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Multiple Wiener Integral

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The notion of *multiple Wiener integral* was introduced first by N. Wiener[1] who termed it *polynomial chaos*. Our definition in the present paper is obtained by a slight modification of Wiener's one, and seems to be more convenient in the point that our integrals of different degrees are orthogonal to each other while Wiener's polynomial chaos has not such a property.

In § 1 we shall define a normal random measure as a generalization of a brownian motion process. In § 2 we shall define multiple Wiener integral and show its fundamental property. In § 3 we shall establish a close relation between our integrals and Hermite polynomials. By making use of this relation we shall give, in § 4, an orthogonal expansion of any L_2-functional of the normal random measure, which proves to be coincident with the expansion given by S. Kakutani[2] for the purpose of the spectral resolution of the shift operator in the L_2 over the brownian motion process. In § 5 we shall treat the case of a brownian motion process, and in this case we shall show that we can define the multiple Wiener integral by the iteration of stochastic integrals.[3]

§ 1. Normal random measure

A system of real random veriables $\xi_\alpha(\omega)$, $\alpha \in A$, ω being a probability parameter, is called normal when the joint destribution of $\xi_{\alpha_1}, \dots, \xi_{\alpha_n}$; $\alpha_1, \dots, \alpha_n \in A$, is always a multivariate Gaussian distribution (including degenerate cases) with the mean vector $(0, \cdots, 0)$.

By making use of Kolmogoroff's theorem[4] of introducing a probability distribution in R^A, we can easily prove the following

1) N. Wiener: The homogeneous chaos, Amer. Journ. Math. Vol. **LV**, No. 4, 1938.

2) S. Kakutani: Determination of the spectrum of the flow of Brownian motion, Proc. Nat. Acad. Sci., U.S.A. **36** (1950), 319–323.

3) K. Itô: Stochastic integral, Proc. Imp. Acad. Tokyo, Vol. **XX**, No. 8, 1944.

4) A. Kolmogoroff: Grundbegriffe der Wahrscheinlichkeitsrechnung, Berlin, 1933. The consistency-condition is well satisfied by virtue of the property of multivariate Gaussian distribution.

Theorem 1. 1 *If* $v_{\alpha\beta}$; $\alpha, \beta \in A$, *satisfies the following two conditions*:

symmetric : $v_{\alpha\beta}=v_{\beta\alpha}$; (1.1)

positive-definite : $\sum x_i \bar{x}_j v_{\alpha_i \alpha_j} \geqq 0$ (*for any* $\alpha_1, \cdots, \alpha_n \in A$ *and for any complex numbers* $x_1, x_2, \cdots, x_n$), *then there exists a normal random system* ξ_α, $\alpha \in A$, *which satisfies* (1.2)

$$v_{\alpha\beta}=\mathfrak{E}(\xi_\alpha \xi_\beta)=\int \xi_\alpha(\omega)\xi_\beta(\omega)d\omega. \tag{1.3}$$

Definition. Let $(T, \boldsymbol{B}, m)$ be a measure space. We denote by $\boldsymbol{B}^*$ the system $\{E ; E \in \boldsymbol{B}, m(E) < \infty\}$. A normal system $\beta(E, \omega)$, $E \in \boldsymbol{B}^*$, is called a *normal random measure* on $(T, \boldsymbol{B}, m)$, if

$$\mathfrak{E}(\beta(E)\ \beta(E'))=m(E \cap E') \text{ for any } E, E' \in \boldsymbol{B}^*. \tag{1.4}$$

Remark. Since we have $m(E \cap E')=m(E' \cap E)$ and $\sum x_i\bar{x}_j m(E_i \cap E_j) = \int |\sum_i x_i C_i(t)|^2 m(dt) \geqq 0$, $C_i(t)$ being the characteristic function of the set E_i, we can see, by Theorem 1.1, the existence of a normal random measure on any measure space $(T, \boldsymbol{B}, m)$.

The following theerem, which can be easily shown, justifies the name of normal random " measure."

Theorem. 1. 2 *Let* $\beta(E)$ *be a normal random measure on* $(T, \boldsymbol{B}, m)$. *If* $E_1, E_2, \cdots$ *are disjoint, then* $\beta(E_1), \beta(E_2), \cdots$ *are independent. Furthermore if* $E=E_1+E_2+\cdots \in \boldsymbol{B}^*$, *then* $\beta(E)=\sum_n \beta(E_n)$ (*in mean convergence*).

Remark. Since $\beta(E_1), \beta(E_2), \cdots$, are independent, then the mean convergence of $\sum \beta(E_n)$ implies the almost certain convergence by virtue of Levy's theorem.[5)]

Hereafter we set the following restriction on the measure m.

Continuity. For any $E \in \boldsymbol{B}^*$ and $\varepsilon > 0$ there exists a decomposition of E:

$$E=\sum_{i=1}^{n} E_i \tag{1.5}$$

such that

$$m(E_i) < \varepsilon,\ i=1, 2, \cdots, n. \tag{1.6}$$

5) P. Lévy: Théorie de l'addition des variariables aléatoires, Paris, 1937.

§ 2. Definition of multiple Wiener integral

By $L^2(T^p)$ we denote the totality of square-summable complex-valued functions defined on the product measure space $(T, \boldsymbol{B}, m)^p$. An elementary function[6] $f(t_1,\cdots,t_p)$ is called *special* if $f(t_1,\cdots,t_p)$ vanishes except for the case that $t_1,\cdots,t_p$ are all different. We shall denote by S_p the totality of special elementary functions.

Theorem 2. 1. *S_p is a linear manifold dense in $L^2(T^p)$.*

Proof. It suffices to show that the characteristic function $c(t_1,\cdots,t_p)$ of any set E of the form:

$$E=E_1\times E_2\times\cdots\times E_p \quad (E_i \in \boldsymbol{B}^*,\ i=1,2,\cdots,p) \tag{2.1}$$

can be approximated (in the L_2-norm) by a special elementary function.

For any $\varepsilon>0$ we can determine, by the *continuity condition*, a set-system $\boldsymbol{F}=\{F_1,\cdots,F_n\}\in\boldsymbol{B}^*$ which satisfies

$\boldsymbol{F}$. 1. $F_1, F_2,\cdots,F_n$ are disjoint,

$\boldsymbol{F}$. 2. $m(F_i)<\varepsilon_1\equiv\varepsilon/\binom{p}{2}\cdot(\sum m(E_i))^{p-1},\ \binom{p}{2}=\dfrac{p(p-1)}{2\cdot 1}$,

$\boldsymbol{F}$. 3. each E_i is expressible as the sum of a subsystem of $\boldsymbol{F}$.

Then $c(t_1,\cdots,t_p)$ is expressible in the form:

$$c(t_1,\cdots,t_p)=\sum \varepsilon_{i_1\ldots i_p} c_{i_1}(t_1)\ldots c_{i_1}(t_p) \tag{2.2}$$

where $\varepsilon_{i_1\ldots i_p}=0$ or 1 and $c_i(t)$ is the characteirstic function of F_i, $i=1, 2,\ldots,n$. We devide $\sum$ into two parts: $\sum'$ and $\sum''$: $\sum'$ corresponds to the indices $\{i_1,\ldots,i_p\}$ which are all different, while $\sum''$ corresponds to the others.

We put

$$f(t_1,\ldots,t_p)=\sum{}' \varepsilon_{i_1\ldots i_p} c_{i_1}(t_1)\ldots c_{i_p}(t_p). \tag{2.3}$$

Then $f\in S_p$ and

$$\|c-f\|^2=\int\cdots\int|c(t_1,\ldots,t_p)-f(t_1,\cdots,t_p)|^2\, m(dt_1)\ldots m(dt_p)$$
$$=\sum{}'' \varepsilon_{i_1\ldots i_p} m(F_{i_1})\ldots m(F_{i_p})$$

6) An elementary function of $(t_1,\ldots,t_p)$ is defined as a linear combination of the characteristic functions of the sets of the form $E_1\times\ldots\times E_p$, $E_i\in\boldsymbol{B}^*$, $i=1\ 2,\ldots,n$.

$$\leqq \binom{p}{2} \sum m(F_i)^2 (\sum m(F_i))^{p-2}$$

$$\leqq \binom{p}{2} \varepsilon_1 (\sum m(F_i))^{p-1} = \binom{p}{2} \varepsilon_1 (\sum m(E_i))^{p-1} < \varepsilon.$$

Now we shall define the multiple wiener integral of $f \in L^2(T^p)$, which we denote by

$$I_p(f) \quad \text{or} \quad \int \cdots \int f(t_1, \ldots, t_p) d\beta(t_1) \ldots d\beta(t_p)$$

Let f be a special elementary function. Then f can be expressible as follows:

$$f(t_1, \ldots, t_p) = a_{i_1 \ldots i_p} \text{ for } (t_1, \ldots, t_p) \in T_{i_1} \times \ldots \times T_{i_p}, \tag{2.4}$$

$$= 0 \qquad \text{elsewhere,}$$

where $T_1, T_2, \ldots, T_n$ are disjoint and $m(T_i) < \infty$, $i = 1, 2, \ldots, n$, and $a_{i_1 \ldots i_p} = 0$ if any two of $i_1, \ldots, i_p$ are equal. We define $I_p(f)$ for such f by

$$I_p(f) = \sum a_{i_1 \ldots i_p} \beta(T_{i_1}) \ldots \beta(T_{i_p}). \tag{2.5}$$

Then we obtain

$$I_p(af + bg) = aI_p(f) + bI_p(g) \tag{I.1}$$

$$I_p(f) = I_p(\tilde{f}), \tag{I.2}$$

where

$\tilde{f}(t_1, \ldots, t_p) = \frac{1}{\underline{|p}} \sum_{(\pi)} f(t_{\pi_1}, \ldots, t_{\pi_p})$, $(\pi) = (\pi_1, \ldots, \pi_p)$ running over all permutations of $(1, 2, \ldots, p)$ ($\underline{|p} = 1 \cdot 2 \cdot \ldots \cdot p$).

$$(I_p(f), I_p(g)) = \underline{|p} \, (\tilde{f}, \tilde{g}), \tag{I.3}$$

where $(I_p(f), I_p(g)) \equiv \mathfrak{E}(I_p(f) \, \overline{I_p(g)}) \equiv \int I_p(f) \cdot \overline{I_p(g)} d\omega$, and

$$(\tilde{f}, \tilde{g}) \equiv \int \cdots \int \tilde{f}(t_1, \ldots, t_p) \, \overline{\tilde{g}(t_1, \ldots, t_p)} m(dt_1) \ldots m(dt_p).$$

$$(I_p(f), I_p(g)) = 0, \text{ if } p \neq q. \tag{I.4}$$

$(I.1)$ is clear. In order to show $(I.2)$ and $(I.3)$ we may assume that f and g are expressible as follows:

$$f(t_1,\dots,t_p)=a_{i_1,\dots,i_p},\ g(t_1,\dots,t_p)=b_{i_1,\dots,i_p}$$

$$\text{for}\quad (t_1,\dots,t_p)\in T_{i_1}\times\cdots\times T_{i_p}$$

and

$$f(t_1,\dots,t_p)=0,\ g(t_1,\dots,t_p)=0\quad\text{elsewhere.}$$

Then we have

$$\begin{aligned}
I_p(f)&=\sum_{i_1<\dots<i_p}\Big(\sum_{(j)\sim(i)}a_{j_1,\dots,j_p}\Big)\,\beta(T_{i_1})\dots\beta(T_{i_p})^{7)}\\
&=\underline{|p}\sum_{i_1<\dots<i_p}\Big(\frac{1}{\underline{|p}}\sum_{(j)\sim(i)}a_{j_1,\dots,j_p}\Big)\,\beta(T_{i_1})\dots\beta(T_{i_p})\\
&=\sum_{i_1,\dots,i_p}\Big(\frac{1}{\underline{|p}}\sum_{(j)\sim(i)}a_{j_1,\dots,j_p}\Big)\,\beta(T_{i_1})\dots\beta(T_{i_p})\\
&=I(\tilde{f}),\quad (\underline{|p}=1\cdot 2\cdot\dots\cdot p)
\end{aligned}$$

which proves $(I.2)$.

$$\begin{aligned}
(I_p(f),\ I_p(g))&=\Big(\sum_{i_1<\dots<i_p}\Big(\sum_{(j)\sim(i)}a_{j_1,\dots,j_p}\Big)\,\beta(T_{i_1})\dots\beta(T_{i_p}),\\
&\qquad\qquad \sum_{i_1<\dots<i_p}\Big(\sum_{(j)\sim(i)}b_{j_1,\dots,j_p}\Big)\,\beta(T_{i_1})\dots\beta(T_{i_p})\Big)\\
&=\sum_{i_1<\dots<i_p}\Big(\sum_{(j)\sim(i)}a_{j_1,\dots,j_p}\Big)\cdot\Big(\sum_{(j)\sim(i)}\bar{b}_{j_1,\dots,j_p}\Big)\,m(T_{i_1})\dots m(T_{i_p})\\
&=\frac{1}{\underline{|p}}\sum_{i_1,\dots,i_p}\Big(\sum_{(j)\sim(i)}a_{j_1,\dots,j_p}\Big)\Big(\sum_{(j)\sim(i)}\bar{b}_{j_1,\dots,j_p}\Big)\,m(T_{i_1})\dots m(T_{i_p})\\
&=\underline{|p}\sum_{i_1,\dots,i_p}\Big(\frac{1}{\underline{|p}}\sum_{(j)\sim(i)}a_{j_1,\dots,j_p}\Big)\Big(\frac{1}{\underline{|p}}\sum_{(j)\sim(i)}b_{j_1,\dots,j_p}\Big)\,m(T_{i_1})\dots m(T_{i_p})\\
&=\underline{|p}\int\dots\int\tilde{f}(t_1,\dots,t_p)\cdot\overline{\tilde{g}(t_1,\dots,t_p)}\,m(dt_1)\dots m(dt_p)\\
&=\underline{|p}\,(\tilde{f},\tilde{g}).
\end{aligned}$$

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7) $(j)\sim(i)$ means that $(j)\equiv(j_1,\dots j_q)$ is a permutation of $(i)\equiv(i_1,\dots,i_p)$.

Thus $(I.3)$ is proved.

By the similar computations we can prove $(I.4)$.
By putting $f=g$ in $(I.3)$, we obtain

$$\|I_p(f)\|^2=\underline{|p}\ \|\tilde{f}\|^2\leqq\underline{|p}\ \|f\|^2, \tag{$I.3'$}$$

the last inequality being obtained by virtue of Schwarz' inequality.

Therefore I_p can be considered as a bounded linear operator from S_p into $L_2(\omega)$, and so it can be extended to an operator from the closure of $S_p(=L^2(T^p)$ by Theorem 2.1) into $L_2(\omega)$ which satisfies also $(I.1)$, $(I.2)$, $(I.3)$, $(I.4)$ and $(I.3')$.

For the later use we denote by L^2_0 the totality of complex numbers and we define as $I_0(c)=c$. Thus $(I.1)$, $(I.2)$, $(I.3)$, $(I.4)$ and $(I.3')$ are true for $p, q=0, 1, 2,\ldots.$

§ 3. Relation between multiple Wiener integrals and Hermite polynomials.

Theorem 3.1. *Let $\varphi_1(t)$, $\varphi_2(t),\ldots,$ $\varphi_n(t)$ be an orthogonal system of real-valued functions in $L^2(T)$ and $H_p(x)$ be the Hermite polynomial of degree p. Then we have*

$$\int\cdots\int\varphi_1(t_1)\cdots\varphi_1(t_{p_1})\cdot\varphi_2(t_{p_1+1})\cdots\varphi_2(t_{p_1+p_2})\cdots$$

$$\times\varphi_n(t_{p_1+\cdots+p_{n-1}+1})\cdots\varphi_n(t_{p_1+\cdots+p_n})d\beta(t_1)\ldots d\beta(t_{p_1+\cdots+p_n})$$

$$=\prod_{\nu=1}^{n}\frac{H_{p_\nu}\left(\frac{1}{\sqrt{2}}\int\varphi_\nu(t)d\beta(t)\right)}{\sqrt{2}^{p_\nu}}.$$

For the proof of this theorem we prepare the following
Theorem 2.2. I, *If $\varphi(t_1,\ldots,t_p)\in L^2(T^p)$ and $\psi(t)\in L^2(T)$, then*

$$\int\cdots\left[|\varphi(t_1,\ldots,t_p)\psi(t_k)|m(dt_k)\right]^2 m(dt_1)\ldots m(dt_{k-1})m(dt_{k-1})\ldots m(dt_p)$$

$$\leqq\|\psi\|^2\cdot\|\varphi\|^2<\infty.$$

Therefore

II. $\varphi\underset{(k)}{\times}\psi(t_1\ldots t_{k-1}t_{k+1}\ldots t_p)\equiv\int\varphi(t_1,\ldots,t_p)\psi(t_k)m(dt_k)$

is a square-summable function of $t_1, \cdots, t_{k-1}, t_{k+1}, \cdots, t_p$, *and it holds*

$$\|\varphi \underset{(k)}{\times} \psi\| \leqq \|\varphi\| \cdot \|\psi\|. \tag{3.3}$$

III. *We have*

$$I_{p+1}(\varphi\psi) = I_p(\varphi) \cdot I_1(\psi) - \sum_{k=1}^{p} I_{p-1}(\varphi \underset{(k)}{\times} \psi) \tag{3.4}$$

Proof. (3.2) is clear by virtue of Schwarz' inequality and (3.3) is also true by the definition of the norm $\|.\|$ in L_2. For the proof of (3.4) we consider firstly the case when φ and ψ are special elementary functions. Then we may express φ and ψ in the form

$$\begin{aligned} \varphi(t_1, \ldots, t_p) &= a_{i_1 \ldots i_p} \quad \text{for } (t_1, \ldots, t_p) \in T_{i_1} \times \ldots \times T_{i_p}, \\ &= 0 \quad \text{elsewhere}, \\ \psi(t) &= b_i \quad \text{for} \quad t_i \in T_i \\ &= 0 \quad \text{elsewhere}, \end{aligned}$$

where $T_1, T_2, \ldots, T_N$ are disjoint and $m(T_i) < \infty$, $i=1, 2, \ldots, N$ and $a_{i_1 \ldots i_p} = 0$ if any two of $i, \ldots, i_p$ are equal.

Put $S = T_1 + \ldots + T_N$, $A = \max |a_i|$, and $B = \max |b_i|$. Then

$$m(S),\ A,\ B < \infty$$

On account of the continuity-condition of m we may assume that

$$m(T_i) < \varepsilon,\ i=1, 2, \ldots, N,$$

for any asigned $\varepsilon > 0$, by subdividing each T_i, if necessary. S, A and B remain invariant by this subdivision.

Now we define a special elementary function χ_ε by

$$\begin{aligned} \chi_\varepsilon(t_1, \ldots, t_p, t) &= a_{i_1 \ldots i_p} b_i, \quad \text{if} \quad (t_1, \ldots, t_p, t) \in T_{i_1} \times \ldots \times T_{i_p} \times T_i, \\ &\qquad \text{and} \quad i \neq i_1, \ldots, i_p. \\ &= 0, \quad \text{if otherwise.} \end{aligned}$$

Then we have

$$I_p(\varphi) \cdot I_1(\psi) = \sum a_{i_1 \ldots i_p} \beta(T_{i_1}) \ldots \beta(T_{i_p}) \sum b_i \beta(T_i)$$

$$= \sum_{i \neq i_1, \dots, i_p} a_{i_1 \dots i_p} b_i \beta(T_{i_1}) \dots \beta(T_{i_p}) \beta(T_i)$$

$$+ \sum_{k=1}^{p} \sum a_{i_1 \dots i_p} b_{i_k} \beta(T_{i_1}) \dots \beta(T_{i_{k-1}}) \beta(T_{i_k})^2 \beta(T_{i_{k+1}}) \dots \beta(T_{i_p})$$

$$= I_{p+1}(\chi_\varepsilon) + \sum_{k=1}^{p} \sum a_{i_1 \dots i_p} b_{i_k} \beta(T_{i_1}) \dots \beta(T_{i_{k-1}}) m(T_{i_k}) \beta(T_{i_{k+1}}) \dots \beta(T_{i_p})$$

$$+ \sum_{k=1}^{p} \sum a_{i_1 \dots i_p} b_{i_k} \beta(T_{i_1}) \dots \beta(T_{i_{k-1}}) (\beta(T_{i_k})^2 - m(T_{i_k})) \beta(T_{i_{k+1}}) \dots \beta(T_{i_p})$$

$$= I_{p+1}(\chi_\varepsilon) + \sum_{k=1}^{p} I_{p-1}(\varphi \underset{(k)}{\times} \psi) + \sum R_k$$

$$\| I_{p+1}(\chi_\varepsilon) - I_{p+1}(\varphi\psi) \|^2 = \lfloor p \; \| \chi_\varepsilon - \varphi \cdot \psi \|^2$$

$$= \sum_{k=1}^{p} \sum a^2_{i_1 \dots i_p} b^2_{i_k} m(T_{i_1}) \dots m(T_{i_{k-1}}) m(T_{i_k})^2 \dots m(T_{i_p})$$

$$\leqq p A^2 B^2 (\sum m(T_i))^{p-1} \cdot (\sum m(T_i))^2$$

$$\leqq \varepsilon p A^2 B^2 (\sum m(T_i))^p = \varepsilon p A^2 B^2 m(S)^p$$

$$\| R_k \|^2 = c \sum a^2_{i_1 \dots i_p} b^2_{i_k} m(T_{i_1}) \dots m(T_{i_{k-1}}) m(T_{i_k})^2 \dots m(T_{i_p})$$

$$\left(c = \frac{1}{2\pi} \int_{-\infty}^{\infty} (x^2 - 1)^2 e^{-x^2/2} dx \right), \quad \leqq \varepsilon c A^2 B^2 m(S)^p.$$

Thus we obtain, as $\varepsilon \longrightarrow 0$, $\quad I_p(\varphi) \cdot I_1(\psi) = I_{p+1}(\varphi\psi) + \sum_{k=1}^{p} I_{p-1}(\varphi k \psi)$.

Let φ and ψ any functions respectively in $L^2(T^p)$ and $L^2(T)$. By virtue of Theorem 2.1 we can find special elementary functions $\varphi_n \in L^2(T^p)$ and $\psi_n \in L^2(T)$ such that $\quad \| \varphi_n - \varphi \| \to 0, \quad \| \psi_n - \psi \| \to 0.$

By the above argument we have

$$I_{p+1}(\varphi_n \psi_n) = I_p(\varphi_n) I_1(\psi_n) - \sum_{k=1}^{p} I_{p-1}(\varphi_n \underset{(k)}{\times} \psi_n). \qquad (3.5)$$

By making use of (3.2), (3.3) and ($I.3'$) (§ 2) we obtain

$$\| I_{p+1}(\varphi_n\psi_n) - I_{p+1}(\varphi\psi) \|_1 = \| I_{p+1}(\varphi_n\psi_n - \varphi\psi) \|_1 \; (\| \cdot \|_1 \text{ being the } L_1\text{-norm})$$

$$\leqq \| I_{p+1}(\varphi_n\psi_n - \varphi\psi) \| \leqq \sqrt{\lfloor p+1} \; \| \varphi_n\psi_n - \varphi\psi \|$$

$$\leqq \sqrt{\lfloor p+1} \; \| \varphi_n(\psi_n - \psi) \| + \sqrt{\lfloor p+1} \; \| (\varphi_n - \varphi)\psi \|$$

$$=\sqrt{\underline{|p+1}}\,\|\varphi_n\|\cdot\|\psi_n-\psi\|+\sqrt{\underline{|p+1}}\,\|\varphi_n-\varphi\|\cdot\|\psi\|.$$

$$\|I_p(\varphi_n)\cdot I_1(\psi_n)-I_p(\varphi)\cdot I_1(\psi)\|_1\leqq\|I_p(\varphi_n)I_1(\psi_n-\psi)\|_1$$
$$+\|I_p(\varphi_n-\varphi)I_1(\psi)\|_1$$
$$\leqq\|I_p(\varphi_n)\|\cdot\|I_1(\psi_n-\psi)\|+\|I_p(\varphi_n-\varphi)\|\cdot\|I_1(\psi)\|$$
$$\leqq\sqrt{\underline{|p}}\,\|\varphi_n\|\cdot\|\psi_n-\psi\|+\sqrt{\underline{|p}}\,\|\varphi_n-\varphi\|\cdot\|\psi\|.$$

$$\|I_{p-1}(\varphi_n\underset{(k)}{\times}\psi)-I_{p-1}(\varphi\underset{(k)}{\times}\psi)\|_1\leqq\|I_{p-1}(\varphi_n\underset{(k)}{\times}\psi_n-\varphi\underset{(k)}{\times}\psi)\|$$
$$\leqq\sqrt{\underline{|p-1}}\,\|\varphi_n\underset{(k)}{\times}\psi_n-\varphi\underset{(k)}{\times}\psi\|$$
$$\leqq\sqrt{\underline{|p-1}}\,\|(\varphi_n-\varphi)\underset{(k)}{\times}\psi_n\|+\sqrt{\underline{|p-1}}\,\|\varphi\underset{(k)}{\times}(\psi_n-\psi)\|$$
$$\leqq\sqrt{\underline{|p-1}}\,\|\varphi_n-\varphi\|\cdot\|\psi_n\|+\sqrt{\underline{|p-1}}\,\|\varphi\|\cdot\|\psi_n-\psi\|.$$

Thus we see that (3.4) is true in the general case by letting n tend to ∞ in (3.5).

Proof of Theorem 3.1. We make use of the mathematical induction with regard to $p_1+\dots+p_n$. The theorem is trivially true in case $p_1+\dots+p_n=0$ or 1. It suffices to show that (3.1) is also true for $p_1+\dots+p_n=p+1$ under the assumption that (3.1) is valid for $p_1+\dots+p_n=p-1$, p. We may suppose that $p_1\geqq 1$ with no loss of generality.

If we put, in Theorem 3.2,

$$\varphi(t_1,\dots,t_p)=\varphi_1(t_1)\dots\varphi_1(t_{p_1-1})\varphi_2(t_{p_1})\dots\varphi_2(t_{p_1+p_2-1})\cdots$$
$$\times\varphi_n(t_{p_1+\dots+p_{n-1}})\cdots\varphi_n(t_{p_1+\dots+p_n-1})$$
$$\psi(t)=\varphi_1(t),$$

we obtain, by the assumption of induction

$$\int\cdots\int\varphi(t_1,\dots,t_p)\varphi_1(t)d\beta(t_1)\dots d\beta(t_p)d\beta(t)$$
$$=\int\cdots\int\varphi(t_1,\dots,t_p)d\beta(t_1)\dots d\beta(t_p)\int\varphi_1(t)d\beta(t)$$
$$-\sum_{k=1}^{p}\int\cdots\int(\varphi\underset{(k)}{\times}\varphi_1)d\beta(t_1)\dots d\beta(t_{k-1})d\beta(t_{k+1})\dots d\beta(t_p)$$

$$= \prod_{\nu=2}^{n} \frac{H_{p_\nu}\left(\frac{1}{\sqrt{2}}\int \varphi_\nu(t)d\beta(t)\right)}{\sqrt{2}^{p_\nu}} \cdot \frac{H_{p_1-1}\left(\frac{1}{\sqrt{2}}\int \varphi_1(t)d\beta(t)\right)}{\sqrt{2}^{p_1-1}}$$

$$\cdot \int \varphi_1(t)d\beta(t) + (p_1-1)$$

$$\times \prod_{\nu=2}^{n} \frac{H_{p_\nu}\left(\frac{1}{\sqrt{2}}\int \varphi_\nu(t)d\beta(t)\right)}{\sqrt{2}^{p_\nu}} \cdot \frac{H_{p_2-2}\left(\frac{1}{\sqrt{2}}\int \varphi_1(t)d\beta(t)\right)}{\sqrt{2}^{p_\nu-2}}$$

$$= \prod_{\nu=1}^{n} \frac{H_{p_\nu}\left(\frac{1}{\sqrt{2}}\int \varphi_\nu(t)d\beta(t)\right)}{\sqrt{2}^{p_\nu}}$$

in considering

$$\int \varphi(t_1,\dots,t_p)\varphi_1(t_k)m(dt_k) = \begin{cases} \varphi_1(t_1)\cdots\varphi_1(t_{k-1})\varphi_1(t_{k+1})\cdots\varphi_1(t_{p_1-1})\varphi_2(t_{p_1}) \\ \quad \cdots\varphi_n(t_{p_1+\dots+p_n-1}) \quad (1 \leqq k \leqq p_1-1) \\ 0 \quad (k \geqq p_1) \end{cases}$$

(which follows from the orthonormality of $\{\varphi_1,\dots,\varphi_n\}$)

and

$$H_{p_1}\left(\frac{x}{\sqrt{2}}\right) = \sqrt{2}\,H_{p_1}\left(\frac{x}{\sqrt{2}}\right) - 2(p_1-1)H_{p_1-2}\left(\frac{x}{\sqrt{2}}\right)$$

(which is the recursion formula of Hermite polynomials).

§4. Orthogonal development of L_2-functionals of β by multiple Wiener integrals.

A mapping from R^{B^*} into the complex number space K is called *B-measurable* if the inverse image of any Borel set in K is a B-measurable set in R^{B^*}, which is a set belonging to the least complete additive class that contains all the Borel cylinder sets in R^{B^*}. A complex-valued random variable $\xi(\omega)$ is a B-measurable function of β if it is expressible in the from:

$$\xi(\omega) = f(\beta(E, \omega),\ E \in \mathbf{B}^*), \tag{4.1}$$

for any ω, f being a B-meassurable mapping from R^{B^*} into K.

Furthermore if

$$||\xi||^2 \equiv \int |\xi(\omega)|^2 d\omega < \infty, \tag{4.2}$$

then we say that $\xi(\omega)$ is an L_2-functional of β.

Theorem 4.1 (R. H. Cameron and W. T. Martin)[8] *Let* $\{\varphi_\alpha(t)\}$ *be a complete orthonormal system. Then any* L_2*-functional* $\xi(\omega)$ *of* β *can be developed as follows:*

$$\xi = \sum_p \sum_{p_1+\cdots+p_n=p} \sum_{\alpha_1,\ldots,\alpha_n} a_{p_1\ldots p_n}^{\alpha_1\ldots\alpha_n} \prod_{\nu=1}^{n} H_{p_\nu}\Big(\frac{1}{\sqrt{2}}\int \varphi_{\alpha_\nu}(t)d\beta(t)\Big).$$

Cameron and Martin has shown this theorem in the case when β is a normal random measure derived from a brownian motion process, but their proof is available for our general case.

Theorem 4.2 *Any* L_2*-functional* ξ *of* β *can be expressible in the form:*

$$\xi = \sum I_p(f_p) = \sum I_p(\tilde{f}_p), \tag{4.3}$$

where f *is given by the following orthogonal development*

$$f_p(t_1,\ldots,t_p) = \sqrt{2} \sum_{p_1+\cdots+p_n=p} \sum_{\alpha_1,\ldots,\alpha_n} a_{p_1\ldots p_n}^{\alpha_1\ldots\alpha_n} \varphi_{\alpha_1}(t_1)\cdots\varphi_{\alpha_1}(t_{p_1})$$

$$\times \varphi_{\alpha_2}(t_{p_1+1})\cdots\varphi_{\alpha_2}(t_{p_1+p_2})\cdots\varphi_{\alpha_n}(t_{p_1+\cdots+p_n+1})\cdots\varphi_{\alpha_n}(t_{p_1+\cdots+p_n}),$$

$\{\varphi_\alpha\}$ *and* $\{a_{p_1\ldots p_n}^{\alpha_1\ldots\alpha_n}\}$ *being the same as those appearing in Theorem* 4.1.

Since $I_p(f_p)$ (or $I_p(\tilde{f}_p)$), $p=0, 1, 2,\ldots$, are orthogonal to each other, (4.3) may be considered as an orthogonal development,

We shall give another method of defining the symmetric functions $\{\tilde{S}\}$ which satisfy $\xi = \sum I_p(\tilde{S}_p)$. Put

$$F(\tilde{h}_p) = \frac{1}{\underline{|p}}(\xi, I_p(\tilde{h}_p)), \tilde{h} \in \tilde{L}^2(T^p),$$

where $\tilde{L}_2(T^p)$ is the totality of symmetric functions in $L^2(T^p)$ which forms a closed linear subspace of $L^2(T^p)$.

8) R. H. Cameron and W. T. Martin: The orthogonal development of non-linear functionals in series of Fourier-Hermite functions.

Then F_p is a bounded linear functional on $\tilde{L}^2(T^p)$, since

$$F_p(a\tilde{h}_p+b\tilde{g}_p)=aF_p(\tilde{h}_p)+bF_p(\tilde{g}_p)$$

$$|F_p(\tilde{h}_p)|\leqq\frac{1}{\underline{|p}}\,\|\xi\|\cdot\|I_p(\tilde{h}_p)\|=\|\xi\|\cdot\|\tilde{h}_p\|.$$

By Riesz-Fischer's theorem in Hilbert space, we can find $\tilde{s}_p\in\tilde{L}^2(T^p)$ such that

$$F_p(\tilde{h}_p)=(\tilde{s}_p,\ \tilde{h}_p).$$

By (4.3) we have $F_p(\tilde{h}_p)=\frac{1}{\underline{|p}}(I_p(\tilde{f}_p),\ I_p(\tilde{h}_p))=(\tilde{f}_p,\tilde{h}_p)$.

Thus we have

$$(\tilde{s}_p,\tilde{h}_p)=(\tilde{f}_p,\tilde{h}_p)\ \text{ for } h_p\in\tilde{L}^2(T^p),$$

which proves $\tilde{s}_p=\tilde{f}_p$.

From the above argument follows at once

Theorem 4.3. *$\xi=\sum I_p(f_p)=\sum I_p(g_p)$ implies $\tilde{f}_p=\tilde{g}_p$.*

§ 5. The case of a brownian motion process.

Let $\beta(t)$, $a<t<b$, be a brownian motion process.
If we put

$$\beta(E)=\int c_E(t)d\beta(t),$$

where $c_E(t)$ is the characteristic function of the set E and the integral is the so-called Wiener integral. Then $\beta(E)$ is a normal random measure on $T=(a,b)$, the measure m on T being the so-called Lebesgue measure, which clearly fulfills the continuity-condition.

Let $f(t_1,\ldots,t_p)\in L^2(T^p)$. Then we can consider

$$I=\int\cdots\int f(t_1,\ldots,t_p)d\beta(t_1)\ldots d\beta(t_p).$$

Theorem 5.1. *The above multiple Wiener integral I is expressible in the form of iterated stochastic intgrals*[9]

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9) loc. cit. 2).

$$I=\lfloor p \int_a^b \left(\int_a^{t_p} \left(\cdots \int_a^{t_3} \left(\int_a^{t_2} f(t_1,\ldots,t_p) d\beta(t_1) \right) d\beta(t_2) \right) \cdots \right) d\beta(t_{p-1}) \right) d\beta(t_p)$$

Proof. If f is a special elementary function, this theorem is easily verified by the definitions. In the general case we can show it by approximating f with a special elementary function and making use of the properties of multiple Wiener integrals and stochastic integral.

Any Wiener functional of the brownian motion process is an L_2-functional of the normal random measure derived from it, and vice versa. Therefore we see that Theorem 4.2 gives an orthogonal development of Wiener functionals.

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