beta_j)+\bar{\sigma}\beta_i\beta_j \tag{F_2}$$

$$\nabla_i\beta_j=\bar{\sigma}\beta_i\alpha_j+\bar{\varkappa}(\qquad ,, \qquad)+\sigma\alpha_i\alpha_j \tag{F_3}$$

$$\alpha_s\alpha^s=-1,\ \beta_s\beta^s=1,\ \alpha_s\beta^s=0 \tag{1·2}$$

and $\overset{a}{\rho}$, $(a=1, \cdots, 4)$; σ, $\bar{\sigma}$; $\varkappa$, $\bar{\varkappa}$ are scalars (real or complex) determined from these equations.

(II) One of five scalars $\overset{a}{\rho}$ and $K\equiv K_{ij}{}^{ji}$ is such that its gradient vector is a linear combination of α_i and β_i.

(III) $$\overset{4}{\rho}-2(\varkappa^2-\bar{\varkappa}^2)\neq 0. \qquad (F_4)$$

(IV) The signature of the fundamental form is give by the type $(---+)$. (g_{ij} is always real).

(α_i, β_i) and ($\overset{a}{\rho}$, σ, $\bar{\sigma}$, $\varkappa$, $\bar{\varkappa}$) are called *characteristic vectors* and *scalars* of the s. s. space-time S_0 and (α_i, $\cdots$, $\bar{\varkappa}$) is generically called *characteristic system*. We shall also call a scalar whose gradient vector is a linear combination of α_i and β_i, *s. s. scalar*. Later in § 3 we shall see that in the standard coordinate system for the c. s. (characteristic system) this definition is equivalent to the ordinary one.

§ 2. Identities concerning c. s.

In this section we shall give some main identities obtained from the definition. From (F_1), (F_2) and (F_3), we have

$$\begin{cases}\sigma=-\alpha^i\beta^j\nabla_i\alpha_j=\alpha^i\alpha^j\nabla_i\beta_j, & \bar{\sigma}=\beta^i\beta^j\nabla_i\alpha^j=-\beta^i\alpha^j\nabla_i\beta_j\\ 2\varkappa=\nabla_s\alpha^s-\bar{\sigma}, & 2\bar{\varkappa}=\nabla_s\beta^s+\sigma\end{cases} \qquad (2\cdot1)$$

$$\begin{cases}\overset{1}{\rho}=2(-K-6\overset{1}{\tau}+3\overset{2}{\tau}+3\overset{3}{\tau}), & \overset{4}{\rho}=-K-2\overset{1}{\tau}+2\overset{2}{\tau}+2\overset{3}{\tau}\\ \overset{2}{\rho}=2(K+3\overset{1}{\tau}-3\overset{2}{\tau}-2\overset{3}{\tau}), & \overset{3}{\rho}=2(K+3\overset{1}{\tau}-2\overset{2}{\tau}-3\overset{3}{\tau})\end{cases} \qquad (2\cdot2)$$

where

$$\begin{cases}\overset{1}{\tau}=\alpha^i\beta^j\alpha^l\beta^mK_{ijlm}=-(\overset{1}{\rho}+\overset{2}{\rho}+\overset{3}{\rho}+2\overset{4}{\rho})/4, & \overset{2}{\tau}=-\alpha^i\alpha^jK_{ij}=-(\overset{1}{\rho}+3\overset{2}{\rho}+\overset{3}{\rho}+6\overset{4}{\rho})/4\\ \overset{3}{\tau}=\beta^i\beta^jK_{ij}=-(\overset{1}{\rho}+\overset{2}{\rho}+3\overset{3}{\rho}+6\overset{4}{\rho})/4, & K=K_i{}^i=-(\overset{1}{\rho}+3\overset{2}{\rho}+3\overset{3}{\rho}+12\overset{4}{\rho})/2\end{cases}$$
$$(2\cdot3)$$

$$K_{jl}=-\frac{1}{4}(\overset{2}{\rho}+\overset{3}{\rho}+6\overset{4}{\rho})g_{jl}+\frac{1}{4}(\overset{1}{\rho}+2\overset{2}{\rho})\alpha_j\alpha_l-\frac{1}{4}(\overset{1}{\rho}+2\overset{3}{\rho})\beta_j\beta_l \qquad (2\cdot4)$$

$$\nabla_{[i}\alpha_{j]}=\sigma P_{ij},\ \nabla_{[i}\beta_{j]}=-\bar{\sigma}P_{ij},\ (P_{ij}=\alpha_{[i}\beta_{j]}). \qquad (2\cdot5)$$

Next we can show that the condition for integrability of (F_2) i. e. $2\nabla_{[h}\nabla_{i]}\alpha_j=K_{hijl}\alpha^l$ is equivalent to

$$\begin{cases} \overset{1}{\tau}-\sigma^2+\bar{\sigma}^2=-\alpha^s\bar{\sigma}_s-\beta^s\sigma_s, \ \gamma^s\sigma_s=\delta^s\sigma_s=\gamma^s\bar{\sigma}_s=\delta^s\bar{\sigma}_s=0 \\ \overset{2}{\rho}+2\overset{4}{\rho}=4(\sigma\bar{\kappa}+\kappa^2+\alpha^s\kappa_s), \ \bar{\kappa}(\bar{\sigma}-\kappa)-\beta^s\kappa_s=0, \ \gamma^s\kappa_s=\delta^s\kappa_s=0 \end{cases} \tag{2·6}$$

where $\sigma_s=\nabla_s\sigma$, $\cdots$ and (γ^i, δ^i) is any set of two unit vectors $(\gamma_s\gamma^s=\delta_s\delta^s=-1)$ which form an orthogonal enupple together with (α^i, β^i). Hence

$$g_{ij}=-\alpha_i\alpha_j-\gamma_i\gamma_j-\delta_i\delta_j+\beta_i\beta_j. \tag{2·7}$$

In the same way from (F_3), we have

$$\overset{3}{\rho}+2\overset{4}{\rho}=4(\bar{\sigma}\kappa-\bar{\kappa}^2-\beta^s\bar{\kappa}_s), \ \kappa(\alpha+\bar{\kappa})+\alpha^s\bar{\kappa}_s=0, \ \gamma^s\bar{\kappa}_s=\delta^s\bar{\kappa}_s=0. \tag{2·8}$$

Therefore σ, $\bar{\sigma}$, κ, $\bar{\kappa}$ are s. s. scalars. If f is any s. s. scalar then $\alpha^s f_s$ and $\beta^s f_s$ where $f_s=\nabla_s f$ are also s. s., therefore $\overset{1}{\tau}$, $(\overset{2}{\rho}+2\overset{4}{\rho})$, $(\overset{3}{\rho}+2\overset{4}{\rho})$, $\overset{2}{\tau}$ and $\overset{3}{\tau}$ are also s. s.

Then we can easily show that under the assumption (I) Bianchi's identity $\nabla_{[n}K_{lm]ij}=0$ is equivalent to

$$\begin{cases} \overset{1}{\rho}_i=-(\alpha^s\overset{1}{\rho}_s)\alpha_i+(\beta^s\overset{1}{\rho}_s)\beta_i-2a\gamma_i-2b\delta_i \\ \overset{2}{\rho}_i=-(\alpha^s\overset{2}{\rho}_s)\alpha_i+\{\overset{1}{\rho}\bar{\kappa}+\overset{2}{\rho}\sigma-\overset{3}{\rho}(\sigma+\bar{\kappa})\}\beta_i+2a\gamma_i+2b\delta_i \\ \overset{3}{\rho}_i=-\{\overset{1}{\rho}\kappa-\overset{3}{\rho}\bar{\sigma}+\overset{2}{\rho}(\bar{\sigma}-\kappa)\}\alpha_i+(\beta^s\overset{3}{\rho}_s)\beta_i+2a\gamma_i+2b\delta_i \\ \overset{4}{\rho}_i=-\overset{2}{\rho}\kappa\alpha_i+\overset{3}{\rho}\bar{\kappa}\beta_i-a\gamma_i-b\delta_i, \ (\overset{a}{\rho}_i=\nabla_i\overset{a}{\rho}). \end{cases} \tag{2·9}$$

Specially when $\overset{1}{\rho}=\overset{2}{\rho}=\overset{3}{\rho}=0$, we have $a=b=0$ and $\overset{4}{\rho}=$const., which is the well known theorem of Schur concerning the space of constant curvature. By (II), $\overset{a}{\rho}$ and K become s. s. and we have $a=b=0$ in (2·9), from which we obtain

$$\begin{cases} \beta^s\overset{2}{\rho}_s=\overset{1}{\rho}\bar{\kappa}+\overset{2}{\rho}\sigma-\overset{3}{\rho}(\sigma+\bar{\kappa}), \ \alpha^s\overset{3}{\rho}_s=\overset{1}{\rho}\kappa-\overset{3}{\rho}\bar{\sigma}+\overset{2}{\rho}(\bar{\sigma}-\kappa) \\ \alpha^s\overset{4}{\rho}_s=\overset{2}{\rho}\kappa, \ \beta^s\overset{4}{\rho}_s=\overset{3}{\rho}\bar{\kappa}. \end{cases} \tag{2·10}$$

Finally we shall add two theorems easily obtained from the definition. Theorem [2·1] *A necessary and sufficient condition that $\alpha^i(\beta^i)$ give a geodesic congruence is given by $\sigma=0$ $(\bar{\sigma}=0)$ and the condition that $\alpha_i(\beta_i)$ be gradient also coincides with this. Further, a necessary and sufficient condition that $\alpha_i(\beta_i)$ be parallel field is given by $\sigma=\bar{\sigma}=\kappa=0$ $(\sigma=\bar{\sigma}=\bar{\kappa}=0)$.*

Theorem [2·2][2] *A necessary and sufficient condition that a s. s. space-time be conformally flat is given by* $\underset{1}{\rho}=0$.

§ 3. Standard form for the c. s.

In this section we shall show that new definition of the s. s. space-time is equivalent to the ordinaay one by showing that ds^2 of the s. s. space-time in the sense of the new definition can be brought into the form (1·1) by a suitable choice of the coordinate system.

From (F_2) and (F_3), we have $\nabla_{[k}P_{ij]}=0$, from which we know that there exist two scalars S and $\bar{S}$ satisfying $P_{ij}=(\nabla_{[i}S)\nabla_{j]}\bar{S}$. Using these scalars we can prove that there exists a coord. system in which

$$\begin{cases} g_{1a}=g^{1a}=\alpha_a=0,\ g_{4b}=g^{4b}=\beta_b=0,\ (a=2,\ 3,\ 4\ ;\ b=1,\ 2,\ 3) \\ g_{11}g^{11}=1=g_{44}g^{44},\ \alpha_1=\alpha_1(x^1,\ x^4)=\sqrt{-g_{11}},\ \beta_4=\beta_4(x^1,\ x^4)=\sqrt{g_{44}} \\ g_{pq}=H^2h_{pq}(x^2,\ x^3),\ H=H(x^1,\ x^4),\ (p,\ q=2,\ 3) \end{cases} \tag{3·1}$$

and σ, $\bar{\sigma}$, $\varkappa$ and $\bar{\varkappa}$ are determined from (F_2) and (F_3) as functions of $(x^1,\ x^4)$. If we put $-dl^2=h_{pq}(x^2,\ x^3)dx^pdx^q$ and use (F_1), we can show that $H_{23}^{\cdot\cdot 23}=1/\Psi$ where $H_{pq}^{\cdot\cdot rs}$ is the curvature tensor of the 2-dimensional space defined by dl^2 and Ψ is a function of $(x^1,\ x^4)$. Hence this space is of non-zero constant curvature by virtue of (III). So by a suitable transformation of $(x^2,\ x^3)$, we have

$$ds^2=-A(r,\ t)dr^2-B(r,\ t)\,(d\theta^2+\sin^2\theta\, d\phi^2)+C(r,\ t)dt^2 \tag{3·2}$$

where we have put $(x^1,\ \cdots,\ x^4)=(r,\ \theta,\ \phi,\ t)$, $g_{11}=-A$, $H^2\Psi=-B$ and $g_{44}=C$. By (IV), we shall assume that $A,\ B,\ C>0$. From (F_2) and (F_3), we have

$$\begin{cases} \alpha_i=\sqrt{A}\,\delta_i^1,\ \beta_i=\sqrt{C}\,\delta_i^4,\ \sigma=-\dot{A}/2A\sqrt{C},\ \bar{\sigma}=-C'/2C\sqrt{A} \\ \varkappa=-B'/2B\sqrt{A},\ \bar{\varkappa}=\dot{B}/2B\sqrt{C} \end{cases} \tag{3·3}$$

where dashes and dots denote partial differentiation with respect to r and t respectively. (F_2) and (F_3) are satisfied by (3·3). For (3·2) it holds that

$$\begin{cases}\alpha\equiv K_{\dot{1}\dot{2}}{}^{12}=K_{\dot{1}\dot{3}}{}^{13}=\{2B''-B'^2/B-A'B'/A-\dot{A}\dot{B}/C\}/4AB\\ \beta\equiv K_{\dot{2}\dot{4}}{}^{24}=K_{\dot{3}\dot{4}}{}^{34}=-\{2\ddot{B}-\dot{B}^2/B-\dot{B}\dot{C}/C-B'C'/A\}/4BC\\ \gamma\equiv K_{\dot{1}\dot{2}}{}^{24}=K_{\dot{1}\dot{3}}{}^{34}=\{2\dot{B}'-\dot{B}B'/B-\dot{A}B'/A-\dot{B}C'/C\}/4BC\\ \xi\equiv K_{\dot{1}\dot{4}}{}^{14}=-\{2(\ddot{A}-C'')+A'C'/A-\dot{A}\dot{C}/C-\dot{A}^2/A+C'^2/C\}/4AC\\ \eta\equiv K_{\dot{2}\dot{3}}{}^{23}=-\{B-B'^2/4A+\dot{B}^2/4C\}/B^2 \quad \text{other } K_{ij}{}^{lm}=0.\end{cases} \tag{3·4}$$

From (F_1), we have

$$\overset{1}{\rho}=4(\xi+\eta-\alpha-\beta),\ \overset{2}{\rho}=4(\alpha-\eta),\ \overset{3}{\rho}=4(\beta-\eta),\ \overset{4}{\rho}=2\eta. \tag{3·5}$$

(F_1) and (III) are satisfied identically by (3·3) and (3·5) except one condition

$$\gamma=0 \text{ i.e. } -2\dot{B}'+\dot{B}C'/C+B'\dot{A}/A+\dot{B}B'/B=0. \tag{3·6}$$

At first sight, since (3·6) must hold, new definition seems more stringent than the usual one. But since it is always possible, by a suitable transformation (real or complex) of (r, t), to make $\bar{g}_{14}=0$ and $\bar{K}_{\dot{1}\dot{2}}{}^{24}=\bar{K}_{\dot{1}\dot{3}}{}^{34}=0$ for ds^2 given by (1·1), both definitions are equivalent to each other. This will also be seen by the fact that there always exists a c. s. in every s. s. space-time in the usual sense. (See § 5). We call (3·3) and (3·5) the *standard form of the c. s.* and the coord. system the *standard coord. system for the c. s.* In connection with this we call the coord. system in which ds^2 takes the form (3·2) a *s. s. coord. system* and the s. s. coord. system in which $\gamma=0$ holds a *standard coord. system for* g_{ij}. Hence in a standard coord. system for g_{ij} a real c. s. is given by (3·3) and (3·5). A s. s. space-time S_0 may have some c. s.'s and some standard coord. systems for g_{ij}.

If we put $F_i=x\alpha_i-\bar{x}\beta_i$, F_i is a gradient vector and in the s. c. s. (standard coord. system) for c. s., putting $F_i=\nabla_i F$, we have $B=pe^{-2r}$ where p is a constant. Normalizing F by $p=1$, we have

$$\overset{4}{\rho}=2(x^2-\bar{x}^2-e^{2r}) \tag{3·7}$$

and we can show that (3·6) is equivalent to the identity

$$\alpha^i\beta^j\nabla_i\nabla_jF=\alpha^j\beta^i\nabla_i\nabla_jF=x\bar{x}=\bar{\sigma}x-\beta_s x^s=-\sigma\bar{x}-\alpha^s\bar{x}_s. \tag{3·8}$$

Similarly

$$\alpha^i\alpha^j\nabla_i\nabla_j F = -\sigma\bar{x} - \alpha^s x_s = x^2 - (\overset{2}{\rho}+2\overset{4}{\rho})/4$$
$$\beta^i\beta^j\nabla_i\nabla_j F = \bar{\sigma}x - \beta^s\bar{x}_s = \bar{x}^2 + (\overset{3}{\rho}+2\overset{4}{\rho})/4 \qquad (3\cdot 9)$$
$$\nabla_s F^s = x^2 - \bar{x}^2 + (\overset{2}{\rho}+\overset{3}{\rho}+4\overset{4}{\rho})/4.$$

§ 4. Freedom of c. s. in a given s. s. space-time

As is easily seen, we have

Theorem [4·1] *Let* $[K]: (\alpha_i, \beta_i, \overset{a}{\rho}, \cdots)$ *be a c. s. of a s. s. space-time* S_0. *Then* (α_i^*, β_i^*) *given by (i)* $\alpha_i^* = \epsilon\alpha_i$, $\beta_i^* = \bar{\epsilon}\beta_i$ *and (ii)* $\alpha_i^* = i\beta_i$, $\beta_i^* = i\alpha_i$ *where* $\epsilon^2 = \bar{\epsilon}^2 = 1$, *are again c. v.'s of* S_0 *and the characteristic scalars corresponding to them are given by (i)* $[K_1]: \overset{a}{\rho}{}^* = \overset{a}{\rho}$, $\sigma^* = \bar{\epsilon}\sigma$, $\bar{\sigma}^* = \epsilon\bar{\sigma}$, $x^* = \epsilon x$, $\bar{x}^* = \bar{\epsilon}\bar{x}$, *and (ii)* $[K_2]: \overset{1}{\rho}{}^* = \overset{1}{\rho}$, $\overset{2}{\rho}{}^* = \overset{3}{\rho}$, $\overset{3}{\rho}{}^* = \overset{2}{\rho}$, $\overset{4}{\rho}{}^* = \overset{4}{\rho}$, $\sigma^* = -i\bar{\sigma}$, $\bar{\sigma}^* = -i\sigma$, $x^* = i\bar{x}$, $\bar{x}^* = ix$, *respectively.*

In the following we shall call the transformation $[K]\to[K_1]$ and $[K]\to[K_2]$, $\epsilon-$ and *$i-$transformations* respectively and study the freedom of the c. s. in an S_0 excluding these transformations. From (2·4), we have

$$(K_i^{\cdot l} - \overset{2}{\tau}\delta_i^l)\alpha^i = 0, \quad (K_i^{\cdot l} - \overset{3}{\tau}\delta_i^l)\beta^i = 0 \qquad (4\cdot 1)$$

which shows that both α^i and β^i are principal directions of the Ricci tensor K_{ij} and $\overset{2}{\tau}$ and $\overset{3}{\tau}$ are corresponding principal invariants i. e. the solutions of $|K_i^{\cdot l} - \nu\delta_i^l| = 0$. In the s. c. s. for the c. s., we have

$$\overset{1}{\nu} = -(2\alpha+\xi) = \overset{2}{\tau}, \quad \overset{2}{\nu} = \overset{3}{\nu} = -(\alpha+\beta+\eta) = -(\overset{2}{\rho}+\overset{3}{\rho}+6\overset{4}{\rho}), \quad \overset{4}{\nu} = -(2\beta+\xi) = \overset{3}{\tau} \qquad (4\cdot 2)$$

where $\overset{1}{\nu}, \cdots, \overset{4}{\nu}$ are principal invariants of K_{ij}.

Case I. When ν's are of the form (a, a, b, c), $(a, b, c \neq)$.

In this case c. s. is determined uniquely to within $\epsilon-$ and $i-$transformations. So $\overset{a}{\rho}$ is determined uniquely and the condition $\overset{1}{\nu} \neq \overset{2}{\nu}$, $\overset{2}{\nu} \neq \overset{4}{\nu}$ and $\overset{1}{\nu} \neq \overset{4}{\nu}$ are equivalent to $\overset{1}{\rho}+2\overset{2}{\rho} \neq 0$, $\overset{1}{\rho}+2\overset{3}{\rho} \neq 0$ and $\overset{2}{\rho} \neq \overset{3}{\rho}$, respectively.

Case II. When ν's are of the form (a, a, b, b), $(a \neq b)$.

We can easily prove that

Theorem [4·2] *A necessary and sufficient condition that*

$$\alpha_i^* = \cosh\omega\alpha_i + \sinh\omega\beta_i, \quad \beta_i^* = \sinh\omega\alpha_i + \cosh\omega\beta_i \qquad (4\cdot 3)$$

where $e^{2\omega}\neq\pm1$, *be again c. v. of* S_0 *is given by*

$$\overset{2}{\rho}=\overset{3}{\rho} \text{ and } \nabla_i\omega=-(\alpha^s\omega_s)\alpha_i+(\beta^s\omega_s)\beta_i,\quad (\omega_s=\nabla_s\omega) \tag{4·4}$$

and ρ^*, x^*,⋯*corresponding to* (α_i^*, β_i^*) *are given by*

$$\overset{a}{\rho}{}^*=\overset{a}{\rho},\ x^*=x\cosh\omega+\bar{x}\sinh\omega,\ \bar{x}^*=x\sinh\omega+\bar{x}\cosh\omega \tag{4·5}$$

$$\sigma^*=(\sigma-\alpha^s\omega_s)\cosh\omega-(\bar{\sigma}+\beta^s\omega_s)\sinh\omega,$$

$$\bar{\sigma}^*=-(\sigma-\alpha^s\omega_s)\sinh\omega+(\bar{\sigma}+\beta^s\omega_s)\cosh\omega.$$

We call this transformation of c. s. $[K]\rightarrow[K^*]$, *ω—transformation.*

Next, let $[K]$ and $[K^*]$ be any two sets of c. s.'s of this S_0 and take the s. c. s. for $[K]$. From the condition $\overset{2}{\rho}=\overset{3}{\rho}$ and $\overset{1}{\rho}+2\overset{2}{\rho}\neq0$, we have (i) When at least one of α_1^*, α_4^*, β_1^*, β_4^*, is not zero, any $[K^*]$ must be obtained from $[K]$ by ϵ—, i—, and ω—transformations and vice versa. (ii) When $\alpha_i^*=(0, \alpha_2^*, \alpha_3^*, 0)$, $\beta_i^*=(0, \beta_2^*, \beta_3^*, 0)$, both (r, t)—space and (θ, ϕ)—space must be of different constant curvature and ds^2 reduces to the form

$$ds^2=-B(d\theta^2+\sin^2\theta d\phi^2)+\{1+(-r^2+t^2)/4R^2\}^{-2}(-dr^2+dt^2),\quad (B\neq R^2) \tag{4·6}$$

where B and R^2 are constants. Therefore when and only when ds^2 is reducible to (4·6), there exist two c. s.'s $[K]$ and $[K^*]$, their c. v.'s together constitute an orthogonal ennuple and any c. s. of S_0 is obtained from either $[K]$ or $[K^*]$ by ϵ—, i—, and ω—transformations. $x=\bar{x}=0$ holds for any c. s. In terms of ρ's this space-time is characterized by

$$\overset{2}{\rho}=\overset{3}{\rho},\ \overset{1}{\rho}+2\overset{2}{\rho}=\text{const.}\neq0,\ (=4(1/B-1/R^2)),$$
$$\overset{2}{\rho}+2\overset{4}{\rho}=0,\ \overset{4}{\rho}=\text{const.}\ (=-2/B\neq0). \tag{4·7}$$

ρ^*'s are obtained from (4·7) by interchanging B and R^2 and remaining members of $[K]$ and $[K^*]$ are also easily obtained. In any s. c. s. for g_{ij} one of $[K]$ and $[K^*]$ is real and the other is complex.

Case III. When ν's are of the form (a, a, a, b), $(a\neq b)$.

In this case we can show that a necessary and sufficient condition that there exist two sets of c. s.'s not transformable by ϵ— and i—transformations is that ds^2 is reducible (including imaginary transformations) to

$$ds^2 = -\frac{e^{2g(t)}}{[1+r^2/4R^2]^2}(dr^2+r^2d\theta^2+r^2\sin^2\theta\, d\phi^2)+dt^2. \qquad (4\cdot8)$$

And in this case $[K^*]$ is obtained from $[K]$ by a motion i. e. a transformation of x^i which keeps the form of g_{ij} invariant. We shall call this way of obtaining $[K^*]$ from $[K]$, *m—transformation.* Thus we know that in this space-time c. s. is determined to within ϵ—, i— and m—transformations. Especially ρ's are determined uniquely and in terms of $\overset{a}{\rho}$ this space-time is characterized by $\overset{1}{\rho}=\overset{2}{\rho}=0$, $\overset{3}{\rho}\neq0$ and $\overset{3}{\rho}_s a^s=\overset{4}{\rho}_s a^s=0$.

When S_0 is not of this type c. s. is determined to within ϵ— and i—transformations. The group of motions for (4·8) is known,(3) so by some m—transformations we can obtain new c. s.'s not s. s. in the usual sense. In any s. s. space-time, since c. s. is defined by tensor conditions $[K^*]$ obtained from one c. s. $[K]$ by any m—transformation again becomes a c. s. of the space-time. In general, however, excluding some special cases like the one mentioned above, the set of $[K]$'s is invariant under m—transformations.

Case IV. When ν's are of the form (a, a, a, a) i. e. when $K_{ij}=(K/4)g_{ij}$.

In this case $\overset{2}{\rho}=\overset{3}{\rho}$ holds and when S_0 is not of constant curvature we can show that a condition in order that there exist two sets of c. s.'s not transformable by ϵ—, and ω—transformations is given by that the space-time is decomposed into two 2-dimensional spaces of the same constant curvature. In this S_0, ds^2 is reducible to

$$ds^2=-R^2(d\theta^2+\sin^2\theta\, d\phi^2)+\{1+(-r^2+t^2)/4R^2\}^{-2}(-dr^2+dt^2) \qquad (4\cdot9)$$

and any c. s. is given by ϵ—, i—, and ω—transformations from two c. s.'s $[K]$ and $[K^*]$ whose c. v.'s together form an orthogonal ennuple and in any s. c. s. for g_{ij} one is real and the other is complex. For all cases ρ's are the same and $x=\bar{x}=0$ holds good.

When S_0 is of constant curvature, ds^2 is reducible to

$$ds^2=-(1-k^2r^2)^{-1}dr^2-r^2(d\theta^2+\sin^2\theta\, d\phi^2)+(1-k^2r^2)dt^2 \qquad (4\cdot10)$$

and any c. s. is obtained from one c. s. by ϵ—, i—, ω— and m—transformations.

Thus we have determined the freedom of c. s. in a s. s. space-time. ρ's are determined uniquely except the case of (4·6).

Hereafter we shall consider only the c. s. which is real in s. c. s. for

g_{ij}. In any s. c. s. for g_{ij} a real c. s. is given by (3·3) and (3·5) and other real ones are obtained from this by at most $\epsilon-$, $\omega-$, and $m-$transformations. By this assumption ρ's are determined uniquely in all S_0 and we have the following table concerning c. s. to within ϵ—transformation:

I. (a, a, b, c) i.e. $(\overset{1}{\rho}+2\overset{2}{\rho}\neq 0,\ \overset{1}{\rho}+2\overset{3}{\rho}\neq 0,\ \overset{2}{\rho}\neq\overset{3}{\rho})$: unique

II. (a, a, b, b) i.e. $(\overset{1}{\rho}+2\overset{2}{\rho}\neq 0,\ \overset{2}{\rho}=\overset{3}{\rho})$: (ω)

III. (a, a, a, b) i.e. $(\overset{1}{\rho}+2\overset{2}{\rho}=0,\ \overset{1}{\rho}+2\overset{3}{\rho}\neq 0)$ or $(\overset{1}{\rho}+2\overset{2}{\rho}\neq 0,\ \overset{1}{\rho}+2\overset{3}{\rho}=0)$

(i) When ds^2 is reducible to (4·8) : β_i is unique $(\overset{1}{\rho}=\overset{2}{\rho}=\bar{\sigma}=\sigma-\bar{x}=\alpha^s\overset{a}{\rho}_s=0)$ for all c.s.'s : (m)

(ii) When ds^2 is reducible to (4·8′) : α_i is unique $(\overset{1}{\rho}=\overset{3}{\rho}=\sigma=\bar{\sigma}-x=\beta^s\overset{a}{\rho}_s=0)$ for all c.s.'s : (m)

(iii) Other cases : unique

IV. (a, a, a, a) i.e. $(\overset{1}{\rho}+2\overset{2}{\rho}=0,\ \overset{2}{\rho}=\overset{3}{\rho})$

(i) When S_0 is of const. curvature : (ω, m)

(ii) When S_0 is not of const. curvature : (ω)

where (4·8′) denotes $ds^2=-dr^2+e^{2h(r)}dm^2$, and $dm^2=-b(t)(d\theta^2+\sin^2\theta\, d\phi^2)+f(t)dt^2$ defines a space of const. curvature. By an imaginary transformation $\sqrt{b}=\bar{r}(1+\bar{r}^2/4R^2)^{-1}$ and $r=i\bar{t}$, (4·8′) becomes (4·8).

§ 5. $S_{\rm I}$ and $S_{\rm II}$. c.s. in a s. s. coordinate system

We denote S_0 whose $B\neq$const. and $B=$const. in s. s. coord. system by $S_{\rm I}$ and $S_{\rm II}$ respectively. This classification has an invariant meaning by virtue of the following theorem.

Theorem [5·1] *A necessary and sufficient condition that S_0 be $S_{\rm II}$ is given by that there exists a c. s. whose $x=\bar{x}=0$, and when this condition is satisfied it holds good for all c. s.'s. In the same way $S_{\rm I}$ is characterized by the existence of a c. s. which does not satisfy $x=\bar{x}=0$.*

The proof is evident. In [5·1] the condition $x=\bar{x}=0$ can be replaced by $F=$const.

Next we shall denote S_0 in which $\overset{2}{\rho}=\overset{3}{\rho}$ and $\overset{2}{\rho}\neq\overset{3}{\rho}$ hold good by S_a and S_b respectively. Then by using the results of § 3 we can easily prove Theorem [5·2] *In S_a any s. s. coord. system is standard for g_{ij}. Moreover a necessary and sufficient condition that S_0 be S_a is given by that $\alpha=\beta$ and $\gamma=0$ hold in any coord. system obtained by any transformation of (r, t) from a s. s. coord. system.* (In such a system g_{14} does not vanish necessarily.)

By solving (F_1), (F_2) and (F_3), we have the following formulae giving c. s. in any s. s. coord. system to within $\epsilon-$, $i-$ and $m-$transformations.

(**a**) When S_0 is S_a. To within $\omega-$transformation

$$\begin{cases} \overset{1}{\rho}=4(\xi+\eta-2\alpha),\ \overset{2}{\rho}=\overset{3}{\rho}=4(\alpha-\eta),\ \overset{4}{\rho}=2\eta,\ F=-(1/2)\log B \\ \alpha_i=\sqrt{A}\,\delta^1_i,\ \beta_i=\sqrt{C}\,\delta^4_i,\ \sigma=-\dot{A}/2A\sqrt{C},\ \bar{\sigma}=-C'/2C\sqrt{A} \\ \varkappa=-B'/2B\sqrt{A},\ \bar{\varkappa}=\dot{B}/2B\sqrt{C} \end{cases} \tag{5·1}$$

where α, β,$\cdots$ are given by (3·4). Specially when S_a is S_{II}, since $\alpha=\beta=\gamma=0$, $\eta=-1/B=$const. $(\neq 0)$, this becomes

$$\begin{cases} \overset{1}{\rho}=4(\xi+\eta),\ \overset{2}{\rho}=\overset{3}{\rho}=-2\overset{4}{\rho}=4/B,\ (\neq 0),\ F=-(1/2)\log B \\ \alpha_i=\sqrt{A}\,\delta^1_i,\ \beta_i=\sqrt{C}\,\delta^4_i,\ \sigma=-\dot{A}/2A\sqrt{C},\ \bar{\sigma}=-C'/2C\sqrt{A},\ \varkappa=\bar{\varkappa}=0. \end{cases} \tag{5·1'}$$

(**b**) When S_0 is S_b. When $\gamma\neq 0$ i.e. when the coord. system is not standard for g_{ij},

$$\begin{cases} \overset{1}{\rho}=4(\xi+\eta-\alpha-\beta),\ \overset{2}{\rho}=M+N,\ \overset{3}{\rho}=M-N,\ \overset{4}{\rho}=2\eta,\ F=-(1/2)\log B \\ M=2(\alpha+\beta-2\eta),\ N=2(\alpha-\beta)/\cosh 2\zeta,\ 2\zeta=\tanh^{-1}\{2\sqrt{C/A}\,\gamma/(\alpha-\beta)\} \\ \alpha_i=(\sqrt{A}\cosh\zeta,\ 0,\ 0,\ \sqrt{C}\sinh\zeta),\ \beta_i=(\sqrt{A}\sinh\zeta,\ 0,\ 0,\ \sqrt{C}\cosh\zeta) \\ \sigma=\bar{M}\cosh\zeta-\bar{N}\sinh\zeta,\ \bar{\sigma}=-\bar{M}\sinh\zeta+\bar{N}\cosh\zeta \\ \bar{M}=(\zeta_1-\dot{A}/2\sqrt{AC})/\sqrt{A},\ \bar{N}=(\zeta_4-C'/2\sqrt{AC})/\sqrt{C},\ (\zeta_i=\partial_i\zeta) \\ \varkappa=\{-(B'/\sqrt{A})\cosh\zeta+(\dot{B}/\sqrt{C})\sinh\zeta\}/2B,\ \bar{\varkappa}=\{-(B'\sqrt{A})\sinh\zeta \\ \qquad +(\dot{B}/\sqrt{C})\cosh\zeta\}/2B. \end{cases} \tag{5·2}$$

When the coord. system is standard for g_{ij}, by putting $\gamma=0$, (5·2) becomes

$$\begin{cases} \overset{1}{\rho}=4(\xi+\eta-\alpha-\beta),\ \overset{2}{\rho}=4(\alpha-\eta),\ \overset{3}{\rho}=4(\beta-\eta),\ \overset{4}{\rho}=2\eta \\ F=-(1/2)\log B,\ \alpha_i=\sqrt{A}\,\delta_i^1,\ \beta_i=\sqrt{C}\,\delta_i^4, \\ \sigma=-\dot{A}/2A\sqrt{C},\ \bar{\sigma}=-C'/2C\sqrt{A},\ \varkappa=-B'/2B\sqrt{A},\ \bar{\varkappa}=\dot{B}/2B\sqrt{C}. \end{cases} \tag{5·2'}$$

In (5·1), (5·1′), and (5·2′) all quantities are real and in (5·2) they are also real when and only when $|\Lambda\leqq 1|$ where $\Lambda=2\sqrt{C/A}\gamma/(\alpha-\beta)$, namely when the six principal invariants of $K_{AB}\equiv K_{ijlm}$ are all real.[4] But when $|\Lambda|>1$, i.e. when K_{AB} has two pairs of conjugate complex principal invariants, $\sinh\zeta$, $\cosh\zeta$, $\sinh 2\zeta,\cdots$ must be interpreted as the symbolical notations of $\sqrt{\{i/\sqrt{\Lambda^2-1}+1\}}/\sqrt{2}$, $\sqrt{\{i/\sqrt{\Lambda^2-1}-1\}}/\sqrt{2}$, $i\sqrt{\Lambda^2-1},\cdots$ respectively. Of course in all cases g_{ij} and consequently K_{ijlm} are all real. The fact that we must treat c. s. of complex values in some special s. s. coord. system of some S_b shows that in such S_b, the s. s. coord. system can not be obtained by a real transformation of (r, t) from the standard one.

Using the above formulae, we have

Theorem [5·3] *A necessary and sufficient condition that S_0 be $S_{\rm II}$ is given by*

$$\overset{2}{\rho}=\overset{3}{\rho}=-2\overset{4}{\rho}=\text{const.}\neq 0,\ (=4/B). \tag{5·3}$$

Hence $S_{\rm II}$ belongs to S_α.

Evidently the line element of $S_{\rm I}$ is transformable into the form

$$ds^2=-A(r,\ t)dr^2-r^2(d\theta^2+\sin^2\theta\, d\phi^2)+C(r,\ t)dt^2 \tag{5·4}$$

by a suitable transformation of (r, t) from a s. s. coord. system. In this coord. system it holds that

$$\begin{cases} \alpha=-A'/2A^2r,\ \beta=C'/2ACr,\ \gamma=-\dot{A}/2ACr,\ \eta=(1-A)/r^2A \\ \xi=-\{2(\ddot{A}-C'')+A'C'/A-\dot{A}\dot{C}/C-\dot{A}^2/A+C'^2/C\}/4AC. \end{cases} \tag{5·5}$$

We shall denote S_0 which belongs to both $S_{\rm I}$ and S_α, by S_{15}. For S_{15}, in the coord. system of (5·4), from $\alpha=\beta$ and $\gamma=0$, we have $A=A(r)$ and $AC=f(t)$. Hence by a suitable transformation of t we have $AC=1$, i. e.

$$S_{15}:\ ds^2=-A(r)dr^2-r^2(d\theta^2+\sin^2\theta\, d\phi^2)+(1/A)dt^2. \tag{5·6}$$

Conversely for (5·6), we have $\overset{2}{\rho}=\overset{3}{\rho}$. Hence

Theorem [5·4] *S_a is either S_{II} or S_{15}. The fundamental form of S_{15} is reducible to (5·6) and vice versa.*

By putting $A=(1-2m/r)^{-1}$ and $=(1-k^2r^2)^{-1}$ in (5·6), we have the space-time of Schwarzschild and de Sitter respectively.

Next, using (F_1), (F_2), (F_3) and the results above obtained we can easily show that

Theorem [5·5] *When S_I is S_a, i.e. S_{15}, in the coord. system of (5·4) A is always static i.e. $A=A(r)$. When S_I is not S_a a necessary and sufficient condition that the fundamental form be reducible to (5·4) in which A is static is given by that $\bar{\varkappa}=0$ holds good for a c. s. and when this condition is satisfied a necessary and sufficient condition that not only A but also C is static is given by $\beta^s \overset{3}{\rho}_s=0$ which is equivalent to $\beta^s\bar{\sigma}_s=0$. When $\bar{\varkappa}=0$ etc. is satisfied for a c. s. they holds good for all c. s.*

From this theorem we can determine whether a s. s. space-time is static or non-static independently of the coord. system. This result will be useful in theoretical physics.

§ 6. Geometric interpretation of the c. s.

As stated in § 4 both α^i and β^i are principal directions of the Ricci tensor and corresponding invariants are $\overset{2}{\tau}$ and $\overset{3}{\tau}$ i.e. $-\overset{2}{\tau}$ and $-\overset{3}{\tau}$ are mean curvatures of S_0 for the directions α^i and β^i respectively. The remaining principal invariants are $-(\overset{2}{\rho}+\overset{2}{\rho}+6\overset{4}{\rho})/4$. From (F_1) we know that $\overset{1}{\tau}$ and $\overset{4}{\rho}/2$ are Riemannian curvatures for the orientations determined by (α^i, β^i) and (γ^i, δ^i) respectively where (γ^i, δ^i) are any set of vectors introduced in § 2.

By putting (1·2)=1, (1·3)=2, ⋯, (3·4)=6, if we renumber the indices, from (F_1), we have

$$\begin{cases}(K_{AB}+g_{AB})P^A=0 \text{ where } K_{AB}\equiv K_{ijlm},\ P^A\equiv P^{ij},\ g_{AB}=2g_{[i[l}g_{j][m]}, \\ (i, j)=A,\ (l, m)=B,\ (A, B=1, \cdots, 6)\end{cases} \tag{6·1}$$

and $g_{AB}P^AP^B=P_AP^A=-1/4$. Hence $2P^A$ is a unit vector which gives a principal direction of K_{AB}, and the corresponding invariant i.e. the root of $|K_A^{\cdot B}-\nu\delta_A^B|=0$ is $-\overset{1}{\tau}$. Using s. c. s. we know that 6 principal invariants are given by

$$(\overset{2}{\rho}+2\overset{4}{\rho})/4, \quad ,, \quad ; (\overset{3}{\rho}+2\overset{4}{\rho})/4, \quad ,, \quad ; -\overset{1}{\tau}; \overset{4}{\rho}/2 \qquad (6\cdot2)$$

and the corresponding principal directions are $(R^A\equiv2\alpha^{[i}\gamma^{j]}, S^A\equiv2\alpha^{[i}\delta^{j]})$, $(\bar{R}^A\equiv2\beta^{[i}\gamma^{j]}, \bar{S}^A\equiv2\beta^{[i}\delta^{j]})$, $2P^A$, $2Q^A\equiv2\gamma^{[i}\delta^{j]}$. In a previous paper the writer tried to classify S_0 in terms of these invariants.(5)

If we put $(\alpha^i, \gamma^i, \delta^i, \beta^i)=\lambda_{h|}{}^i$, $(h=1, \cdots, 4)$ and calculate the coefficients of rotation γ_{lhk} we have

$$\begin{cases}\gamma_{411}=-\gamma_{141}=\sigma, \ \gamma_{144}=-\gamma_{414}=\bar{\sigma} \\ \gamma_{212}=-\gamma_{122}=\gamma_{313}=-\gamma_{133}=\kappa, \ \gamma_{242}=-\gamma_{422}=\gamma_{343}=-\gamma_{433}=\bar{\kappa}.\end{cases} \qquad (6\cdot3)$$

By considering the geometrical meaning of γ_{lhk} we obtain a geometrical interpretation of $\bar{\sigma}$, $\dot{\bar{\sigma}}$, κ, and $\bar{\kappa}$.

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Notes

(1) J. Eiesland, Trans. Amer. Math. Soc., 27, (1925), 213.

(2) This theorem was read at the annual meating of the Math. Soc. of Japan held in Nov. 1, 1948. We can prove this by showing that $\overset{1}{\rho}=0$ is equivalent to the vanishing of the conformal curvature tensor.

(3) H. Takeno, Jour. Sci. Hiroshima Univ. 11, (1942), 228.

(4), (5) H. Takeno, Jour. Sci. Hiroshima Univ. 12, (1942), 131.