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On the Algebraic Structure of Group Rings

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§ 1. Introduction

1. Let $\mathfrak{G}$ be a group of finite order g. If K is any given field of characteristic 0, the group ring Γ of $\mathfrak{G}$ with regard to K is a semisimple algebra. By Wedderburn's theorems, Γ is a direct sum of simple algebras A_i;

$$(1) \qquad \Gamma = \mathsf{A}_1 \oplus \mathsf{A}_2 \oplus \cdots \oplus \mathsf{A}_s.$$

Each A_i is isomorphic to a complete matric algebra of a certain degree q_i over a division algebra Δ_i;

$$(2) \qquad \mathsf{A}_i \cong [\Delta_i]_{q_i}.$$

The center Z_i of A_i may also be considered as the center of Δ_i. It is an extension field of finite degree r_i over K. Since Δ_i then is a central simple algebra over Z_i, its rank over Z_i is the square of a natural integer m_i. Then A_i has the rank $r_i q_i^2 m_i^2$ over K. We shall call the numbers m_i the *Schur indices* of $\mathfrak{G}$, since they first occurred in the work of *I. Schur* on representations of $\mathfrak{G}$ by linear transformations.

2. The theory of representations of groups of finite order was developed originally by *Frobenius* for the case that the coefficients of the representing linear transformations belong to an algebraically closed field of characteristic 0. The case of an arbitrary field K of characteristic 0 was considered by *I. Schur*.[1] We quote the main results.

Every representation of $\mathfrak{G}$ is completely reducible. Two representations of $\mathfrak{G}$ are similar, if and only if they have the same character. It is then sufficient to consider the irreducible representations of $\mathfrak{G}$ in K and their characters. These irreducible representations $\mathfrak{T}_1, \mathfrak{T}_2, \cdots, \mathfrak{T}_s$ are in one-to-one correspondence to the simple algebras $\mathsf{A}_1, \mathsf{A}_2, \cdots, \mathsf{A}_s$ in (1).

If $\overline{\mathsf{K}}$ is the algebraic closure of K, then $\mathfrak{T}_i$ breaks up in $\overline{\mathsf{K}}$ into r_i

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1) Schur [1], [2]. The connections with the theory of algebras are given in Brauer [1], [2]. See also Albert [1]; van der Waerden [1], Chapter XVII, [2]; Weyl [1], Chapters III and X.

distinct absolutely irreducible representations $\mathfrak{F}_i, \mathfrak{F}_{i'}, \mathfrak{F}_{i''}, \cdots$, each appearing with the same multiplicity m_i. Here, r_i and m_i are the same numbers which appeared in **1.** Thus, if the character of $\mathfrak{F}_j$ is denoted by χ_j, the character of $\mathfrak{T}_i$ is given by

$$m_i\,(\chi_i + \chi_i' + \cdots).$$

We now speak of m_i as the Schur index of each of the characters $\chi_i, \chi_{i'}, \cdots$ with regard to K.[2]

The r_i characters $\chi_i, \chi_{i'}, \cdots$ form a full family of absolutely irreducible characters of $\mathfrak{G}$ which are algebraically conjugate with regard to K. Conversely, each such family of characters appears in one and only one irreducible representation $\mathfrak{T}_i$ of $\mathfrak{G}$ in K. Thus, if the characters of $\mathfrak{G}$ (in the classical sense) are known, it remains only to determine the Schur indices m_i in order to have a complete theory of representations of $\mathfrak{G}$ in K. We then also know the number s of terms in (1) and the numbers q_κ in (2) because $q_i m_i$ is equal to the degree of $\mathfrak{F}_i$. Furthermore, the centers Z_i are known, since Z_i is isomorphic over K to the field $K(\chi_i)$ obtained from K by adjunction of all values $\chi_i(G)$, $G \in \mathfrak{G}$.

According to a result of *Schur*, the index m_i can also be characterized in the following manner. The representation $\mathfrak{F}_i$ can be written in certain extension fields Ω of K. In the language of the theory of algebras, these fields Ω are the splitting fields of A_i. It is clear that a splitting field Ω must contain the character χ_i. If a splitting field Ω has finite degree over $K(\chi_i)$, this degree is divisible by m_i. On the other hand, there exist splitting fields of exact degree m_i over $K(\chi_i)$. Thus, m_i is the minimal value of the degrees of splitting fields Ω over $K(\chi_i)$.

If we are able to determine the Schur index of χ_i with regard to an arbitrary field, we can decide whether or not a field $\Omega \supseteq K(\chi_i)$ is a splitting field of A_i. This will be so, if and only if χ_i has the Schur index 1 with regard to Ω.

3. The different characterizations of the Schur index do not provide a method to determine m_i, and this whole question remains open in Schur's

2) In the case of fields of characteristic $p \neq 0$, it follows from Wedderburn's theorem on division algebras over finite fields that all Schur indices are 1, cf. Brauer [2]. However Γ is no longer semisimple in this case.

theory.[3] It is the purpose of the present paper to show that the problem can be reduced to the case where the group is a soluble group of a very special type ($\mathfrak{E}$). Only groups of type ($\mathfrak{E}$) have to be considered which are subgroups of the given group $\mathfrak{G}$. The groups of type ($\mathfrak{E}$) shall be treated in a subsequent paper.

Though no use of class field theory is made in this investigation, it is perhaps pertinent to remark that the group theoretical methods used were first developed in connection with a problem which arose in class field theory. Thus, in an indirect way, we have benefitted from *Takagi*'s fundamental work.

Notation

4. The order of the given group $\mathfrak{G}$ will be denoted by g. For G_1, $G_2 \in \mathfrak{G}$, we write $G_1 \sim G_2$, if G_1 and G_2 are conjugate in $\mathfrak{G}$. If $\mathfrak{A}$ is a subset of $\mathfrak{G}$, we shall denote by $\mathfrak{N}(\mathfrak{A})$ the normalizer of $\mathfrak{A}$, i.e. the subgroup of $\mathfrak{G}$ consisting of those elements G for which $\mathfrak{A}G = G\mathfrak{A}$. In particular, this will be done, if $\mathfrak{A}$ consists of one element A, we then write $\mathfrak{N}(A)$. The order of $\mathfrak{N}(A)$ will be $n(A)$.

If $\psi = \psi(U)$ is a character of a group $\mathfrak{U}$, restriction of the argument U to a subgroup $\mathfrak{V}$ of $\mathfrak{U}$ yields a character of $\mathfrak{V}$ for which we use the notation $\psi(\mathfrak{V})$. By an irreducible character of a group, we always mean an absolutely irreducible character, that is, a character which is irreducible in the algebraically closed field.

The letter P will be used for the field of rational numbers and ε will stand for a primitive g-th root of unity. The Galois group of $\mathsf{P}(\varepsilon)$ with regard to P is denoted by $\mathfrak{L}$. Each $\sigma \in \mathfrak{L}$ carries ε into a power of ε; we set

$$\sigma: \quad \varepsilon \to \varepsilon^{\nu(\sigma)} \tag{3}$$

Here, $\nu(\sigma)$ is an integer determined (mod g) and prime to g. The correspondence $\sigma \to \nu(\sigma)$ defines an isomorphism of $\mathfrak{L}$ on the multiplicative group of integers prime to g (mod g).

Each character ψ of a subgroup $\mathfrak{U}$ of $\mathfrak{G}$ lies in $\mathsf{P}(\varepsilon)$. An element

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3) For instance, these characterizations do not show that all m_i are equal to 1, if K contains the g-th roots of unity.

$\sigma \in \mathfrak{L}$ carries ψ into a character ψ^σ. If we write $\psi(U)$ as sum of characteristic roots for $U \in \mathfrak{U}$, we see that

$$\psi^\sigma(U) = \psi(U^{\nu(\sigma)}). \tag{4}$$

If ψ is irreducible, so is ψ^σ.

If p is a fixed prime, the *p-part* of a rational integer r is the highest power p^ρ of p dividing r. Similarly, we speak of the $\mathfrak{p}$-part of algebraic integers for suitable prime ideals $\mathfrak{p}$. If $\mathfrak{U}$ is a group of finite order, a fixed p-Sylow subgroup of $\mathfrak{U}$ will often be denoted by $\mathfrak{U}_p$. In particular, $\mathfrak{L}_p$ will always stand for the unique p-Sylow subgroup of the group $\mathfrak{L}$. Thus, $\mathfrak{L}_p$ consists of those $\sigma \in \mathfrak{L}$ for which $\nu(\sigma)$ in (3) belongs to an exponent (mod g) which is a power of p.

An element G of $\mathfrak{G}$ will be said to be *p-regular*, if the order of G is prime to p.

§ 2. Group of type ($\mathfrak{E}$)

5. Let $\chi = \chi(G)$ denote an irreducible character of the group $\mathfrak{G}$. In order to determine the Schur index m of χ with regard to a given field K of characteristic 0, it is sufficient to determine the p-part m_p of m for every prime number p. Since m divides the degree of χ, we have $m_p = 1$, if p does not divide the order g of $\mathfrak{G}$. Our method will be based on the following remark:

(2A) *Let K^* be a maximal subfield of $\mathsf{K}(\chi, \varepsilon)$ over $\mathsf{K}(\chi)$ such that the degree $[\mathsf{K}^* : \mathsf{K}(\chi)]$ is not divisible by the prime p. If ξ is an irreducible character of a subgroup $\mathfrak{G}^*$ of $\mathfrak{G}$ such that ξ lies in K^* and that ξ appears in $\chi(\mathfrak{G}^*)$ with a multiplicity v prime to p, then the p-part μ_p of the index μ of ξ with regard to $\mathsf{K}(\chi)$ is equal to the p-part m_p of the index m of χ with regard to K.*

Proof:[4] There exists a representation of $\mathfrak{G}$ in $\mathsf{K}(\chi)$ whose character θ is $m\chi$. Then $\theta(\mathfrak{G}^*)$ contains ξ with the multiplicity mv and hence $\mu | mv$. Since $(v, p) = 1$, we have $\mu_p \mid m_p$.

On the other hand, there exists a representation of $\mathfrak{G}^*$ in K^* with the character $\mu\xi$. The induced representation of $\mathfrak{G}$ then lies in K^* and its character contains χ with the multiplicity μv. Thus the index of χ

4) For the method used here, cf. Schur [1].

with regard to K^* divides μv and this implies that $m \mid \mu v[K^*:K(\chi)]$. Since the last two factors here are prime to p, $m_p|\mu_p$. This proves (**2A**).

6. It will be shown below that there always exist subgroups $\mathfrak{G}^*$ of a very special type ($\mathfrak{E}$) such that for a suitable character ξ of $\mathfrak{G}^*$ the assumptions of (**2A**) are satisfied. We now study subgroups of this type ($\mathfrak{E}$).

If p is a given prime number we shall say that a group $\mathfrak{H}$ is of *type* ($\mathfrak{E}$) (for p), if $\mathfrak{H}$ contains a normal cyclic subgroup $\mathfrak{A}=\{A\}$ of order a prime to p, such that $\mathfrak{H}/\mathfrak{A}$ is a p-group. It is clear that all such groups $\mathfrak{H}$ are soluble. If $\mathfrak{P}$ is a Sylow subgroup $\mathfrak{H}_p$ of $\mathfrak{H}$, we have

(5) $$\mathfrak{H}=\mathfrak{A}\mathfrak{P}.$$

For each $X\in\mathfrak{H}$, we must have an equation

$$XAX^{-1}=A^{\lambda}$$

where λ is an integer prime to a which is determined (mod a). The mapping $X\to\lambda$ is a homomorphism of $\mathfrak{H}$ on a multiplicative group Λ of residue classes of integers (mod a). The kernel of this homomorphism is the normalizer $\mathfrak{H}_0$ of A in $\mathfrak{H}$. If $\mathfrak{P}_0=\mathfrak{P}\cap\mathfrak{H}_0$, then the product $\mathfrak{A}\mathfrak{P}_0$ is direct and

(6) $$\mathfrak{H}_0=\mathfrak{A}\times\mathfrak{P}_0.$$

Since $\mathfrak{H}_0$ is normal in $\mathfrak{H}$, $\mathfrak{P}_0$ is normal in $\mathfrak{P}$. We have

(7) $$\mathfrak{H}/\mathfrak{H}_0\cong\mathfrak{P}/\mathfrak{P}_0\cong\Lambda.$$

For given p we shall call a group an *elementary group*, if it is the direct product of a p-group with a cyclic group of an order prime to p. We now have

(**2B**) *A group $\mathfrak{H}$ of type* ($\mathfrak{E}$) *for p contains an elementary normal subgroup $\mathfrak{H}_0$ such that $\mathfrak{H}/\mathfrak{H}_0$ is an abelian p-group. Groups of type* ($\mathfrak{E}$) *can be defined by this condition.*

We show next

(**2C**) *The degrees of the irreducible representations of a group $\mathfrak{H}$ of type* ($\mathfrak{E}$) *for p are all powers of p.*

Proof: The corresponding statement is certainly true for $\mathfrak{H}_0$ since $\mathfrak{H}_0$ is the direct product of a p-group with a cyclic group. If φ is an irreducible character of a group $\mathfrak{U}$ and $\mathfrak{B}$ a normal subgroup of index p, then $\varphi(\mathfrak{B})$

is either irreducible or it breaks up into p irreducible constituents of equal degrees. The statement is obtained if this is applied successively to the groups of a composition series leading from $\mathfrak{H}$ to $\mathfrak{H}_0$.

7. In order to construct subgroups $\mathfrak{H}$ of type ($\mathfrak{E}$) of a given group $\mathfrak{G}$, we pick a p-regular element A of $\mathfrak{G}$. Since $\mathfrak{P}$ in (5) must be a p-group contained in the normalizer $\mathfrak{N}(\mathfrak{A})$ of $\mathfrak{A}=\{A\}$ in $\mathfrak{G}$, we obtain the maximal subgroups of type ($\mathfrak{E}$) of $\mathfrak{G}$ by choosing $\mathfrak{P}$ as Sylow group $\mathfrak{N}(\mathfrak{A})_p$ of $\mathfrak{N}(\mathfrak{A})$ and taking $\mathfrak{H}=\mathfrak{A}\mathfrak{P}$.

We shall have to work only with these maximal subgroups of type ($\mathfrak{E}$); subgroups $\mathfrak{H}$ and $\mathfrak{H}^*$ which are conjugate in $\mathfrak{G}$ are equivalent for our purpose. Hence it will not matter, which Sylow group of $\mathfrak{N}(\mathfrak{A})$ is chosen for $\mathfrak{P}$. We may also replace A by a conjugate element in $\mathfrak{G}$. Thus, for given p, the number of groups $\mathfrak{H}$ to be considered is equal to the number l of classes of p-regular conjugate elements in $\mathfrak{G}$. If, for each of these l groups $\mathfrak{H}$, we know how the character breaks up into irreducible characters of $\mathfrak{H}$, we can decide at once which of these $\mathfrak{H}$ can be used for $\mathfrak{G}^*$ in **(2A)**. Our principal result is that such groups $\mathfrak{H}$ always exist. However, this will be proved only at the end of § 4.

We add here a few simple remarks concerning maximal subgroups of $\mathfrak{G}$ of type ($\mathfrak{E}$).

(2D) *If $\mathfrak{H}=\mathfrak{A}\mathfrak{P}$ is a maximal subgroup of $\mathfrak{G}$ of type* ($\mathfrak{E}$), $\mathfrak{A}=\{A\}$, *then $\mathfrak{H}$ contains a Sylow group $\mathfrak{N}(A)_p$ of the normalizer $\mathfrak{N}(A)$ of A in $\mathfrak{G}$. We may take $\mathfrak{P}_0$ for $\mathfrak{N}(A)_p$.*

Proof: If we use the same notation as in **6**, then $\mathfrak{P}_0$ will be contained in a Sylow subgroup $\mathfrak{P}_1$ of $\mathfrak{N}(A)$ and $\mathfrak{P}_1$ in turn is contained in a Sylow subgroup $\mathfrak{P}^*$ of $\mathfrak{N}(\mathfrak{A})$. Since $\mathfrak{P}$ too is a Sylow-subgroup of $\mathfrak{N}(\mathfrak{A})$, both $\mathfrak{P}$ and $\mathfrak{P}^*$ are conjugate in $\mathfrak{N}(\mathfrak{A})$, say, $\mathfrak{P}=N^{-1}\mathfrak{P}^*N$ with $N\epsilon\mathfrak{N}(\mathfrak{A})$. Hence $N^{-1}\mathfrak{P}_1N\subseteq\mathfrak{P}$. As $N^{-1}\mathfrak{P}_1N\subseteq\mathfrak{N}(A)$, it follows that $N^{-1}\mathfrak{P}_1N$ belongs to $\mathfrak{P}\cap\mathfrak{N}(A)$. This intersection lies in $\mathfrak{P}_0$. Thus the order of $\mathfrak{P}_0$ is at least equal to the order of $\mathfrak{P}_1$. Hence $\mathfrak{P}_0=\mathfrak{P}_1$, and this proves **(2D)**.

(2E) *If A is conjugate in $\mathfrak{G}$ to A^λ and if λ belongs to an exponent* (mod a) *which is a power of p, then A and A^λ are conjugate with regard to the Sylow subgroup $\mathfrak{P}$ of $\mathfrak{N}(\mathfrak{A})$.*

Proof: If $GAG^{-1}=A^\lambda$ with $G\epsilon\mathfrak{G}$, it follows that $G^jAG^{-j}=A^{\lambda^j}$. If j is congruent to 1 modulo a sufficiently high power of p, this becomes $G^jAG^{-j}=A^\lambda$. We may impose on j the further condition that it is divisible by all prime powers dividing g and prime to p. Then the order of G^j is a power of p. Since G^j belongs to $\mathfrak{N}(\mathfrak{A})$, it belongs to a conjugate $N\mathfrak{P}N^{-1}$ of the Sylow subgroup $\mathfrak{P}$; $N\in\mathfrak{N}(\mathfrak{A})$. For $X=N^{-1}G^jN$, we have $XAX^{-1}=A^\lambda$, $X\in\mathfrak{P}$, as stated.

In the case of a maximal subgroup $\mathfrak{H}=\{A\}\mathfrak{P}$ of $\mathfrak{G}$ of type $(\mathfrak{E})$, the set Λ in (7) can now be characterized by the condition that it consists of the λ (mod a) such that

(I) λ is prime to a and belongs to an exponent (mod a) which is a power of p.

(II) The elements A and A^λ are conjugate in $\mathfrak{G}$.

For each $\lambda\in\Lambda$, we can choose an $X_\lambda\in\mathfrak{P}$ such that

$$X_\lambda AX_\lambda^{-1}=A^\lambda. \tag{8}$$

These X_λ form a complete residue system of $\mathfrak{P}$ (mod $\mathfrak{P}_0$) and hence of $\mathfrak{H}$ (mod $\mathfrak{H}_0$). For $\lambda,\mu\in\Lambda$, we have

$$X_\lambda X_\mu=X_{\lambda\mu}P_{\lambda,\mu} \tag{9}$$

with $P_{\lambda,\mu}\in\mathfrak{P}_0$. (The indices here are to be taken mod a).

§ 3. Association of the characters of $\mathfrak{G}$ with p-regular elements

8. Let $\chi_1,\chi_2,\cdots,\chi_k$ denote the irreducible characters of $\mathfrak{G}$. Suppose that a fixed prime p has been chosen. We wish to associate each χ_i with some p-regular element A of $\mathfrak{G}$ in a fashion which will enable us to show later that the corresponding maximal subgroup $\mathfrak{H}$ of type $(\mathfrak{E})$ has a character ξ satisfying the assumptions of **(2A)**.

Let $\mathfrak{K}_1,\mathfrak{K}_2,\cdots,\mathfrak{K}_k$ be the classes of conjugate elements of $\mathfrak{G}$ and let G_j be a representative element of $\mathfrak{K}_j$. Then $\mathfrak{K}_j$ consists of $g/n(G_j)$ elements. The p-regular elements among $G_1,G_2,\cdots,G_k$ will be denoted by $A_1,A_2,\cdots,A_l$. For each $A_\varkappa$, we define the section $S(A_\varkappa)$ as the set of those classes $\mathfrak{K}_j$ which contain elements $A_\varkappa P$ such that P belongs to a Sylow group $\mathfrak{N}(A_\varkappa)_p$. Each class $\mathfrak{K}_i$ belongs to one and only one of the sections $S(A_1)$, $S(A_2)$, $\cdots$, $S(A_l)$. Thus, if $S(A_\varkappa)$ consists of $h(A_\varkappa)$ classes $\mathfrak{K}_j$,

$$k=\sum_{\varkappa=1}^{l}h(A_\varkappa).$$

We start from the determinant

$$D=|\chi_i(G_j)|\,;\quad (i,j=1,\ 2,\ \cdots,\ k).$$

It follows from the orthogonality relations for characters that

$$D=\prod_{j=1}^{k} n(G_j)^{\frac{1}{2}} \tag{10}$$

Now use the Laplace expansion of the determinant D with regard to the l sections. In order to have a convenient way of writing the formula, we introduce the following notation. Let $Z(A_\varkappa)$ denote the set of $h(A_\varkappa)$ indices j for which $\mathfrak{K}_j\in S(A_\varkappa)$, taken in some fixed order. If Y is an ordered set of $h(A_\varkappa)$ indices i, $1\leqq i\leqq k$, we set

$$D(Y, Z(A_\varkappa))=|\chi_i(G_j)|\,;\qquad (i\in Y,\ j\in Z(A_\varkappa)). \tag{11}$$

Let $\mathfrak{S}$ denote the symmetric group of all permutations of 1, 2, $\cdots$, k and let $\mathfrak{N}$ denote the subgroup which permutes the elements of each $Z(A_\varkappa)$ among themselves. Then

$$D=\sum_{\pi}{}'\varphi(\pi)\prod_{\varkappa=1}^{l} D(Z(A_\varkappa)\pi,\ Z(A_\varkappa)) \tag{12}$$

where π ranges over a complete residue system of $\mathfrak{S}$ (mod $\mathfrak{N}$), and where $\varphi(\pi)=+1$ for even π, $\varphi(\pi)=-1$ for odd π. If we denote the product in (12) by $T(\pi)$,

$$D=\sum_{\pi}{}'\varphi(\pi)T(\pi)\,; \tag{12*}$$

then $\varphi(\pi)T(\pi)$ remains unchanged, if π is replaced by another element of the same residue class.

Chose a fixed prime ideal divisor $\mathfrak{p}$ of p in the field $\mathrm{P}(\varepsilon)$. As shown previously,[5] the determinants (11) are divisible by a certain power $\mathfrak{p}^*(A_\varkappa)$ of $\mathfrak{p}$ which is defined by the condition that its square $\mathfrak{p}^*(A_\varkappa)^2$ is the $\mathfrak{p}$-part of $\prod_j n(G_j)$ where the product is extended over $j\in Z(A_\varkappa)$. If we set

$$\mathfrak{p}^*=\prod_{\varkappa=1}^{l}\mathfrak{p}^*(A_\varkappa),$$

$$T(\pi)\equiv 0 \quad (\text{mod } \mathfrak{p}^*). \tag{13}$$

9. On the other hand, as shown by (10), D is not divisble by $\mathfrak{p}\mathfrak{p}^*$. If we succeed in distributing the terms of the sum (12*) into disjoint sets

5) Brauer [3].

such that the number of terms in each set is a power of p and that any two terms belonging to the same set are congruent modulo $\mathfrak{p}\mathfrak{p}^*$, it follows that there must exist a set consisting of only one term, such that for this term and all $\varkappa$

(14) $$D(Z(A_\varkappa)\pi,\ Z(A_\varkappa)) \not\equiv 0 \quad (\mathrm{mod}\ \mathfrak{p}\mathfrak{p}^*(A_\varkappa))$$

Every element σ of the Galois group $\mathfrak{L}$ effects a permutation σ^* of the k characters, $\chi_i \to \chi_i^\sigma$. We write $\chi_i^\sigma = \chi_{i\sigma^*}$ where $i\sigma^*$ stands for one of the indices 1, 2, $\cdots$, k. It follows from (4) that

(15) $$\chi_{i\sigma^*}(G) = \chi_i^\sigma(G) = \chi_i(G^{\nu(\sigma)}).$$

Since $\nu(\sigma)$ is prime to g, the mapping $G^{\nu(\sigma)} \to G$ is a permutation of the elements of $\mathfrak{G}$ which maps a class of conjugate elements $\mathfrak{K}_j$ on a class of conjugate elements $\mathfrak{K}_{\bar{j}}$. Let $\bar{\sigma}$ denote this permutation of $\mathfrak{K}_1, \mathfrak{K}_2, \cdots, \mathfrak{K}_k$; we set $\bar{j} = j\bar{\sigma}$. For $G = G_{j\bar{\sigma}}$, (15) implies

(16) $$\chi_{i\sigma^*}(G_{j\bar{\sigma}}) = \chi_i(G_j).$$

Substitution of this in (11) yields

(17) $$D(Y, Z(A_\varkappa)) = D(Y\sigma^*, Z(A_\varkappa)\bar{\sigma}).$$[6]

The permutation $\bar{\sigma}$ will carry the section $S(A_\varkappa)$ into a section $S(A_{\varkappa'})$ where $A_{\varkappa'}$ is determined by the condition $A_{\varkappa'}^{\nu(\sigma)} \sim A_\varkappa$. We can then set

(18) $$Z(A_\varkappa)\bar{\sigma} = Z(A_{\varkappa'})\tau$$

with $\tau \in \mathfrak{R}$. It is seen easily that

(19) $$\bar{\sigma}\mathfrak{R} = \mathfrak{R}\bar{\sigma}.$$

Now, (17) for $Y = Z(A_\varkappa)\pi$ becomes

$$D(Z(A_\varkappa)\pi,\ Z(A_\varkappa)) = D(Z(A_\varkappa)\pi\sigma^*,\ Z(A_{\varkappa'})\tau) = D(Z(A_{\varkappa'})\tau\bar{\sigma}^{-1}\pi\sigma^*,\ Z(A_{\varkappa'})\tau),$$

Here, the factor τ can be removed, since it appears both in the rows and in the columns. If $\varkappa$ ranges from 1 to l, so does $\varkappa'$ and multiplication over $\varkappa$ yields

(20) $$T(\pi) = T(\bar{\sigma}^{-1}\pi\sigma^*)$$

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6) If τ is a permutation of a certain set X, and if X_0 is a subset of X, we write $X_0\tau$ for the set obtained from X_0 by application of τ.

where $T(\pi)$ is the same as in (12*).

This holds for all $\sigma\in\mathfrak{L}$. We now restrict σ to the Sylow group $\mathfrak{L}_p$ of $\mathfrak{L}$. If p is odd, both $\bar{\sigma}$ and σ^* are even and hence $\varphi(\pi)=\varphi(\bar{\sigma}^{-1}\pi\sigma^*)$. In any case, we have

$$(21) \qquad \varphi(\pi)\equiv\varphi(\bar{\sigma}^{-1}\pi\sigma^*) \pmod{p}$$

since for $p=2$ both sides are ± 1.

We call two permutations π, $\pi'\in\mathfrak{S}$ equivalent, if there exists a $\sigma\in\mathfrak{L}_p$ such that $\mathfrak{R}\pi'=\mathfrak{R}\bar{\sigma}^{-1}\pi\sigma^*$. Because of (19), this is an equivalence relation. By (20), (21), and (13),

$$\varphi(\pi)T(\pi)\equiv\varphi(\pi')T(\pi') \pmod{\mathfrak{p}\mathfrak{p}^*}$$

for equivalent π, π'. For a fixed $\pi_0\in\mathfrak{S}$, the $\sigma\in\mathfrak{L}_p$ with $\mathfrak{R}\pi_0=\mathfrak{R}\bar{\sigma}^{-1}\pi_0\sigma^*$ form a subgroup $\mathfrak{L}_p^0$ of $\mathfrak{L}_p$. Then the number of terms of (12*) for which π is equivalent to π_0 is equal to $(\mathfrak{L}_p:\mathfrak{L}_p^0)$, that is, it is a power of p.

If we collect the terms of (12*) which belong to equivalent permutations, we now see that the conditions set down at the beginning of **9** are satisfied. Hence it is possible to choose a permutation $\pi=\pi_0$ such that (14) holds and that $\mathfrak{L}_p=\mathfrak{L}_p^0$. Hence $\mathfrak{R}\pi=\mathfrak{R}\bar{\sigma}^{-1}\pi\sigma^*$ for all $\sigma\in\mathfrak{L}_p$ and this yields $\pi\sigma^*\in\mathfrak{R}\bar{\sigma}\pi$. Now (18) shows that if $i\in Z(A_\varkappa)\pi$, then $i\sigma^*\in Z(A_{\varkappa'})\pi$. We associate with $A_\varkappa$ the $h(A_\varkappa)$ characters χ_i with $i\in Z(A_\varkappa)\pi$, $(\varkappa=1, 2, \cdots, l)$.

We have thus shown

(3A) *Let $A_1, A_2, \cdots, A_l$ represent the different classes of p-regular conjugate elements in $\mathfrak{G}$. If the section of $A_\varkappa$ consists of $h(A_\varkappa)$ classes, we can associate $h(A_\varkappa)$ irreducible characters of $\mathfrak{G}$ with $A_\varkappa$, $(\varkappa=1, 2, \cdots, l)$, such that each character χ_i of $\mathfrak{G}$ is associated with exactly one $A_\varkappa$ and that the following two conditions (α), (β) hold*

(α) *If χ_i is associated with $A_\varkappa$, then, for $\sigma\in\mathfrak{L}_p$, χ_i^σ is associated with that element $A_{\varkappa'}$ for which $A_{\varkappa'}^{\nu(\sigma)}\sim A_\varkappa$.*

(β) *If χ_i ranges over the characters associated with $A_\varkappa$, and if G_j ranges over the representatives of the classes of the section of $A_\varkappa$, we have*

$$(22) \qquad |\chi_i(G_j)| \not\equiv 0 \pmod{\mathfrak{p}\mathfrak{p}^*(A_\varkappa)}$$

where $\mathfrak{p}$ is a prime ideal divisor of p in the field $\mathrm{P}(\varepsilon)$ and $\mathfrak{p}^*(A_\varkappa)^2$ is the $\mathfrak{p}$-part of $\prod_j n(G_j)$.

The following statement is a special case of (α) :

(3B) *Suppose that χ_i is associated with A_κ. For $\sigma\in\mathfrak{L}_p$, the character χ_i^σ is associated with A_κ, if and only if $A_\kappa\sim A_\kappa^{\nu(\sigma)}$. In particular, if $\chi_i^\sigma=\chi_i$ for $\sigma\in\mathfrak{L}_p$, then $A_\kappa\sim A_\kappa^{\nu(\sigma)}$.*

§ 4. Proof of the main result

10. Let $A=A_\kappa$ be one of the elements $A_1, \cdots, A_l$. With A there are associated $h(A)$ characters χ_i and for suitable choice of A, any given irreducible character of $\mathfrak{G}$ appears among the χ_i. Changing the notation, we may assume that the characters χ_i, $i=1, 2, \cdots, h(A)$, are associated with A. If A is now fixed, we construct a corresponding maximal subgroup $\mathfrak{H}$ of $\mathfrak{G}$ of type $(\mathfrak{E})$ as described in **7.** By **(2D)**, and (6), $\mathfrak{H}_0$ is a direct product of $\mathfrak{A}=\{A\}$ and a Sylow subgroup $\mathfrak{P}_0=\mathfrak{N}(A)_p$.

Let $\vartheta_1, \vartheta_2, \cdots, \vartheta_t$ denote the irreducible character of $\mathfrak{P}_0$. Each irreducible character ψ of $\mathfrak{H}_0$ then is a product of a linear character ζ of $\mathfrak{A}$ and one of the ϑ_i

$$\psi(A^\tau P)=\zeta(A)^\tau\vartheta_i(P) \qquad (\text{for } P\in\mathfrak{P}_0).$$

Since each $\chi_i(\mathfrak{H}_0)$ must break up into characters ψ, we can set

$$\chi_i(AP)=\sum_{j=1}^{t} z_{ij}\vartheta_j(P) \qquad (\text{for } P\in\mathfrak{P}_0) \tag{23}$$

where the z_{ij} are algebraic integers, $z_{ij}\in\mathrm{P}(\varepsilon)$.

11. The elements $\sigma\in\mathfrak{L}_p$, for which $A\sim A^{\nu(\sigma)}$, form a subgroup $\mathfrak{L}_p^*$. If we apply $\sigma\in\mathfrak{L}_p^*$ to (23) and use the same notation as in (15), we find

$$\chi_i(AP)^\sigma=\chi_{i\sigma^*}(AP)=\sum_{j=1}^{t} z_{i\sigma^*,j}\vartheta_j(P). \tag{24a}$$

On the other hand, by (4), $\chi_i(AP)^\sigma=\chi_i(A^{\nu(\sigma)}P^{\nu(\sigma)})$. For $\sigma\in\mathfrak{L}_p^*$, the exponent $\nu(\sigma)$ satisfies the conditions (I), (II) in **7.** Hence $\nu(\sigma)\in\Lambda$. By (8), $A^{\nu(\sigma)}=X_{\nu(\sigma)}AX_{\nu(\sigma)}^{-1}$ and, consequently

$$\chi_i(AP)^\sigma=\chi_i(X_{\nu(\sigma)}AX_{\nu(\sigma)}^{-1}P^{\nu(\sigma)})=\chi_i(AX_{\nu(\sigma)}^{-1}P^{\nu(\sigma)}X_{\nu(\sigma)}).$$

Now (23) yields

$$\chi_i(AP)^\sigma=\sum_{j=1}^{t} z_{ij}\vartheta_j(X_{\nu(\sigma)}^{-1}P^{\nu(\sigma)}X_{\nu(\sigma)}).$$

Since, for fixed σ, the mapping $P\to X_{\nu(\sigma)}^{-1}PX_{\nu(\sigma)}$ is an automorphism of $\mathfrak{P}_0$, the expression

$$\vartheta_{j'}(P)=\vartheta_j(X_{\nu(\sigma)}^{-1}P^{\nu(\sigma)}X_{\nu(\sigma)})=\vartheta_j(X_{\nu(\sigma)}^{-1}PX_{\nu(\sigma)})^\sigma$$

is again an irreducible character of $\mathfrak{P}_0$. Hence we have a permutation σ' of $\vartheta_1, \vartheta_2, \cdots, \vartheta_t$; we set $j'=j\sigma'$. Furthermore, it follows from (9) that for $\sigma_1, \sigma_2 \in \mathfrak{L}_p^*$, we have $(\sigma_1\sigma_2)'=\sigma_1'\sigma_2'$; the σ' form a representation of $\mathfrak{L}_p^*$ by permutations. We can now write

(24b) $$\chi_i(AP)^\sigma=\sum_{j=1}^{t} z_{ij}\vartheta_{j\sigma'}(P)$$

and on comparing this with (24a), we obtain

(25) $$z_{ij}=z_{i\sigma^*,\,j\sigma'}\quad (1\leq i\leq h(A),\ 1\leq j\leq t;\ \sigma\in\mathfrak{L}_p^*)$$

Let X denote any set of $h(A)$ indices j. We can then form the minor $W(X)$ of the $(h(A)\times t)$-matrix (z_{ij}) which contains the columns $j\in X(A)$.[7] We shall consider X as an unordered set. Then $W(X)$ is determined only apart from a $\pm$ sign. The method applied in an earlier investigation together with (22) yields

(26) $$\sum_X W(X)^2 \not\equiv 0 \qquad (\mathrm{mod}\ \mathfrak{p}).^{8)}$$

On the other hand, (25) gives

$$W(X\sigma')=\pm W(X) \qquad (\text{for } \sigma\in\mathfrak{L}_p^*).$$

Now an argument similar to that used in **9** in connection with the sum (12*) shows that there must exist a minor $W(X)\not\equiv 0$ (mod $\mathfrak{p}$) such that σ' permutes the corresponding ϑ_j among themselves. Taking the ϑ_j in suitable order, we may assume that $W(X)$ occupies the first $h(A)$ columns.

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7) As shown in Brauer [3], we have $t\geqq h(A)$.

8) We use the formulas (23), (24) in Brauer [4]. The determinant Δ there is the same as the determinant (22) of the present paper. If U in (23) of the previous paper is specialized suitably, we obtain a formula

$$aW(X)=\pm\beta M(X).$$

Here, $M(X)$ is the minor of degree $t-h(A)$ of Θ_1^* which contains the characters ϑ_j with $j\notin X$. The numbers α, β are algebraic integers with $(\beta, \mathfrak{p})=1$ which do not depend on X and which can be given explicitly. On the other hand, the $\mathfrak{p}$-part of

$$|\Theta_1^{*\prime}\Theta_1^*|=\sum M(X)^2$$

has been determined in Brauer [3], pp. 59-61. It follows from (22) of the present paper that this si the same as the $\mathfrak{p}$-part of α_*^2 and this gives the desired result.

The square matrix

$$(z_{ij}), \quad (i, j=1, 2, \cdots, h(A)),$$

now has the following properties : (a) The coefficients are algebraic integers of a certain number field. (b) The determinant is not divisible by a prime ideal divisor $\mathfrak{p}$ of p. (c) There exist two permutation representations $\{\sigma^*\}$ and $\{\sigma'\}$ of a certain p-group $\mathfrak{L}_p^*$ such that application of σ^* to the rows and of σ' to the columns maps each coefficient z_{ij} on an equal one, cf. (25). A simple lemma[9] states that we then may arrange the columns in such an order that

$$z_{ii} \not\equiv 0 \pmod{\mathfrak{p}} \tag{27}$$

for $i=1, 2, \cdots, h(A)$ and that the two equations $i\sigma^*=i$, $i\sigma'=i$ imply each other.

12. Let $\chi=\chi_i$ be a fixed character associated with A and let K^* have the same significance as in **(2A)**. The Galois group $\mathfrak{M}$ of $\mathsf{K}^*(\varepsilon)$ with regard to K^* may be considered as a subgroup of $\mathfrak{L}_p$. Since χ_i lies in K^*, we have $\chi_{i\sigma^*}=\chi_i$ for $\sigma\in\mathfrak{M}$ and **(3B)** shows that $\mathfrak{M}\subseteq\mathfrak{L}_p^*$. Furthermore, the last statement in **11** gives

$$i\sigma'=i \qquad (\text{for } \sigma\in\mathfrak{M}). \tag{28}$$

If we break up $\chi_i(\mathfrak{H})$ into irreducible characters of $\mathfrak{H}$, characters which are algebraically conjugate with regard to K^* appear with the same multiplicity. We write the formula in the form

$$\chi_i(\mathfrak{H})=\sum v_\nu(\xi_\nu+\xi_\nu'+\xi_\nu''+\cdots) \tag{29}$$

where the $\xi_\nu, \xi_\nu', \xi_\nu'', \cdots$ are irreducible characters of $\mathfrak{H}$ and where in each parenthesis characters have been collected which are algebraically conjugate with regard to K^*. Hence the number of characters in each parenthesis is a power of p, and each character in the parenthesis containing ξ_ν has the form ξ_ν^σ with $\sigma\in\mathfrak{M}$.

Replace for a moment $\mathfrak{G}$ by $\mathfrak{H}$. For $\xi_\nu(AP)$, we must have formulas analogous to (23), say

$$\xi_\nu(AP)=\sum \tilde{z}_{\nu j}\vartheta_j(P).$$

Applying (24b) in this case, we have

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9) The proof of the lemma is not difficult. It will be given in the continuation of the paper.

(30) $$\xi_\nu(AP)^\sigma=\sum_j z_{\nu,j}\,\vartheta_{j\sigma'}(P).$$

In particular, for $\sigma\in\mathfrak{M}$, the term $\vartheta_i(P)$ appears with the coefficient $z_{\nu i}$ because of (28). Substitute (30) in (29) for the element $AP\in\mathfrak{H}_0\subseteq\mathfrak{H}$. On comparing the coefficient of $\vartheta_i(P)$ in $\chi_i(AP)$ here and in (23), we have

$$z_{ii}=\sum_\nu v_\nu(z_{\nu i}+z_{\nu i}+\cdots).$$

Now, (27) shows that there must appear at least one ξ_ν in (29) such that $v_\nu\not\equiv 0 \pmod p$ and that there is only one term in its bracket. The latter statement means that ξ_ν belongs to K^*.

We have now shown that ξ_ν satisfies the conditions of **(2A)** and hence **(2A)** can be used to find the p-part of the index m of χ. The result **(2C)** yields a slight simplification: We must have $\mu_p=\mu$, since the degree of ξ_ν is a power of p.

We thus have

Theorem: *If χ is an irreducible character of the group $\mathfrak{G}$, if K is a field of characteristic* 0, *then for every prime p there exists a subgroup $\mathfrak{H}$ of type* ($\mathfrak{E}$) *and an irreducible character ξ of $\mathfrak{H}$ such that the p-part of the Schur index of χ with regard to K is equal to the Schur index μ of ξ with regard to $\mathsf{K}(\chi)$.*

If we take $\mathsf{K}=\mathsf{P}$ and determine the character ξ in this case, the same character ξ can be used for every field of characteristic 0. Hence we have the

Remark: The character ξ in the Theorem can be chosen independent of the field K.

As already remarked, the selection of ξ can be made if we know how $\chi(\mathfrak{H})$ breaks up into irreducible characters of $\mathfrak{H}$ for every maximal subgroup of type ($\mathfrak{E}$) of $\mathfrak{G}$.

Thus, the whole problem of the Schur indices has been reduced to the case where the group is of type ($\mathfrak{E}$).

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