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Affine and Projective Geometries of System of Hypersurfaces

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§ 1. *Introduction.* J. Douglas [1]* studied affine and projective geometries of a space of K-spreads, K-spreads being given by a system of partial differential equations of the form

$$\frac{\partial^2 x^i}{\partial u^\beta \partial u^\gamma}+H^i_{\beta\gamma}(x\,;p)=0, \qquad \left(p^i_\alpha=\frac{\partial x^i}{\partial u^\alpha}\right), \tag{1·1}$$

$$(i, j, k, \ldots\ldots =1, 2, \ldots\ldots, N;\ \alpha, \beta, \gamma, \ldots\ldots=1, 2, \ldots\ldots, K)$$

where $H^i_{\beta\gamma}=H^i_{\gamma\beta}$ are homogeneous function system of p with respect to the lower indices β and γ.

The problem to determine a privileged projective connection with respect to which the system of K-dimensional flat subspaces coincides exactly with that of K-spreads given by (1·1) was studied by S. S. Chern [3], Chih-Ta Yen [6] and the present authors [4], [5].

In a space of K-spreads, we consider that the elements of the space are the points and the K-dimensional linear spaces at each point. The point is represented by its coordinates (x^i) and the K-dimensional linear space by K linearly independent contravariant vectors (p^i_α) contained in it. But, in a space of $(n-1)$-spreads, the hyperplane element may be represented by a covariant vector (u_i). Thus the geometry of $(n-1)$-spreads may be studied by a method some what different from the general one. The case of $(n-1)$-spreads was once treated by M. Hachtroudi [2], but we shall retake this case and study it by a method shown in [5].

Suppose that there be given, in an N-dimensional space X_N referred to a coordinate system (x^i), a system of hypersurfaces in such a way that there exists one and only one hypersurface passing through N points given in general position, and belonging to the system. Such a system of hypersurfaces is given by a finite equation of the form

$$f(x^1, x^2, \ldots\ldots, x^N; a^1, a^2, \ldots\ldots, a^N)=0, \tag{1·2}$$

* See the Bibliography at the end of the paper.

where a's are N essential parameters determining each hypersurface. We shall call such hypersurfaces hyperplanes of the space.

Now, the system of hyperplanes is represented by

(R_1) the coordinate system (x^i) in the space;

(R_2) the function $f(x\,;a)$ appearing in the equation of the system of hyperplanes;

(R_3) the essential parameters (a^i) determining each hyperplane.

But, the system of hyperplanes must be a configuration invariant under

(T_1) the transformation of coordinates $\bar{x}^i=\bar{x}^i(x)$;

(T_2) the transformation of factor $\bar{f}=\alpha f$;

(T_3) the transformation of essential parameters $\bar{a}^i=\bar{a}^i(a)$.

The transformation of the essential parameters having no effect in the following discussions, we shall consider only (T_1) and (T_2) and study the properties invariant under these transformations.

The geometry of system of hyperplanes is called affine one when we consider only a constant α in (T_2), while it is called projective one when we consider a general $\alpha(x\,;a)$ in (T_2). The main problem in the present paper is to determine a projective connection with respect to which the system of projectively flat hypersurfaces will coincide with the given system of hyperplanes.

§ 2. *Affine geometry of hyperplanes.* If we effect a transformation of the factor $f(x\,;a)=\alpha\cdot f_0(x\,;a)$, α being a constant and $f_0(x\,;a)$ a specified function, and put

$$u_i\equiv\frac{\partial f}{\partial x^i}=\alpha\cdot\frac{\partial f_0}{\partial x^i}, \tag{2·1}$$

we have

$$f(x\,;a)=0,\quad u_i dx^i=0,\quad du_j=\alpha\cdot\frac{\partial^2 f_0}{\partial x^j\partial x^k}dx^k$$

along a hyperplane. Solving α and a from $f_0(x\,;a)=0$ and $u_i=\alpha\frac{\partial f_0}{\partial x^i}$ as functions of x and u, and substituting these into the last equations, we have

$$\omega\equiv u_i dx^i=0,\quad \omega_j\equiv du_j-\Gamma_{jk}(x\,;u)dx^k=0, \tag{2·2}$$

as Pfaffian equtions of the system of hyperplanes, where the functions $\Gamma_{jk}(x\,;u)$ are symmetric with respect to j and k and homogeneous of degree one with respect to u.

The u_i being components of a covariant vector, the transformation law of the functions Γ_{jk} under (T_1) is given by

$$\bar{\Gamma}_{jk}=\frac{\partial x^b}{\partial \bar{x}^j}\frac{\partial x^c}{\partial \bar{x}^k}\Gamma_{bc}+\frac{\partial^2 x^a}{\partial \bar{x}^j \partial \bar{x}^k}u_a. \tag{2·3}$$

Differentiating (2·3) with respect to $\bar{u}_i$, we find

$$\bar{\Gamma}^i_{jk}=\frac{\partial \bar{x}^i}{\partial x^a}\left(\frac{\partial x^b}{\partial \bar{x}^j}\frac{\partial x^c}{\partial \bar{x}^k}\Gamma^a_{bc}+\frac{\partial^2 x^a}{\partial \bar{x}^j \partial \bar{x}^k}\right), \tag{2·4}$$

where we have put $\bar{\Gamma}^i_{jk}=\bar{\Gamma}_{jk}|^i=\partial\bar{\Gamma}_{jk}/\partial\bar{u}_i$ and $\Gamma^a_{bc}=\Gamma_{bc}|^a=\partial\Gamma_{bc}/\partial u_a$. The equation (2·4) shows that Γ^a_{bc} are components of an affine connection.

In the following, we shall denote the partial differentiation with respect to x^i by a comma and that with respect to u_i by a vertical stroke.

Now the Pfaffian equations (2·2) must be completely integrable, so that, ω' and ω_j' must vanish if we take account of $\omega=0$ and $\omega_j=0$, from which we have

$$R_{jkl}u_m+R_{jlm}u_k+R_{jmk}u_l=0, \tag{2·5}$$

where

$$R_{jkl}=\Gamma_{jk,l}+\Gamma^a_{jk}\Gamma_{al}-\Gamma_{jl,k}-\Gamma^a_{jl}\Gamma_{ak}. \tag{2·6}$$

We observe here that R_{jkl}, being homogeneous functions of degree one with respect to u_i, satisfy the identities:

$$R_{jkl}+R_{jlk}=0, \quad R_{jkl}+R_{klj}+R_{ljk}=0, \quad u_iR^i{}_{jkl}=R_{jkl}, \tag{2·7}$$

where

$$R^i{}_{jkl}=R_{jkl}|^i=(\Gamma^i_{jk,l}+\Gamma^i_{jk}|^a\Gamma_{al})-(\Gamma^i_{jl,k}+\Gamma^i_{jl}|^a\Gamma_{ak})+\Gamma^a_{jk}\Gamma^i_{al}-\Gamma^a_{jl}\Gamma^i_{ak}. \tag{2·8}$$

Differentiating (2·5) with respect to u_m and contracting, we find

$$R^i_{jkl}-\frac{1}{N-1}(R^a{}_{jka}\delta^i_l-R^a{}_{jla}\delta^i_k)=0. \tag{2·9}$$

It is readily seen that if we have (2·9), then the condition (2·5) will be satisfied. Consequently we have the

Theorem I. The necessary and sufficient condition that the Pfaffian equations (2·2) *be completely integrable is that we have* (2·5) *or* (2·9).

The tensor charactor of R_{jkl} and $R^i{}_{jkl}$ being easily obtained by calculating the integrability conditions of (2·4), we call R_{jkl} and $R^i{}_{jkl}$ affine curvature tensors of our affine space of hyperplanes.

When there exists a coordinate system in which the function f takes a linear form with respect to x^i, we say that our space of hyperplanes is affinely flat. In order that it may be the case, we must have $\bar{\Gamma}_{jk}=0$ in some coordinate system $(\bar{x}^i)$ and hence the equations

$$\frac{\partial^2 \bar{x}^i}{\partial x^b \partial x^c}=\frac{\partial \bar{x}^i}{\partial x^a}\Gamma^a_{bc} \tag{2·10}$$

obtained by putting $\bar{\Gamma}^i_{jk}=0$ in (2·4), must admit N independent solutions $\bar{x}^i=\bar{x}^i(x)$. Thus the functions Γ^i_{jk} must be independent of u_i and moreover we must have $R^i{}_{jkl}=0$ as integrability conditions of (2·10). Conversely, if we have $\Gamma^i_{jk}|^l=0$ and $R^i{}_{jkl}=0$, then the differential equations (2·10) admit N independent solutions $\bar{x}^i=\bar{x}^i(x)$ and, in the coordinate system $(\bar{x}^i)$, the components $\bar{\Gamma}^i_{jk}$ of the affine connection are all zero and consequently $\bar{\Gamma}_{jk}$ are functions of $\bar{x}^i$ only. But the functions $\bar{\Gamma}_{jk}$ must be homogeneous of degree one with respest to $\bar{u}_i$ and consequently it must vanish identically. Thus we have the

Theorem II. The necessary and sufficient condition that the affine space of hyperplanes be affinely flat is that $\Gamma^i_{jk}|^l=0$ and $R^i{}_{jkl}=0$.

§ 3. *Projective geometry of hyperplanes.* We next consider the projective geometry of hyperplanes. If we effect the transformation of the factor $\bar{f}=a(x\,;a)f$, we have

$$\frac{\partial \bar{f}}{\partial x^i}=a\frac{\partial f}{\partial x^i},\quad \frac{\partial^2 \bar{f}}{\partial x^j \partial x^k}=a\cdot\frac{\partial^2 f}{\partial x^j\partial x^k}+\frac{\partial a}{\partial x^j}\frac{\partial f}{\partial x^k}+\frac{\partial a}{\partial x^k}\frac{\partial f}{\partial x^j}$$

along the hyperplane $f=0$, and consequently, denoting by

$$\bar{u}_i dx^i=0,\quad d\bar{u}_j-\bar{\Gamma}_{jk}(x\,;\bar{u})=0,\quad \left(\bar{u}_i=\frac{\partial \bar{f}}{\partial x^i}\right) \tag{3·1}$$

the equations corresponding to (2·2), we find

$$\bar{\Gamma}_{jk}(x\,;\bar{u})=\Gamma_{jk}(x\,;\bar{u})+\frac{\partial \log a}{\partial x^j}\bar{u}_k+\frac{\partial \log a}{\partial x^k}\bar{u}_j.$$

Thus, if we choose functions $H_i(x\,;u)$ which are homogeneous of degree zero with respect to u_i and satisfy

$$\frac{\partial \log a}{\partial x^i}=H_i(x\,;u) \tag{3·2}$$

on $f=0$, the above equation may be wiitten as

$$\bar{\Gamma}_{jk}(x\,;u)=\Gamma_{jk}(x\,;u)+H_j u_k+H_k u_j, \tag{3·3}$$

which may be considered as giving a projective change of Γ_{jk}.

Now, we shall examine the integrability conditions of (3·2). If we put

$$\lambda\equiv d\log a-H_i(x\,;u)\,dx^i=0,$$

then λ' must vanish when we take account of (2·2), from which

$$\Lambda_{jk}u_l+\Lambda_{kl}u_j+\Lambda_{lj}u_k=0, \tag{3·4}$$

where

$$\Lambda_{jk}=H_{j,k}+H_j|^a\Gamma_{ak}-H_{k,j}-H_k|^a\Gamma_{aj}. \tag{3·5}$$

If the functions Γ_{jk} are transformed into $\bar{\Gamma}_{jk}$ by a projective change (3·3), then the tensor R_{jkl} will be transformed into the tensor

$$\bar{R}_{jkl}=R_{jkl}-\varphi_{jk}u_l+\varphi_{jl}u_k+u_j(\varphi_{kl}-\varphi_{lk}), \tag{3·6}$$

where

$$\varphi_{jk}=H_{j,k}+H_j|^a\Gamma_{ak}-H_a\Gamma^a_{jk}-H_jH_k-\frac{1}{2}u_jH_k|^aH_a \tag{3·7}$$

So that, we can easily see that, if Γ_{jk} satisfy the integrability conditions (2·5), then the $\bar{\Gamma}_{jk}$ will satisfy the corresponding ones providing that H_j satisfy (3·4)

Now, if the functions Γ_{jk} are transformed into $\bar{\Gamma}_{jk}$ following (3·3), the functions Γ^i_{jk} are transformed into

$$\bar{\Gamma}^i_{jk}=\Gamma^i_{jk}+\delta^i_jH_k+\delta^i_kH_j+H_j|^iu_k+H_k|^iu_j. \tag{3·8}$$

This may be regarded as giving the projective change of affine connection Γ^i_{jk}.

From (3·8), we have by contractiou,

$$\bar{\Gamma}^a_{ak}=\Gamma^a_{ak}+(N+1)H_k+H_a|^a u_k, \tag{3·9}$$

from which

$$\bar{\Gamma}^a_{ab}|^b=\Gamma^a_{ab}|^b+2NH_a|^a. \tag{3·10}$$

To eliminate $H_a|^a$, substituting (3·10) into (3·9), we find

$$\bar{L}_k=L_k+H_k, \tag{3·11}$$

where

$$\bar{L}_k=\frac{1}{N+1}\left(\bar{\Gamma}^a_{ak}-\frac{1}{2N}\bar{\Gamma}^a_{ab}|^b u_k\right),\quad L_k=\frac{1}{N+1}\left(\Gamma^a_{ak}-\frac{1}{2N}\Gamma^a_{ab}|^b u_k\right), \tag{3·12}$$

both being homogeneous functions of degree zero with respect to u_i. To eliminate H_k, substituting (3·11) into (3·3), we find that the functions

$$*\Pi_{jk}=\Gamma_{jk}-L_j u_k-L_k u_j \tag{3·13}$$

are invariant under the projective change of Γ_{jk}. So that, we can easily see that

$$*\Pi^i_{jk}=*\Pi_{jk}|^i=(\Gamma_{jk}-L_j u_k-L_k u_j)|^i \tag{3·14}$$

gives components of a projective connection and

$$P^i{}_{jk}{}^l=(\Gamma_{jk}-L_j u_k-L_k u_j)|^i|^l \tag{3·15}$$

components of a projective tensor. We observe here that the projective connection $*\Pi^i_{jk}$ and the projective tensor $P^i{}_{jk}{}^l$ satisfy the identities:

$$*\Pi^i_{jk}=*\Pi^i_{kj},\quad *\Pi^a_{ja}=*\Pi^a_{aj}=0,\quad P^i{}_{jk}{}^l=P^i{}_{kj}{}^l=P^l{}_{jk}{}^i,\quad u_i P^i{}_{jk}{}^l=0,\quad P^a{}_{ja}{}^l=0, \tag{3·16}$$

$*\Pi^a_{ja}=0$ being the consequence of $L_a|^a=\frac{1}{2N}\Gamma^a_{ab}|^b$.

The functions L_k defined by (3·12) are not components of a vector, but the functions $L_k|^i$ are components of a tensor and its transformation law under the projective change of Γ_{jk} is given by

$$\bar{L}_k|^i=L_k|^i+H_k|^i. \tag{3·17}$$

Thus if we put

$$\Pi^i_{jk}=\Gamma^i_{jk}-L_j|^i u_k-L_k|^i u_j, \tag{3·18}$$

then, Π^i_{jk} are components of an affine connection, and its law of transformation under the projective change of Γ_{jk} is given by

$$\bar{\Pi}^i_{jk}=\Pi^i_{jk}+\delta^i_j H_k+\delta^i_k H_j. \tag{3·19}$$

we observe here that

$$\Pi^a_{ja}=\Pi^a_{aj}=(N+1)L_j, \quad u_i\Pi^i_{jk}=\Gamma_{jk}, \quad \Pi^i_{jk}+u_a(\Pi^a_{jk}|^i)=\Gamma^i_{jk}. \tag{3·20}$$

Now, if we define a curvature tensor $\Pi^i{}_{jkl}$ by

$$\Pi^i{}_{jkl}=(\Pi^i_{jk,l}+\Pi^i_{jk}|^b\Pi^a_{bl}u_a)-(\Pi^i_{jl,k}+\Pi^i_{jl}|^b\Pi^a_{bk}u_a)+\Pi^a_{jk}\Pi^i_{al}-\Pi^a_{jl}\Pi^i_{ak}, \tag{3·21}$$

we have

$$u_i\Pi^i{}_{jkl}=R_{jkl}, \tag{3·22}$$

and consequently we can easily write down the integrability conditions (2·5) in terms of Π^i_{jk}.

The transformation law of $\Pi^i{}_{jkl}$ under the projective change (3·3) or (3·19) is

$$\bar{\Pi}^i{}_{jkl}=\Pi^i{}_{jkl}-H_{jk}\delta^i_l+H_{jl}\delta^i_k+\delta^i_j(H_{kl}-H_{lk})+\Pi^i_{jk}|^a H_a u_l-\Pi^i_{jl}|^a H_a u_k, \tag{3·23}$$

where

$$H_{jk}=H_{j,k}+H_j|^b\Pi^a_{bk}u_a-H_i\Pi^i_{jk}-H_jH_k+H_j|^a H_a u_k. \tag{3·24}$$

But, as we have, from (3·15),

$$P^i{}_{jk}{}^a=\Pi^i_{jk}|^a-\delta^i_j L_k|^a-\delta^i_k L_j|^a,$$

the equations (3·23) may be written as

$$\begin{aligned}\bar{\Pi}^i{}_{jkl}=\Pi^i{}_{jkl}&-(H_{jk}+L_j|^a H_a u_k)\delta^i_l+(H_{jl}+L_j|^a H_a u_l)\delta^i_k \\ &+\delta^i_j(H_{kl}+L_k|^a H_a u_l-H_{lk}-L_l|^a H_a u_k)+P^i{}_{jk}{}^a H_a u_l-P^i{}_{jl}{}^a H_a u_k.\end{aligned} \tag{3·25}$$

Contracting over i and l in this equation, we have, by virtue of (3·16),

$$H_{jk}+L_j|^a H_a u_k=-\frac{1}{N^2-1}(N\bar{\Pi}^a{}_{jka}+\bar{\Pi}^a{}_{kja})+\frac{1}{N^2-1}(N\Pi^a{}_{jka}+\Pi^a{}_{kja}), \tag{3·26}$$

and consequently, substituting (3·26) into (3·25), we find

$$\bar{P}^{i}{}_{jkl}=P^{i}{}_{jkl}+P^{i}{}_{jk}{}^{a}H_{a}u_{l}-P^{i}{}_{jl}{}^{a}H_{a}u_{k}, \tag{3·27}$$

where

$$P^{i}{}_{jkl}=\Pi^{i}{}_{jkl}-\frac{1}{N^{2}-1}(N\Pi^{a}{}_{jka}+\Pi^{a}{}_{kja})\delta^{i}_{l}+\frac{1}{N^{2}-1}(N\Pi^{a}{}_{jla}$$

$$+\Pi^{a}{}_{lja})\delta^{i}_{k}+\frac{1}{N-1}\delta^{i}_{j}(\Pi^{a}{}_{kla}-\Pi^{a}{}_{lka}), \tag{3·28}$$

$\bar{P}^{i}{}_{jkl}$ being given by a similar exprssion.

Differentiating (3·27) partially with respect to u_l and contracting, we find

$$\bar{P}^{i}{}_{jkm}|^{m}=P^{i}{}_{jkm}|^{m}-(N-2)P^{i}{}_{jk}{}^{a}H_{a}-(P^{i}{}_{jm}{}^{a}H_{a})|^{m}u_{k}. \tag{3·29}$$

Substituting (3·29) into (3·27), we find that

$$L^{i}{}_{jkl}=P^{i}{}_{jkl}-\frac{1}{N-2}P^{i}{}_{jkm}|^{m}u_{l}+\frac{1}{N-2}P^{i}{}_{jlm}|^{m}u_{k} \tag{3·30}$$

is a purely projective tensor.

Now, to obtain the projective invariants, we can proceed as follows. We know that the functions $*\Pi_{jk}$ defined by (3·13) and

$$*\Pi^{i}_{jk}=*\Pi_{jk}|^{i}=\Gamma^{i}_{jk}-\delta^{i}_{j}L_{k}-\delta^{i}_{k}L_{j}-L_{j}|^{i}u_{k}-L_{k}|^{i}u_{j} \tag{3·31}$$

are both invariant under the projective change of Γ_{jk}. From the transformation law of Γ^{i}_{jk} under the coordinate transformation and (3·12), we can find out that of L_k:

$$\bar{L}_{j}=\frac{\partial x^{b}}{\partial\bar{x}^{j}}L_{b}+\bar{\theta}_{j},\quad\left(\bar{\theta}_{j}=\frac{1}{N+1}\frac{\partial\log\Delta}{\partial\bar{x}^{j}},\quad\Delta=\left|\frac{\partial x}{\partial\bar{x}}\right|\right) \tag{3·32}$$

which shows that L_j is not a vector, but $L_j|^{i}$ a mixed tensor.

From the transformation law of Γ^{i}_{jk} and that of L_j, we can find out that of $*\Pi^{i}_{jk}$:

$$\frac{\partial^{2}x^{a}}{\partial\bar{x}^{j}\partial\bar{x}^{k}}=\frac{\partial x^{a}}{\partial\bar{x}^{i}}*\bar{\Pi}^{i}_{jk}-\frac{\partial x^{b}}{\partial\bar{x}^{j}}\frac{\partial x^{c}}{\partial\bar{x}^{k}}*\Pi^{a}_{bc}+\frac{\partial x^{a}}{\partial\bar{x}^{j}}\bar{\theta}_{k}+\frac{\partial x^{a}}{\partial\bar{x}^{k}}\bar{\theta}_{j}. \tag{3·33}$$

The integrability conditions of these equations may be written as

$$*\bar{\Pi}^i{}_{jkl}=\frac{\partial\bar{x}^i}{\partial x^a}\frac{\partial x^b}{\partial\bar{x}^j}\frac{\partial x^c}{\partial\bar{x}^k}\frac{\partial x^d}{\partial\bar{x}^l}*\Pi^a{}_{bcd}+\bar{\theta}_{jk}\delta^i_l-\bar{\theta}_{jl}\delta^i_k-*\bar{\Pi}^i_{jk}|^s\bar{\theta}_s\bar{u}_l+*\bar{\Pi}^i_{jl}|^s\bar{\theta}_s\bar{u}_k, \tag{3·34}$$

where

$$*\Pi^i{}_{jkl}=(*\Pi^i_{jk,l}+*\Pi^i_{jk}|^t*\Pi^s_{tl}u_s)-(*\Pi^i_{jl,k}+*\Pi^i_{jl}|^t*\Pi^s_{tk}u_s)+*\Pi^a_{jk}*\Pi^i_{al}-*\Pi^a_{jl}*\Pi^i_{ak}, \tag{3·35}$$

$*\bar{\Pi}^i{}_{jkl}$ being defined by a similar expression, and

$$\bar{\theta}_{jk}=\bar{\theta}_{j,k}-\bar{\theta}_i*\bar{\Pi}^i_{jk}-\bar{\theta}_j\bar{\theta}_k. \tag{3·36}$$

From (3·34), we obtain, by contraction,

$$*\bar{\Pi}^a{}_{jka}=\frac{\partial x^b}{\partial\bar{x}^j}\frac{\partial x^c}{\partial\bar{x}^k}*\Pi^a{}_{bca}+(N-1)\bar{\theta}_{jk}. \tag{3·37}$$

So, substituting (3·37) into (3·34), we find

$$*\bar{W}^i{}_{jkl}=\frac{\partial\bar{x}^i}{\partial x^a}\frac{\partial x^b}{\partial\bar{x}^j}\frac{\partial x^c}{\partial\bar{x}^k}\frac{\partial x^d}{\partial\bar{x}^l}*W^a{}_{bcd}-*\bar{\Pi}^i_{jk}|^s\bar{\theta}_s\bar{u}_l+*\bar{\Pi}^i_{jl}|^s\bar{\theta}_s\bar{u}_k, \tag{3·38}$$

where

$$*W^i{}_{jkl}=*\Pi^i{}_{jkl}-\frac{1}{N-1}(*\Pi^a{}_{jka}\delta^i_l-*\Pi^a{}_{jla}\delta^i_k), \tag{3·39}$$

$*\bar{W}^i{}_{jkl}$ being defined by a similar expression. Thus we can see that $*W^i{}_{jkl}$ is projective invàriant but not a tensor.

We shall now form a tensor starting from (3·34). Substituting $\bar{\theta}_s=\bar{L}_s-\frac{\partial x^b}{\partial\bar{x}^s}L_b$ into (3·34) and using the relations $u_i*\Pi^i_{jk}=*\Pi_{jk}=\Gamma_{jk}-L_ju_k-L_ku_j$, we fined

$$*\bar{B}^i{}_{jkl}=\frac{\partial\bar{x}^i}{\partial x^a}\frac{\partial x^b}{\partial\bar{x}^j}\frac{\partial x^c}{\partial\bar{x}^k}\frac{\partial x^d}{\partial\bar{x}^l}*B^a{}_{bcd}+\bar{\theta}_{jk}\delta^i_l-\bar{\theta}_{jl}\delta^i_k, \tag{3·40}$$

where

$$*B^i{}_{jkl}=(*\Pi^i_{jk,l}+*\Pi^i_{jk}|^a\Gamma_{al})-(*\Pi^i_{jl,k}+*\Pi^i_{jl}|^a\Gamma_{ak})+*\Pi^a_{jk}*\Pi^i_{al}-*\Pi^a_{jl}*\Pi^i_{ak}, \tag{3·41}$$

$*\bar{B}^{i}{}_{jkl}$ being given by a similar expression. Eliminating $\bar{\theta}_{jk}$ from (3·40), we can see that

$$*P^{i}{}_{jkl}=*B^{i}{}_{jkl}-\frac{1}{N-1}*B^{a}{}_{jka}\delta^{i}_{l}+\frac{1}{N-1}*B^{a}{}_{jla}\delta^{i}_{k} \tag{3·42}$$

are components of a tensor which we shall call the curvature tensor of our projective geometry of hyperplanes. But we should remark here that the components $*P^{i}{}_{jkl}$ of this curvature tensor are, under the projective change of Γ_{jk}, transformed into

$$*\bar{P}^{i}{}_{jkl}=*P^{i}{}_{jkl}+*\Pi^{i}_{jk}|^{a}H_{a}u_{l}-*\Pi^{i}_{jl}|^{a}H_{a}u_{k}, \tag{3·43}$$

which shows that this tensor is not invariant under such transformation.

Now, eliminating H_a from (3·43) by using the homogeneity properties of the functions H_a, $*\Pi^{i}_{jk}$ and $*P^{i}{}_{jkl}$, we can see that a tensor defined by

$$*L^{i}{}_{jkl}=*P^{i}{}_{jkl}-\frac{1}{N-2}*P^{i}{}_{jka}|^{a}u_{l}+\frac{1}{N-2}*P^{i}{}_{jla}|^{a}u_{k} \tag{3·44}$$

is invariant under a projective change of Γ_{jk}.

Now, we shall rewrite the equations (2·1) under the form

$$u_{i}dx^{i}=0,\quad du_{j}-(*\Pi_{jk}+L_{j}u_{u}+L_{u}u_{j})dx^{k}=0 \tag{3·46}$$

in terms of $*\Pi_{jk}$ and L_j. If we put

$$\Omega_{jk}=(L_{j,k}+L_{j}|^{a}\Gamma_{ak})-(L_{k,j}+L_{k}|^{a}\Gamma_{aj}),$$

then we have

$$\begin{aligned}&\Omega_{jk}u_{l}+\Omega_{kl}u_{j}+\Omega_{lj}u_{k}\\&=-\frac{1}{N+1}[(R^{a}{}_{jka}u_{l}-R^{a}{}_{jla}u_{k})+(R^{a}{}_{kla}u_{j}-R^{a}{}_{kja}u_{l})+(R^{a}{}_{lja}u_{k}-R^{a}{}_{lka}u_{j})]\\&=-\frac{N-1}{N+1}[R_{jkl}+R_{klj}+R_{ljk}]=0\end{aligned}$$

by virtue of (2·7) and (2·9), and consequently we can conclude that the equation

$$d\log\beta+L_{j}dx^{j}=0$$

is integrable on $f=0$. Thus, effecting the transformation of the factor $\bar{f}=\beta f$, the equations (3·46) take the form

$$u_i dx^i=0, \quad du_j-{}^*\Pi_{jk}(x\,;u)dx^k=0. \tag{3·47}$$

It is evident that the transformation $\bar{f}=\alpha f$ preserving the form (3·47) must be one for which α is a constant. We shall remark here that, under a transformation of coordinates, the equations (3·47) will be trasformed into those of the form

$$\bar{u}_i d\bar{x}^i=0, \quad d\bar{u}_j-[{}^*\bar{\Pi}_{jk}(\bar{x}\,;\bar{u})+\bar{\theta}_j\bar{u}_k+\bar{\theta}_k\bar{u}_j]\ d\bar{x}^k=0.$$

The integrability conditions of (3·47) are given by

$$^*\Pi_{jkl}u_m+{}^*\Pi_{jlm}u_k+{}^*\Pi_{jmk}u_l=0, \quad ({}^*\Pi_{jkl}=u_i{}^*\Pi^i{}_{jkl}) \tag{3·48}$$

or

$$u_i{}^*W^i{}_{jkl}={}^*\Pi_{jkl}-\frac{1}{N-1}({}^*\Pi^a{}_{jka}u_l-{}^*\Pi^a{}_{jla}u_k)=0. \tag{3·49}$$

It is to be noted that the equations (2·5), (2·9, (3·48) and (3·49) are all equivalent to each other.

If, by a suitable transformation of factor and that of coordinates, we can reduce the function f to a linear function of x^i, we say that our space of hyperplanes is projectively flat.

Theorem III. The necessary and sufficient condition that the space be projectively flat is that the functions $^*\Pi^i_{jk}$ *do not contain the variables* u_i.

For, if our space is projectively flat, then there exists a suitable factor and a coordinate system with repect to which we have $\Gamma_{jk}=\Gamma^i_{jk}={}^*\Pi_{jk}={}^*\Pi^i_{jk}=0$, from which ${}^*\Pi^i_{jk}|^l=0$. But ${}^*\Pi^i_{jk}|^l$ being a projective tensor, ${}^*\Pi^i_{jk}|^l=0$ holds for any factor and coordinate system. Thus, the functions $^*\Pi^i_{jk}$ must be independent of u_i.

Conversely, if the functions $^*\Pi^i_{jk}$ are independent of u_i, then the components $^*W^i{}_{jkl}$ of the projective invariant are also independent of u_i and consequently we have, from (3·49),

$$^*W^i{}_{jkl}={}^*\Pi^i{}_{jkl}-\frac{1}{N-1}({}^*\Pi^a{}_{jka}\delta^i_l-{}^*\Pi^a{}_{jla}\delta^i_k)=0,$$

where

$$*\Pi^{i}{}_{jkl}={}^*\Pi^{i}_{jk,l}-{}^*\Pi^{i}_{jl,k}+{}^*\Pi^{a}_{jk}{}^*\Pi^{i}_{al}-{}^*\Pi^{a}_{jl}{}^*\Pi^{i}_{ak}.$$

Thus, the equations

$$\frac{\partial^2 x^a}{\partial\bar{x}^j\partial\bar{x}^k}=-\frac{\partial x^b}{\partial\bar{x}^j}\frac{\partial x^c}{\partial\bar{x}^k}{}^*\Pi^a_{bc}+\frac{\partial x^a}{\partial\bar{x}^j}\bar{\theta}_k+\frac{\partial x^a}{\partial\bar{x}^k}\bar{\theta}_j$$

obtained from (3·33) putting ${}^*\bar{\Pi}^i_{jk}=0$ are completely integrable, and consequently there exists a coordinate system in which ${}^*\bar{\Pi}^i_{jk}=0$. In such a coordinate system, the equations (3·47) take the form

$$u_i dx^i=0,\quad du_j-(L_j u_k+L_k u_j)dx^k=0,$$

and consequently, if we effect a suitable transformation of the factor, these become $u_i dx^i=0$, $du_j=0$. Thus the theorem is proved.

§ 4. *Space of hyperplane elements with projective connection.* Let us consider a space, whose elements are points (x^i) and hyperplane elements (u_i) passing through each point, and at each element of which is associated an arbitrary projective space of N dimensions called tangent space at this element, and suppose that, between two projective tangent spaces attached to the nearby elements (x^i, u_i) and (x^i+dx^i, u_i+du_i), there is always given a projective correspondence. We shall call such a space space of hyperplane elements with projective connection.

The projective connection is expressed analytically by the equations of the form

$$dA_0=\omega_0^0 A_0+\omega_0^i A_i,\qquad dA_j=\omega_j^0 A_0+\omega_j^i A_i, \tag{4·1}$$

where $[A_0, A_1, \ldots\ldots, A_n]$ is a frame of reference taken in each tangent space. A_0 coinciding with the contact point (x^i) and ω's are Pfaffian forms with respect to (x^i) and (u_i). It will be easily seen that the Pfaffian forms ω_0^i do not contain the differentials du_i and consequently that we can take a frame of reference in such a way that we have $\omega_0^i=dx^i$. Such a frame of reference will be called semi-natural frame of reference. With respect to a semi-natural frame of reference, the equations (4·1) take the form

$$\left.\begin{aligned}&dA_0=(\phi_i dx^i+\phi^i du_i)A_0+dx^i A_i,\\&\quad dA_j=(\omega^0_{jk}dx^k+\omega_j^{0k}du_k)A_0+(\omega^i_{jk}dx^k+\omega_j^{ik}du_k)A_i.\end{aligned}\right\} \tag{4·2}$$

Here, the hyperplane element being represented by $u_i dx^i=0$, the u_i

are homogeneous coordinates of the hyperplane element, so the equations (4·2) must be invariant under the change $u_i \to a u_i$, and consequently, the functions ϕ_i, ω^0_{jk} and ω^i_{jk} are homogeneous functions of degree zero, ϕ^i, ω^{0k}_j and ω^{ik}_j those of degree -1 with respect to u_i and satisfy the equations

$$\phi^i u_i = 0, \quad \omega^{0k}_j u_k = 0, \quad \omega^{ik}_j u_k = 0. \tag{4·3}$$

The projective connection being completely determined by the Pfaffian forms ω^0_j and $\omega^i_j - \delta^i_j \omega^0_0$, we put

$$\Pi^0_{jk} = \omega^0_{jk}, \quad \Pi^{0k}_j = \omega^{0k}_j, \quad \Pi^i_{jk} = \omega^i_{jk} - \delta^i_j \phi_k, \quad \Pi^{ik}_j = \omega^{ik}_j - \delta^i_j \phi^k \tag{4·4}$$

and call them components of projective connection. They have the properties similar to those ω's

Now, we take an infinitesimal parallelogram formed by (x^i, u_i), $(x_i + \underset{1}{d}x^i, u_i + \underset{1}{d}u_i)$, $(x^i + \underset{1}{d}x^i + \underset{2}{d}x^i + \underset{21}{dd}x^i, u_i + \underset{1}{d}u_i + \underset{2}{d}u_i + \underset{21}{dd}u_i)$, $(x^i + \underset{2}{d}x^i, u_i + \underset{2}{d}u_i)$ and consider the projective development along the infinitesimal circuit, then the infinitesimal change of A_0 will be given by

$$\underset{21}{dd}A_0 - \underset{12}{dd}A_0 = \Omega^0_0 A_0 + \Omega^i_0 A_i, \tag{4·5}$$

where

$$\left.\begin{aligned} \Omega^0_0 = {} & (\phi_{k,l} - \phi_{l,k} + \omega^0_{kl} - \omega^0_{lk}) \underset{1}{d}x^k \underset{2}{d}x^l \\ & + (\phi_k|^l - \phi^l{}_{,k} + \omega^{0l}_k)(\underset{1}{d}x^k \underset{2}{d}u_l - \underset{2}{d}x^k \underset{1}{d}u_l) + (\phi^k|^l - \phi^l|^k) \underset{1}{d}u_k \underset{2}{d}u_l, \\ \Omega^i_0 = {} & (\Pi^i_{jk} - \Pi^i_{kj}) \underset{1}{d}x^j \underset{2}{d}x^k + \Pi^{ik}_j (\underset{1}{d}x^j \underset{2}{d}u_k - \underset{2}{d}x^j \underset{1}{d}u_k). \end{aligned}\right\} \tag{4·6}$$

If the point A_0 will return to the initial position after a projective development along an infinitesimal circuit neglecting the infinitesimal quantities higher than the second order, we say that our space has no torsion. The necessary and sufficient condition that our space have no torsion is that

$$\Pi^i_{jk} = \Pi^i_{kj}, \quad \Pi^{ik}_j = 0. \tag{4·7}$$

We shall assume in the following that our space has always no torsion.

Now, the semi-natural frame of reference is not uniquely determined, but it may be subjected to a transformation of the form

$$\bar{A}_0 = A_0, \quad \bar{A}_j = \lambda_j A_0 + A_j, \tag{4·8}$$

where the functions λ_j are supposed to be homogeneous of degree zero with respect to u_i. Considering the hyperplane determined by N points $A_1, A_2, \ldots\ldots, A_n$ as the hyperplane at infinity, we shall call (4·8) transformation of the hyperplane at infinity. The transformation law of the components of projective connection under this transformation of hyperplane at infinity is found to be

$$\left.\begin{aligned}&\bar{\Pi}^0_{jk}=\Pi^0_{jk}+\lambda_{j,k}-\lambda_i\Pi^i_{jk}-\lambda_j\lambda_k, \quad \bar{\Pi}^{0k}_j=\Pi^{0k}_j+\lambda_j|^k,\\ &\bar{\Pi}^i_{jk}=\Pi^i_{jk}+\delta^i_j\lambda_k+\delta^i_k\lambda_j.\end{aligned}\right\} \qquad (4\cdot9)$$

If we effect a transformation of coordinates $\bar{x}^i=\bar{x}^i(x)$, then the frame of reference $[\bar{A}_0, \bar{A}_1, \ldots\ldots, \bar{A}_n]$ which is semi-natural with respect to new coordinate system and has the same hyperplane at infinity as that of the old one given by

$$\bar{A}_0=A_0, \quad \bar{A}_j=\frac{\partial x^b}{\partial \bar{x}^j}A_b. \qquad (4\cdot10)$$

We call this transformation simply transformation of coordinates. The transformation law of the components of projective connection under coordinate transformation is found to be

$$\left.\begin{aligned}&\bar{\Pi}^0_{jk}=\frac{\partial x^b}{\partial \bar{x}^j}\frac{\partial x^c}{\partial \bar{x}^k}\Pi^0_{bc}, \qquad \bar{\Pi}^{0k}_j=\frac{\partial x^b}{\partial \bar{x}^j}\frac{\partial \bar{x}^k}{\partial x^c}\Pi^{0c}_b,\\ &\bar{\Pi}^i_{jk}=\frac{\partial \bar{x}^i}{\partial x^a}\left(\frac{\partial x^b}{\partial \bar{x}^j}\frac{\partial x^c}{\partial \bar{x}^k}\Pi^a_{bc}+\frac{\partial^2 x^a}{\partial \bar{x}^j\partial \bar{x}^k}\right)\end{aligned}\right\} \qquad (4\cdot11)$$

Thus Π^0_{jk} and Π^{0k}_j are components of tensors and Π^i_{jk} are those of affine connection under a coordinate transformation of the space.

§ 5. *Differential equations of projectively flat hypersurfaces.* Let us consider, in our space, a hypersurface defined by $f(x)=0$, and assume that the point $P=A_0+p^iA_i\left(u_ip^i=0, \ u_i=\dfrac{\partial f}{\partial x^i}\right)$ contained in the hyperplane element is always contained in the hyperplane element when we effect a projective development along the hypersurface $f(x)=0$, then such a hypersurface is called projectively flat hypersurface. Now, as we have

$$P+dP=(1+\omega^0_0+p^j\omega^0_j)A_0+(p+dx^i+dp^i+p^j\omega_j)A_i,$$

the necessary and sufficient condition that the point $P+dP$ be contained in the hyperplane element is that

$$u_i(dp^i + p^j\omega_j^i) = 0$$

be satisfied by any p^i and dx^i satisfying $u_ip^i=0$ and $u_idx^i=0$, from which

$$u_idx^i=0, \quad du_j - \Pi^i_{jk}u_idx^k = u_j\psi_kdx^k, \tag{5·1}$$

or

$$u_idx^i=0, \quad du_j - (\Pi^i_{jk}u_i + \psi_ju_k + \psi_ku_j)dx^k = 0. \tag{5·2}$$

These are the equations of projectively flat hypersurfaces, where ψ_j are arbitrary functions homogeneous of degree zero with respect to u_i.

From the form of (5·2), we can see that they are invariant in the form under (i) the transformation of the hyperplane at infinity, (ii) the transformation of coordinates and (iii) the change of the factor. We shall remark here that, if we effect a suitable transformation of the hyperplane the equations (5·2) take the form

$$u_idx^i=0, \quad du_j - \Pi^i_{jk}u_idx^k = 0. \tag{5·3}$$

§ 6. *The main problem.* Let us suppose that there is given, in an N-dimensional space, a system of hypersurfaces having the properties as explained in § 2 and § 3.

Then, our main problem is to determine a projective connection in a space of hyperplane elements whose system of projectively flat hypersurfaces will coincide exactly with the given system of hypersurfaces.

The differential equations of hyperplanes take the form

$$u_idx^i=0, \quad du_j - \Pi^i_{jk}u_idx^k = 0 \tag{6·1}$$

with respect to a suitable semi-natural frame of reference. So, to solve our main problem, comparing (2·2) and (6·1), we put the first condition:

$$\Gamma^i_{jk} = \Pi^i_{jk} + \Pi^{0i}_j u_k + \Pi^{0i}_k u_j. \tag{6·2}$$

As we have, from (6·2),

$$\Gamma_{jk} = \Gamma^i_{jk}u_i = \Pi^i_{jk}u_i, \tag{6·3}$$

the equations (2·2) and (6·1) coincide and the condition (6·2) is consistent.

Moreover, as is explained in § 3, the projective change of Γ^i_{jk}:

$$\bar{\Gamma}^i_{jk}=\Gamma^i_{jk}+\delta^i_j H_k+\delta^i_k H_j+H_j|^i u_k+H_k|^i u_j \tag{6·4}$$

may be regarded as a transformation of the hyperplane at infinity by the functions H_j, where H_j are functions homogeneous of degree zero with respect to u_i and satisfy (3·4)

For, during the transformation of the hyperplane at infinity defined by H_j, the functions Π^i_{jk} and Π^{0i}_j are transformed into $\bar{\Pi}^i_{jk}$ and $\bar{\Pi}^{0i}_j$ respectively by the formulae

$$\bar{\Pi}^i_{jk}=\Pi^i_{jk}+\delta^i_j H_k+\delta^i_k H_j, \quad \bar{\Pi}^{0i}_j=\Pi^{0i}_j+H_j|^i,$$

and consequently we have

$$\bar{\Pi}^i_{jk}+\bar{\Pi}^{0i}_j u_k+\bar{\Pi}^{0i}_k u_j=\Pi^i_{jk}+\Pi^{0i}_j u_k+\Pi^{0i}_k u_j+\delta^i_j H_k+\delta^i_k H_j+H_j|^i u_k+H_k|^i u_j. \tag{6·5}$$

But, if we effect a transformation of the hyperplane at infinity defined by an arbitrary functions λ_j, then the functions defined by (6·2) will be transformed into

$$\bar{\Gamma}^i_{jk}=\Gamma^i_{jk}+\delta^i_j\lambda_k+\delta^i_k\lambda_j+\lambda_j|^i u_k+\lambda_k|^i u_j, \tag{6·6}$$

where λ_j do not necessarily satisfy the condition (3·4). So that, we can not generally regard the transformation of the hyperplane at infinity as the transformation of the factor. We can do it when and only when the functions λ_j defining it satisfy the codition (3·4).

We are now going to determine all the coefficients of the projective connection in terms of the given functions Γ_{jk} only.

§ 7. *Projective curvature tensors.* We shall effect a projective development along an infinitesimal parallelogram which was considered in § 4 and calculate the variations of the points A_j. By a straightforward calculation, we find

$$\underset{21}{dd}A_j-\underset{12}{dd}A_j=\Omega^0_j A_0+\Omega^i_j A_i. \tag{7·1}$$

But, following E. Cartan, the curvature tensors of the space are given by Ω^0_j and $\Omega^i_j-\delta^i_j\Omega^0_0$, thus putting

$$\delta u_j=du_j-\Pi^i_{jk}u_i dx^k, \tag{7·2}$$

$$\Omega^0_j = P^0{}_{jkl}\underset{1}{dx^k}\underset{2}{dx^l} + P^0{}_{jk}{}^l(\underset{1}{dx^k}\underset{2}{\delta u_l} - \underset{2}{dx^k}\underset{1}{\delta u_l}) + P^0{}_j{}^{kl}\underset{1}{\delta u_k}\underset{2}{\delta u_l}, \tag{7·3}$$

$$\Omega^i_j - \delta^i_j\Omega^0_0 = P^i{}_{jkl}\underset{1}{dx^k}\underset{2}{dx^l} + P^i{}_{jk}{}^l(\underset{1}{dx^k}\underset{2}{\delta u_l} - \underset{2}{dx^k}\underset{1}{\delta u_l}), \tag{7·4}$$

we find

$$\begin{aligned} P^0{}_{jkl} = & (\Pi^0_{jk,l} + \Pi^0_{jk}|^b \Pi^a_{bl}u_a) - (\Pi^0_{jl,k} + \Pi^0_{jl}|^b \Pi^a_{bk}u_a) \\ & + \Pi^a_{jk}(\Pi^0_{al} + \Pi^{0b}_a \Pi^c_{bl}u_c) - \Pi^a_{jl}(\Pi^0_{ak} + \Pi^{0b}_a \Pi^c_{bk}u_c) \\ & - (\Pi^{0a}_{j,k} + \Pi^{0a}_j|^b \Pi^c_{bk}u_c)\Pi^d_{al}u_d + (\Pi^{0a}_{j,l} + \Pi^{0a}_j|^b \Pi^c_{bl}u_c)\Pi^d_{ak}u_d, \end{aligned} \tag{7·5}$$

$$P^0{}_{jk}{}^l = \Pi^0_{jk}|^l - (\Pi^{0l}_{j,k} + \Pi^{0l}_j|^b \Pi^a_{bk}u_a) + \Pi^{0a}_j|^l \Pi^b_{ak}u_b + \Pi^a_{jk}\Pi^{0l}_a, \tag{7·6}$$

$$P^0{}_j{}^{kl} = \Pi^{0k}_j|^l - \Pi^{0l}_j|^k, \tag{7·7}$$

$$\begin{aligned} P^i{}_{jkl} = & (\Pi^i_{jk,l} + \Pi^i_{jk}|^b \Pi^a_{bl}u_a) - (\Pi^i_{jl,k} + \Pi^i_{jl}|^b \Pi^a_{bk}u_a) + \Pi^a_{jk}\Pi^i_{al} - \Pi^a_{jl}\Pi^i_{ak} \\ & + (\Pi^0_{jk} + \Pi^{0b}_j \Pi^a_{bk}u_a)\delta^i_l - (\Pi^0_{jl} + \Pi^{0b}_j \Pi^a_{bl}u_a)\delta^i_k \\ & - \delta^i_j\{(\Pi^0_{kl} + \Pi^{0b}_k \Pi^a_{bl}u_a) - (\Pi^0_{lk} + \Pi^{0b}_l \Pi^a_{bk}u_a)\}, \end{aligned} \tag{7·8}$$

$$P^i{}_{jk}{}^l = \Pi^i_{jk}|^l - \delta^i_j \Pi^{0l}_k - \delta^i_k \Pi^{0l}_j. \tag{7·9}$$

As is easily seen, there exist the following identities satisfied by these curvature tensors:

$$P^0{}_{jk}{}^l u_l = 0, \quad P^0{}_j{}^{kl} u_k = 0, \quad P^0{}_j{}^{kl} u_l = 0, \quad P^i{}_{jk}{}^l u_l = 0. \tag{7·10}$$

Now, if we effect a transformation of the hyperplane at infinity (4·8), these curvature tensors are transformed respectively into

$$\bar{P}^0{}_{jkl} = P^0{}_{jkl} - \lambda_i P^i{}_{jkl} + (P^0{}_{jk}{}^a - \lambda_i P^i{}_{jk}{}^a)\lambda_a u_l - (P^0{}_{jl}{}^a - \lambda_i P^i{}_{jl}{}^a)\lambda_a u_k, \tag{7·11}$$

$$\bar{P}^0{}_{jk}{}^l = P^0{}_{jk}{}^l - \lambda_i P^i{}_{jk}{}^l + P^0{}_j{}^{al}\lambda_a u_k, \tag{7·12}$$

$$\bar{P}^0{}_j{}^{kl} = P^0{}_j{}^{kl}, \tag{7·13}$$

$$\bar{P}^i{}_{jkl} = P^i{}_{jkl} + P^i{}_{jk}{}^a\lambda_a u_l - P^i{}_{jl}{}^a\lambda_a u_k, \tag{7·14}$$

$$\bar{P}^i{}_{jk}{}^l = P^i{}_{jk}{}^l. \tag{7·15}$$

Thus we can see that the tensor $P^0{}_j{}^{kl}$ and $P^i{}_{jk}{}^l$ are both purely projective.

§ 8. *Determination of coefficients of projective connection.* Now, we shall proceed to solve our main problem, that is to say, to determine all the coefficients of the projective connection in terms of Γ_{jk}. For this, we put first a purely projective condition:

$$P^a{}_{ja}{}^l=0. \tag{8·1}$$

Putting (6·2) into the expression of $P^i{}_{jk}{}^l$ and calculating (8·1), we find

$$(\Gamma^a_{aj}-\Pi^{0a}_a u_j)|^i-(N+1)\Pi^{0i}_j=0,$$

from which, contracting with respect to the indices i and j, we find $\Pi^{0a}_a=\dfrac{1}{2N}\Gamma^a_{ab}|^b$, and consequently

$$\Pi^{0i}_j=L_j|^i=\frac{1}{N+1}\Big(\Gamma^a_{aj}-\frac{1}{2N}\Gamma^a_{ab}|^b u_j\Big)|^i. \tag{8·2}$$

The Π^{0i}_j being thus determined, we have, from (6·2),

$$\Pi^i_{jk}=\Gamma^i_{jk}-L_j|^i u_k-L_k|^i u_j. \tag{8·3}$$

Substituting (8·2) and (8·3) into (7·9), we shall have the expressions for $P^i{}_{jk}{}^l$ in terms of Γ^i_{jk}, but this expression will be condensed in the form

$$P^i{}_{jk}{}^l=(\Gamma_{jk}-L_j u_k-L_k u_j)|^i|^l. \tag{8·4}$$

The functions Π^{0i}_j and Π^i_{jk} being thus determined, to determine the functions Π^0_{jk}, we put the condition

$$P^a{}_{jka}=0. \tag{8·5}$$

This condition is, as is seen from (7·14), (8·1) and $u_i P^i{}_{jk}{}^l=P^l{}_{jk}{}^i u_i=0$, invariant under the transformation of the hyperplane at infinity, that is, the condition is a purely projective one. Following E. Cartan, we shall call the space of hyperplane elements with normal projective connection, the space whose projective connection satisfies the condition (8·1) and (8·5).

From (7·8) and (8·5), we find

$$\Pi^0_{jk}+\Pi^{0b}_j\Pi^a_{bk}u_a=-\frac{1}{N^2-1}(N\Pi^a{}_{jka}+\Pi^a{}_{kja}), \tag{8·6}$$

where we have put

$$\Pi^{i}{}_{jkl}=(\Pi^{i}_{jk,l}+\Pi^{i}_{jk}|^{b}\Pi^{a}_{bl}u_{a})-(\Pi^{i}_{jl,k}+\Pi^{i}_{jl}|^{b}\Pi^{a}_{bk}u_{a})+\Pi^{a}_{jk}\Pi^{i}_{al}-\Pi^{a}_{jl}\Pi^{i}_{ak}, \quad (8\cdot7)$$

Substitutiog (8·6), into (7·8), we find

$$P^{i}{}_{jkl}=\Pi^{i}{}_{jkl}-\frac{1}{N^2-1}(N\Pi^{a}{}_{jka}+\Pi^{a}{}_{kja})\delta^{i}_{l} \quad (8\cdot8)$$

$$+\frac{1}{N^2-1}(N\Pi^{a}{}_{jla}+\Pi^{a}{}_{lja})\delta^{i}_{k}+\frac{1}{N+1}\delta^{i}_{j}(\Pi^{a}{}_{kla}-\Pi^{a}{}_{lka}).$$

Substituting (8·2), (8·3) and (8·6) into (7·5), (7·6) and (7·7), we can find the expressions for other curvature tensors $P^{0}{}_{jkl}$, $P^{0}{}_{jk}{}^{l}$ and $P^{0}{}_{j}{}^{kl}$.

Thus we have succeeded in determining the projective connection with respect to a semi-natural frame of reference in such a way that the equations (2·1) will be exactly those for the system of projectively flat hypersurfaces.

§ 9. *Natural frame of reference.* We have completely determined the coefficients of a projective connection of a space of hyperplane elements, but they are determined with respect to a semi-natural frame of reference, so that we can yet effect a transformation of the hyperplane at infinity. Among these semi-natural frames of reference, we shall choose a special one called natural frame of reference. If the functions Π^{i}_{jk} satisfy the condition

$$*\Pi^{a}_{ja}={}^{*}\Pi^{a}_{ja}=0, \quad (9\cdot1)$$

the semi-natural frame of reference will be called the natural frame of reference. In the following, we shall add an asterisk to the quantities expressed wih respect to the natural frame of reference.

The transformation of the hyperplane at infinity which carries a semi-natural frame of reference $[A_0, A_j]$ into the natural one $[*A_0, *A_j]$ is given by

$$*A_0=A_0, \quad *A_j=-\frac{1}{N+1}\Pi^{a}_{aj}A_0+A_j, \quad (9\cdot2)$$

or, according to (3·20),

$$*A_0=A_0, \quad *A_j=-L_jA_0+A_j. \quad (9\cdot3)$$

The coefficients of the projective connection with respect to the natural frame of reference are given by

$$*\Pi^0_{jk}=\Pi^0_{jk}-L_{j,k}+L_i\Pi^i_{jk}-L_jL_k, \tag{9·4}$$

$$*\Pi^{0i}_j=0, \tag{9·5}$$

$$*\Pi^i_{jk}=\Pi^i_{jk}-\delta^i_jL_k-\delta^i_kL_j. \tag{9·6}$$

The functions $*\Pi^i_{jk}$ given by (9·6) are also written as

$$*\Pi^i_{jk}=\Gamma^i_{jk}-\delta^i_jL_k-\delta^i_kL_j-L_j|^iu_k-L_k|^iu_j=(\Gamma_{jk}-L_ju_k-L_ku_j)|^i, \tag{9·7}$$

and coincide well with the coefficients of the connection of the projective geometry of hyperplanes discussed in § 3.

The curvature quantities with respect to the natural frame of reference are given by

$$*P^0{}_{jkl}=(*\Pi^0_{jk,l}+*\Pi^0_{jk}|^{a}*\Pi_{al})-(*\Pi^0_{jl,k}+*\Pi^0_{jl}|^{a}*\Pi_{ak})$$
$$+*\Pi^a_{jk}*\Pi^0_{al}-*\Pi^a_{jl}*\Pi^0_{ak}, \tag{9·8}$$

$$*P^0{}_{jk}{}^l=*\Pi^0_{jk}|^l, \tag{9·9}$$

$$*P^0{}_j{}^{kl}=0, \tag{9·10}$$

$$*P^i{}_{jkl}=*\Pi^i{}_{jkl}+*\Pi^0_{jk}\delta^i_l-*\Pi^0_{jl}\delta^i_k-\delta^i_j(*\Pi^0_{kl}-*\Pi^0_{lk}), \tag{9·11}$$

$$*P^i{}_{jk}{}^l=*\Pi^i_{jk}|^l, \tag{9·12}$$

where

$$*\Pi^i{}_{jkl}=(*\Pi^i_{jk,l}+*\Pi^i_{jk}|^{a}*\Pi_{al})-(*\Pi^i_{jl,k}+*\Pi^i_{jl}|^{a}*\Pi_{ak})+*\Pi^a_{jk}*\Pi^i_{al}-*\Pi^a_{jl}*\Pi^i_{ak}, \tag{9·13}$$

The $*\Pi^i{}_{jkl}$ satisfies

$$*\Pi^a{}_{akl}=0, \quad *\Pi^a{}_{jka}=*\Pi^a{}_{kja} \tag{9·14}$$

Remembering that our projective connection is normal and consequently we have $*P^a{}_{jka}=0$, we can obtain, from (9·11),

$$*\Pi^0_{jk}=-\frac{1}{N-1}*\Pi^a{}_{jka}, \tag{9·15}$$

$$*P^i{}_{jkl}=*\Pi^i{}_{jkl}-\frac{1}{N-1}*\Pi^a{}_{jka}\delta^i_l+\frac{1}{N-1}*\Pi^a{}_{jla}\delta^i_k. \tag{9·16}$$

The $*P^i_{jkl}$ which is nothing but $*W^i_{jkl}$ in § 3, satisfy

$$*P^a_{akl}=0, \quad *P^a_{jka}=0, \quad u_i*P^i_{jkl}=0. \tag{9·17}$$

Now, we shall consider the relation between the curvature quantities $*P^i_{jkl}$ and P^i_{jkl}. From (7·14) and (9·3) we have

$$*P^i_{jkl}=P^i_{jkl}-P^i_{jk}{}^aL_au_l+P^i_{jl}{}^aL_au_k,$$

but remembering that $P^i_{jk}{}^l=*P^i_{jk}{}^l=*\Pi^i_{jk}|^l$, we can rewrite the above expression in the form

$$P^i_{jkl}=*P^i_{jkl}+*\Pi^i_{jk}|^aL_au_l-*\Pi^i_{jl}|^aL_au_k. \tag{9·18}$$

If we use the relation

$$*\Pi_{jk}=\Gamma_{jk}-L_ju_k-L_ku_j, \tag{9·19}$$

obtained from (9·8), then the P^i_{jkl} becomes

$$P^i_{jkl}=B^i_{jkl}-\frac{1}{N-1}B^a_{jka}\delta^i_l+\frac{1}{N-1}B^a_{jla}\delta^i_k, \tag{9·20}$$

where

$$B^i_{jkl}=(*\Pi^i_{jk,l}+*\Pi^i_{jk}|^a\Gamma_{al})-(*\Pi^i_{jl,k}+*\Pi^i_{jl}|^a\Gamma_{ak})+*\Pi^a_{jk}*\Pi^i_{al}-*\Pi^a_{jl}*\Pi^i_{ak}. \tag{9·21}$$

The epuation (9·20) shows us that the P^i_{jkl} coincide will with the curvature tensor (3·42) given for the projective geometry of hyperplanes in § 3.

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