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Integration of Fokker-Planck's Equation with a Boundary Condition

Kôsaku YOSIDA

1. Introduction. We consider Fokker-Planck's equation[1)]

$$\frac{\partial f(t,x)}{\partial t}=Af(t,x),\ t\geqq 0, \tag{1}$$

$$(Af)(x)=\frac{1}{\sqrt{g(x)}}\frac{\partial^2}{\partial x^i\partial x^j}(\sqrt{g(x)}\,b^{ij}(x)f(x))$$
$$+\frac{1}{\sqrt{g(x)}}\frac{\partial}{\partial x^i}(-\sqrt{g(x)}\,a^i(x)f(x))$$

in a connected region R of an n-dimensional orientable Riemannian space with the metric $ds^2=g_{ij}(x)dx^idx^j$. As usual, the volume element in R is defined by $dx=\sqrt{g(x)}dx^1dx^2\cdots dx^n$, $g(x)=det(g_{ij}(x))$. We assume that the contravariant tensor $b^{ij}(x)$ be such that $b^{ij}(x)\xi_i\xi_j>0$ in R (for $\sum_i \xi_i^2>0$). The $a^i(x)$ obeys, by the coordinate change $x\rightarrow\bar{x}$, the transformation rule

$$\bar{a}^i(\bar{x})=\frac{\partial\bar{x}^i}{\partial x^k}a^k(x)+\frac{\partial^2\bar{x}^i}{\partial x^k x^s}b^{ks}(x). \tag{2}$$

These propeities of the coefficients $a^i(x)$ and $b^{ij}(x)$ are connected with the probabilistic meaning of the equation (1).

We assume that $g_{ij}(x)$, $a^i(x)$ and $b^{ij}(x)$ are infinitely differentiable functions of the coordinates $x=(x^1,x^2,\cdots,x^n)$. The purpose of the present note is to consider a certain natural boundaiy condition on the boundary ∂R of R for the probability density $f(t,x)$ at the time moment $t>0$ and to discuss, for this boundary condition, the stochastic integrability (in the sense to be explained in § 3) of the equation (1). As in the previous papers, our treatment and the method of proof relies upon the theory of semi-group of linear operators,[2)] which is, so to speak, an operator-theo-

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1) A. Kolmogoroff: Zur Theorie der stetigen zufälligen Prozess, Math. Ann., **108** (1933), 149–160. K. Yosida: An extension of Fokker-Planck's equation, Proc. Japan Acad., **25** (1949), (9), 1–3.

2) E. Hille: Functional Analysis and Semi-groups, New York (1948). K. Yosida: On the differentiability and the representation of one-parameter semi-group of linear operators, Journ. Math. Soc. Japan, **1** (1949), 1, 15–21, and K. Yosida: An operator-theoretical treatment of temporally homogeneous Markoff process, ibid., **1** (1949), 1, 244–235.

retical adaptation of the Laplace transform method in partial differential equations.

2. Green's formula and the boundary condition. Let A' be the formally adjoint operator of A:

(3) $$(A'h)(x)=b^{ij}(x)\frac{\partial^2 h}{\partial x^i \partial x^j}+a^i(x)\frac{\partial h}{\partial x^i}.$$

By partial integration, we obtain the Green's formula:

(4) $$\int_G (h(x)(Af)(x)-f(x)(A'h)(x))dx=$$
$$\int_{\partial G}\sqrt{g(x)}b^{ij}(x)\,(h(x)\frac{\partial f}{\partial x^j}-f(x)\frac{\partial h}{\partial x^j})\,\Pi_i(x)dS+$$
$$\int_{\partial G}\Big(\frac{\partial\sqrt{g(x)}b^{ij}(x)}{\partial x^j}-\sqrt{g(x)}a^i(x)\Big)\Pi_i(x)f(x)h(x)dS,$$

where $\Pi_i(x)$ is $\cos(n, x^i)$, n being outer normal at the point x of the boundary ∂G of the connected domain $G\subseteq R$, and dS denotes hypersurface area on ∂G. If $b^{ij}(x)\Pi_i(x)\Pi_j(x)>0$ at $x\in\partial G$, we may define the outer transversal direetion ν at x by

(5) $$\frac{dx^i}{\sqrt{g(x)}b^{ij}(x)\,\Pi_j(x)}=d\nu \quad (i=1,2,\cdots,n)$$

so that

(6) $$\sqrt{g(x)}b^{ij}(x)\Big(h(x)\frac{\partial f}{\partial x^j}-f(x)\frac{\partial h}{\partial x^j}\Big)\Pi_i(x)dS=$$
$$\Big(h(x)\frac{\partial f}{\partial \nu}-f(x)\frac{\partial h}{\partial \nu}\Big)dS.$$

We consider A to be an additive operator defined for the totality $D(A)$ of infinitely differentiable functions $f(x)$ on R which vanish outside some compact set (depending upon $f(x)$) and which also satisfy the boundary condition on ∂R:

(7) $$\sqrt{g(x)}b^{ij}(x)\frac{\partial f}{\partial x^j}\Pi_i(x)+\Big(\frac{\partial\sqrt{g(x)}b^{ij}(x)}{\partial x^j}$$
$$-\sqrt{g(x)}a^i(x)\Big)\Pi_i(x)f(x)=0.$$

$D(A)$ is surely dense in the Banach space $L_1(R)$ of integrable (with respect to dx) functions on R, metrized by the norm $\|f\|=\int_R|f(x)|dx$. Thus A

may be considered as an additive operator defined for $D(A) \subseteqq L_1(R)$ with values in $L_1(R)$.

3. Stochastic integration of (1) with the boundary condition (7). We first prove

Lemma 1. Let $f(x) \in D(A)$ be positive (negative) in a connected domain $G \subseteqq R$ such that $f(x)$ vanishes on $\partial G - \partial R$, viz. on the part of ∂G not contained in ∂R. Then we have, for any positive number m, the inequality

(8) $$\int_G (f(x) - m^{-1}(Af)(x))dx \geqq \int_G f(x)dx > 0 \left(\leqq \int_G f(x)dx < 0\right).$$

Proof. By (4)–(7), we have

$$\int_G (Af)(x)dx = \int_{G-\partial R} \frac{\partial f}{\partial \nu} dS \leqq 0 \quad (\geqq 0).$$

Corollary. For any $f(x) \in D(A)$, we have

(9) $$\|f - m^{-1}Af\| \geqq \|f\| \text{ for } m > 0, \text{ and for } f \in D(A).$$

Proof. Let $h(x)$ be $=1, -1$ or 0 according as $f(x)$ is >0, <0 or $=0$. Since the conjugate space $L_1(R)^*$ of $L_1(R)$ is the space of all the essentially bounded measurable functions $k(x)$ with the norm $\|k\|^* =$ essential sup $|k(x)|$, we have, by the above lemma,

$$\|f - m^{-1}Af\| \geqq \int_R h(x)(f(x) - m^{-1}(Af)(x))dx = \int_R |f(x)|dx$$
$$- m^{-1} \sum_i \int_{P_i} (Af)(x)dx + m^{-1} \sum_j \int_{N_j} (Af)(x)dx,$$

where $P(N)$ is connected domain in which $f(x) > 0$ (<0) such that $f(x)$ vanishes on the boundary $\partial P(\partial N)$. Q. E. D.

Thus, there exists the bounded additive inverse

(10) $$(I - m^{-1}\tilde{A})^{-1},$$

where I and $\tilde{A}$ respectively denote identity operator and the smallest closed extension of A. Thus we have the

Lemma 2. The resolvent I_m (= everywhere defined inverse $(I - m^{-1}\tilde{A})^{-1}$) exists if and only if the range $\{(I - m^{-1}A)f;\ f \in D(A)\}$ of the operator $(I - m^{-1}A)$ is dense in $L_1(R)$. Moreover, if the resolvent I_m exists, it is a transition operator, viz.

(11) $f(x) \geqq 0$ and $f \in L_1(R)$ imply $(I_m f)(x) \geqq 0$ and $\int_R (I_m f)(x)dx = \int_R f(x)dx$.

Proof. The first part of the lemma is evident. Let the resolvent I_m

exist, For any $g(x)\geqq 0$ of $L_1(R)$, there exists a sequence $\{f_k(x)\}\subseteqq D(A)$ such that

$$\text{strong} \lim_{k\to\infty} f_k=f \text{ exists and strong} \lim_{k\to\infty} (f-m^{-1}Af_k)=f-m^{-1}\tilde{A}f=g.$$

On the other hand, by the boundary condition (7), we have

$$\int_R (f_k(x)-m^{-1}(Af_k)(x))dx=\int_R f_k(x)dx.$$

Hence, in the limit $k\to\infty$, we have

$$\int_R g(x)dx=\int_R f(x)dx \text{ and } \int_R |g(x)|dx\geqq \int_R |f(x)|dx \text{ (by (9))}.$$

Therefore $f(x)\geqq 0$ almost everywhere and $\|g\|=\|f\|$. Q. E. D.

By the semi-group theory, there exists a one-parameter semi-group of transition operators T_t satisfying the conditions:

(12) $T_tT_s=T_{t+s}$, $(s, t\geqq 0)$, $T_0=$the identity,

strong $\lim_{t\to t_0} T_tf=T_{t_0}f$, $f\epsilon L_1(R)$,

strong $\lim_{\delta\to 0} \frac{T_{t+\delta}-T_t}{\delta}f=\tilde{A}T_tf$ for f in the domain $D(\tilde{A})$ of $\tilde{A}$,

if and only if the resolvents I_m (for $m>0$) exist as transition operators. This T_t is, in fact, defined by

(13) $T_tf=\text{strong} \lim_{m\to\infty} (I-m^{-1}t\tilde{A})^{-m}f$, $f\epsilon L_1(R)$.

The existence of this semi-group is just the stochastic integrability mentioned in the introduction. That $T_tf(t>0)$, $f\epsilon L_1(R)$, satisfies the boundary condition (7) in a limiting sense may be seem fiom (13) and the definition of the smallest closed extention $\tilde{A}$ of A.

4. The theorem. Since the conjugate space of $L_1(R)$ is the space of essentially bounded measurable functions, we see that the stochastic integrability of (1) is equivalent to the non-existence of bounded measurable function $h(x)$ such that

(14) $\|h\|^*>0$ and $\int_R h(x)(f(x)-m^{-1}(Af)(x))dx=0$ for all $f\epsilon D(A)$.

Thus, if we define the distribution $H(f)$ in the sense of L. Schwartz:

(15) $$H(f)=\int_R h(x)f(x)dx,$$

we see that H satisfies the elliptic differential equation (in the sense of the distribution)[3]

(16) $$A'H=mH.$$

3) L. Schwartz: Théorie des distributions, 1, Paris (1950).

Hence,[4] if $n \geqq 2$, there exists a function $\tilde{h}(x)$ infinitely differentiable in R such that

(17) $$(A'\tilde{h})(x) = m\tilde{h}(x) \text{ in } R,\ H(f) = \int_R \tilde{h}(x) f(x) dx.$$

Surely $\tilde{h}(x)$ is equal to $h(x)$ almost everywhere, and so does not vanish identically. Let $\{R_k\}$ be a monotone increasing sequence of connected domains $\subseteqq R$ such that the boundary ∂R_k tends, as $k \to \infty$, to the boundary ∂R very smoothly. Then we have, by the Green's formula (4), (14) and (17),

(18) $$\int_{\partial R_k} \sqrt{g(x)}\, b^{ij}(x) \Big(\tilde{h}(x) \frac{\partial f}{\partial x^j} - f(x) \frac{\partial \tilde{h}}{\partial x^j}\Big) \Pi_i(x) dS +$$
$$\int_{\partial R_k} \Big(\frac{\partial \sqrt{g(x)}\, b^{ij}(x)}{\partial x^j} - \sqrt{g(x)}\, a^i(x)\Big) \Pi_i(x) f(x) \tilde{h}(x) dS = 0, f \in D(A).$$

By the boundedness of $\tilde{h}(x)$ and the boundary condition of $f(x) \in D(A)$, we have

(19) $$\lim_{k \to \infty} \int_{\partial R_k} \sqrt{g(x)}\, b^{ij}(x) f(x) \frac{\partial \tilde{h}}{\partial x^j} \Pi_i(x) dS = 0,\ f \in D(A).$$

Therefore we have the

Theorem. Let the dimension n of R be $\geqq 2$. Then the stochastic integrability of (1) with the boundary condition (7) is equivalent to the non-existence, for $m > 0$, of bounded solution $\tilde{h}(x) \not\equiv 0$ of

(20) $$(A'\tilde{h})(x) = m\tilde{h}(x) \text{ in } R$$

satisfying the boundary candition (19).

Remark. The above condition of the stochastic integrability is satisfied in the case of compact Riemannian spach R, as is shown by the following argument. At a maximizing (minimizing) point x_0 of $\tilde{h}(x)$ we have $(A'\tilde{h})(x_0) \leqq (\geqq) 0$, so that a continuous solution $\tilde{h}(x)$ of (20) cannot have either a positive maximum or a negative minimum. Other applications of the Theorem to concrete examples will be published elsewhere.

Nagoya University

4) L. Schwartz: loc. cit. Cf. also an early paper by K. Kodaira: Harmonic tensor fields in Riemannian manifolds, Ann. of Math., **50** (1949), 587–655.

5) Cf. K. Yosida: Integration of Fokker-Planck's equation in a compact Riemannian space, Arkiv för Matematik, **1**, (1949), 9, 1–3.