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Some Remarks on Relatively Free Homotopy.

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Consider an arcwise connected topological space Z and select one of its points $*$ as a base point. Suppose furthermore that there is given an arcwise connected subspace Y of Z containing the base point $*$. Given a point $*'$ of Y, which may or may not be distinct from $*$, a path component of Y, i. e. a homotopy class of paths from $*$ to $*'$, induces an isomorphism between two n-th relative homotopy groups $\pi_n(Z, Y, *)$ and $\pi_n(Z, Y, *')$, attached to two points $*$, $*'$ respectively. If in particular $* = *'$, every element of the fundamental group $\pi_1(Y, *)$ induces an automorphism of the group $\pi_n(Z, Y, *)$, and therefore, algebraically speaking, the former may be regarded as a group of operators on the latter. Now I shall define a homotopy group $\sigma_n(Z, Y, *)$ for every integer $n \geqq 3$, containing subgroups isomorphic to $\pi_n(Z, Y, *)$ and $\pi_1(Y, *)$, in which the operation of $\pi_1(Y, *)$ on $\pi_n(Z, Y, *)$ forms an inner automorphism. As is seen later, an element of the group σ_n can be represented by a continuous mapping belonging to Z^{E^n} which transforms $S^{n-1} = \dot{E}^n$ into Y and to different points on S^{n-1} into the base point $*$. (E^n means an n-dimensional cube, see foot note) The pair (Z, Y) is usually called "relatively n-simple," if $\alpha^\xi = \alpha$ for any element ξ of $\pi_1(Y, *)$ and any α belonging to $\pi_n(Z, Y, *)$, and it is well known that in such a pair of spaces a base point $*$ can be arbitrarily selected in Y, in the sense that the isomorphism between two groups $\pi_n(Z, Y, *)$ and $\pi_n(Z, Y, *')$ attached to an arbitrarily chosen point $*'$ in Y is determined indepently of the path connecting $*$ to $*'$. Therefore the simplicity of a pair of spaces may be considered as an intrinsic property of the pair. A pair (Z, Y) which is relatively n-simple is characterized by the purely algebraic relation in σ_n : $\sigma_n(Z, Y, *)$ is isomorphic to the direct product of two groups $\pi_n(Z, Y, *)$ and $\pi_1(Y, *)$. This paper will contain these and some other remarks obtained by applying M. Abe's arguments in (1) to the case of relative homotopy groups.

1. *Definition of* $\sigma_n(Z, Y, *)$ *for* $n \geqq 3$.

Let $e(x_0)$, $1 \geqq x_0 \geqq 0$, be a $*$-based loop in Y. Denote by σ_n the

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1) $E^n = x^n\ (x_0, x_1, \cdots\cdots, x_{n-1})$; $1 \geqq x_1 \geqq 0$, $n-1 \geqq i \geqq 0$,
$x^n{}_i = (x_i\ , x_{i+1} \cdots\cdots x_{n-1})$

collection of all the Z-valued functions of the n-dimensional cube E^n satisfying the following conditions :[1]

i) $f(\bar{x}_0, x_1^n)$, for $1 \geqq \bar{x}_0 \geqq 0$, represents an element of $\pi_{n-1}(Z, Y, e(\bar{x}_0))$,
ii) $f(0, x_1^n) = f(1, x_1^n) = *$

Such a mapping f may also be described as follows;

$$\begin{aligned} f(x^n) &= * \quad \text{when } x_0(x_0-1)=0, \\ &= e(x_0) \text{ when } (x_{n-1}-1)\prod_{i=1}^{n-2} x_i(x_i-1)=0, \\ &\in Y \quad \text{when } \prod_{i=0}^{n-1} x_i(x_i-1)=0. \end{aligned}$$

Two such functions f and g, belonging to σ_n, are multiplied together according to the rule:

$$\begin{aligned} f\cdot(xg^n) &= f(2x_0, x_1^n) \qquad \text{when } \frac{1}{2} \geqq x_0 \geqq 0, \\ &= g(2x_0-1, x_1^n) \quad \text{when } 1 \geqq x_0 \geqq \frac{1}{2}, \end{aligned}$$

and the resulting function $f\cdot g$ is again a member of the collection σ_n. The elements of σ_n are classified by the homotopy concept, and the multiplication in σ_n induces a multiplication in the set of homotopy classes. Thus the classes of elements of σ_n together with the multiplication defined between them constitute a group, which I designate by $\sigma_n(Z, Y, *)$. As an immediate consequence of the definition, we remark that the identity of the group may be represented by a mapping, which transforms E^n into Y, such that $e(x^0)$, $1 \geqq x_0 \geqq 0$, can be shrunk in Y into the base point $*$. For convenience' sake K^n is referred to as the point set $\{x^n;\ x_0^2 + \cdots\cdots + x_{n-1}^2 \leqq 1\}$ and then the boundary $\dot{K}^n$ of K^n is of course an $(n-1)$-dimensional sphere S^{n-1}. Now consider a mapping φ of E^n onto K^n such that $\varphi(x^n(0, x_1^n)) = p_0$, $\varphi(x^n(1, x_1^n)) = p_1$, where p_0 and p_1 are two distinct points on $\dot{K}^n$; all the points of the same partial coordinate x_0 on the faces $(x_{n-1} -1) \prod_{i=0}^{n-2} x_i(x_i-1)=0$ are mapped continuously by φ to a point of the arc C on S^{n-1} joining p_0 to p_1; and the interior of E^n into the interior of K^n. (See figure 1, $n=3$) Then we have a mapping $\bar{f}$ of K^n into Z such that $f(x^n) = \bar{f}\varphi(x^n)$ and designate by $\bar{\sigma}_n$ the set of all the mappings which transform K^n into Z, S^{n-1} into Y, and two points on S^{n-1} into $*$. It is easy to see that two function spaces σ_n and σ_n are homeomorphic by the correspondence φ. For the reasons that an element of σ_n can be grasped in an intuitive manner and also be compared quite clearly with a representative of an element of the relative homotopy group $\pi_n(Z, Y, *)$, it seems advantageous to refer to the function space σ_n. As is well known, an element of $\pi_n(Z, Y, *)$ may be represented by a mapping which transforms K^n, into Z, S^{n-1} into Y, and the arc C on S^{n-1} joining p_0 to p_1 into $*$. The set of all such mappings will be denoted by Π_n. In order to avoid confusion we agree that the homotopic relation in Π_n is described by the symbol$\approx$, while in case of such a relation in σ_n or σ_n the symbol$\sim$will be used.

2. *Algebraic structure of* $\sigma_n(Z, Y, *)$.

First we shall prove that $\sigma_n(Z, Y, *)$ contains a subgroup $\pi_n(Z, Y, *)$ isomorphic to $\pi_n(Z, Y, *)$, and then that the factor group of $\sigma_n(Z, Y, *)$ by $\bar{\pi}_n(Z, Y, *)$ is isomorphic to the group $\bar{\pi}_1(Y, *)$, where $\bar{\pi}_1(Y, *)$ denotes a subgroup of $\sigma_n(Z, Y, *)$ isomorphic to $\pi_1(Y, *)$.

It is obvious that for two mappings f and g belonging to $\Pi_n, f\sim g$, if $f\approx g$. In order to prove the first assertion it is sufficient to show that if $f\sim g$, then $f\approx g$. Since $f\sim g$, there exists a mapping $h(x, s)$ belonging to $ZK^n\times \overset{s}{\breve{I}}$ such that $h(x, o)=f(x)$ and $h(x, 1)=g(x)$ for $x \in K^n$.

Furthermore $h(x, s) \in Y$, if $x \in S^{n-1}$ and $s \in \overset{s}{\breve{I}}$,

$$h(p_0, s)=h(p_1, s)=* \text{ for } s \in \overset{s}{\breve{I}}$$

As a point set $\{C\times (0)+C\times (1)+p_0\times \overset{s}{\breve{I}}\}$ is a deformation retract of $c\times \overset{s}{\breve{I}}$, a deformation D_t can be defined. (See figure 2) Let $\{s^{n-1}\times (0)+s^{n-1}\times (1)+c\times \overset{s}{\breve{I}}\}$ be denoted by T, then a mapping $\varphi(x, s, t)$ of $\{S^{n-1}\times \overset{s}{\breve{I}}\times (0)+T\times \overset{s}{\breve{I}}\}$ into Y is defined as follows;

$$\varphi(x, s, o)=h(x, s) \qquad \text{when } x \in S^{n-1},\ s \in \overset{s}{\breve{I}},$$

$$\varphi(x, o, t)=h(x, o)=f(x) \text{ when } x \in S^{n-1},\ t \in \overset{t}{\breve{I}},$$

$$\varphi(x, 1, t)=h(x, 1)=g(x) \text{ when } x \in S^{n-1},\ t \in \overset{s}{\breve{I}},$$

$$\varphi(x, s, t)=h(D_t(x, s)) \quad \text{when } x \in C,\ s \in \overset{s}{\breve{I}}, \text{ and } t \in \overset{t}{\breve{I}},$$

then the continuity of the mapping φ is verified from the following considerations. As an immediate consequence of the definition of D_t, we have

$\varphi(x, o, t)=h(D_t(x, 0))=h(x, 0)=f(x)=*$, $\varphi(x, 1, t)=g(x)=*$ if $x \in C$ and $t \in \overset{t}{\breve{I}}$, and $\varphi(x, s, 0)=h(D_0(x, s))=h(x, s)$ if $x \in C$ and $s \in \overset{s}{\breve{I}}$. It should be noted that $\varphi(x, s, 1)=h(D_1(x, s))=*$ when $x \in C$ and $s \in \overset{s}{\breve{I}}$. Since T is a subcomplex of $S^{n-1}\times \overset{s}{\breve{I}}$, $\{S^{n-1}\times \overset{s}{\breve{I}}\times (0)+T\times \overset{t}{\breve{I}}\}$ may be regarded as a deformation retract of $S^{n-1}\times \overset{s}{\breve{I}}\times \overset{t}{\breve{I}}$ so that φ defined on $\{S^{n-1}\times \overset{s}{\breve{I}}\times (0)+T\times \overset{s}{\breve{I}}\}$ can be extended continuously to a mapping of $S^{n-1}\times \overset{s}{\breve{I}}\times \overset{t}{\breve{I}}$ into Y. This extended mapping φ can be extended again in the following manner:

$$\Psi\equiv\varphi \quad \text{on } S^{n-1}\times \overset{s}{\breve{I}}\times \overset{t}{\breve{I}},$$

$$\Psi(x, o, t)=f(x) \quad \text{when } x \in K^n,\ t \in \overset{t}{\breve{I}},$$

$$\Psi(x, 1, t)=g(x) \quad \text{when } x \in K^n,\ t \in \overset{t}{\breve{I}},$$

$$\Psi(x, s, o)=h(x, s) \quad \text{when } x \in K^n,\ s \in \overset{s}{\breve{I}},$$

thus Ψ is defined on the complex $\{K^n\times \overset{s}{\breve{I}}\times (0)+S^{n-1}\times \overset{s}{\breve{I}}\times \overset{t}{\breve{I}}+K^n\times (0)\times \overset{t}{\breve{I}}+K^n\times (1)\times \overset{t}{\breve{I}}\}=\{S^{n-1}\times \overset{t}{\breve{I}}+K^n\times (0)+K^n\times (1)\}\times \overset{t}{\breve{I}}+K^n\times \overset{s}{\breve{I}}\times (0)$ which is a deformation retract of $K^n\times \overset{s}{\breve{I}}\times \overset{t}{\breve{I}}$. Therefore Ψ can be extended to a mapping of $K^n\times \overset{t}{\breve{I}}\times \overset{t}{\breve{I}}$ into Z, which we denote by the same letter Ψ. Now the partial mapping $\Psi|K^n\times \overset{s}{\breve{I}}\times (1)=\chi(x, s)$ is such that $\chi(x, 0)=f(x)$, $\chi(x, 1)=g(x)$, and $\chi(x, s)=*$ if $x \in C$, $s \in \overset{s}{\breve{I}}$, and therefore the first assertion is established.

The next part of our assertion was $\sigma_n(Z, Y, *)|\bar{\pi}_n(Z, Y, *)\cong\bar{\pi}_1(Y, *)$. To every mapping $f \in \sigma_n$, let there correspond an element f^φ defined by the rule $f^\varphi(x_0)\equiv f(x_0, 0,\cdots\cdots, 0)$. Then f^φ represents an element of $\pi_1(Y,*)$. As we can easily verify that $f\sim g\rightarrow f^\varphi\sim g^\varphi$ and $(f\cdot g)^\varphi=f^\varphi\cdot g^\varphi$, φ induces a homomorphism Φ of $\sigma_n(Z, Y, *)$ into $\pi_1(Y, *)$. Next a correspondence ψ: $a\rightarrow a^\psi$, where a is a representative of an element ξ of $\pi_1(Y)$, is defined by the rule $a^\psi(x^n)\equiv a(x_0)$, and a^ψ represents an element of $\sigma_n(Z, Y, *)$. As in case of φ it is easily verified that ψ induces a homomorphism Ψ of $\pi_1(Y, *)$ into $\sigma_n(Z, Y, *)$. Moreover $(a^\psi)^\varphi=a$, so that Φ is a homomorphism of σ_n onto π_1 and as $\Phi\Psi=1$, Ψ is an isomorphism of π_1 into σ_n. Hence it follows that $\sigma_n(Z, Y, *)$ contains a subgroup $\bar{\pi}_1(Y, *)$

isomorphic to $\pi_1(Y, *)$. Furthmore, it is easy to see that the kernel of Φ is contained in $\pi_n(Z, Y, *)$ and conversely $\Phi(\overline{\pi}_n(Z, Y, *))=1$, so that our assertion is completely proved.

3. *Remarks on relatively free homotopy.*

By using the structure of the group $\sigma_n(Z, Y, *)$, we shall give some remarks on relatively free homotopy. First we prove $\alpha^{\xi}=\bar{\xi}\ \alpha\ \bar{\xi}^{-1}$, where $\alpha \in \pi_n(Z, Y, *)$, $\xi \in \pi_1(Y. *)$ and $\bar{\xi}=\Psi(\xi)$ just used in the proof in the last paragraph. From the definition of α^{ξ}, two mappings f, g representing α and α^{ξ} respectively, are relatively free homotopic with respect to the path $e(x_n)$ so that a mapping $F(x^{n+1})$ of $E^n \times I$ into Z can be defined as follows:

$$F(x^n, 1)=f(x^n), \qquad F(x^n, 0)=g(x^n), \text{ when } x \in E^n,$$

$$F(x^{n+1}) \in Y \qquad \text{when } x^n \in \dot{E}^n,$$

$$F(x^{n+1})=e(x_n) \quad \text{when } (x_{n-1}-1) \prod_{i=0}^{n-2} x_i(x_i-1)=0.$$

Denote a system of curves drawn on the face $x^{n+1}(x_0, 0\cdots\cdots 0, x_n)$ as in figure 3 by a system of parametric equations, $x_0=\varphi_t(s)$ and $x_n=\psi_t(s)$, where for a fixed t, $1 \geqq t \geqq 0$, $x^{n+1}(\varphi_t(s), 0\cdots, 0\ \psi_t(s))$ forms a curve according as s varies from 0 to 1. Define $\dot{F}(\varphi_t(s), x_1, \cdots\cdots, x_{n-1}, \psi_t(s))= h_t(s, x_1, \cdots, x_{n-1})$, then

$$h_0(s, x_1^n)=F(\varphi_0(s), x_1^n, \psi_0(s))=F(x_0, x_1^n, 0)=g(x^n)$$

$$h_1(s, x_1^n)=F(\varphi_1(s), x_1^n, \psi_1(s))=\begin{cases} F(0, x_1^n, x_n) & \text{if } \frac{1}{3} \geqq s \geqq 0, \\ F(x^n, 1) & \text{if } \frac{2}{3} \geqq s \geqq \frac{1}{3}, \\ F(1, x_1^n, x_n) & \text{if } 1 \geqq s \geqq \frac{2}{3}. \end{cases}$$

Since $F(x^n, 1)=f(x^n)$, $F(0, x_1^n, x_n)=e(x_n)$, and $F(1, x_1^n, x_n)=e(x_n)$, it is obvious that $\bar{\xi}\ \alpha\ \bar{\xi}^{-1}$. Moreover we see that h_t belongs to σ_n, from the following considerations

$$h_t(0, x_1^n)=F(\varphi_t(0), x_1^n, \psi_t(0))=F(0, x_1^n, 0)=g(0, x_1^n)=*,$$

$$h_t(1, x_1^n)=F(\varphi_t(1), x_1^n, \psi_t(1))=(1, x_1^n, 0)=g(1, x_1^n)=*,$$

$$h_t(\bar{s}, x_1^n)=F(\varphi_t(\bar{s}), x_1^n, \Psi_t(\bar{s}))=e(\Psi_t(\bar{s})) \text{ when } (x_{n-1}-1)\prod_{i=1}^{n-2} x_i(x_i-1)=0,$$

$$h_t(\bar{s}, x_1^n) \in Y \qquad \text{when } \prod_{i=1}^{n-1} x_i(x_i-1)=0.$$

Thus it is concluded that $g \sim h_1 = a\ \alpha\ a^{-1}$, namely $\alpha^{\xi} = \xi\ \alpha\ \xi^{-1}$, and the proof is completed.

If $\alpha^{\xi} = \alpha$ for any ξ of $\pi_1(Y, *)$, then $\alpha = \xi\ \alpha\xi^{-1}$ so that an element belonging to $\bar{\pi}_n(Z, Y, *)$ commutes with every element of $\sigma_n(Z, Y, *)$. Thus it follows that $\bar{\pi}_n(Z, Y, *)$ lies in the center of σ_n and that $\sigma_n(Z, Y, *)$ may be said to be isomorphic to the direct product of $\bar{\pi}_n(Z, Y, *)$ and if $\bar{\pi}_1(Y, *)$ (Z, Y) is relatively n-simple. Conversely it is also proved that (Z, Y) is relatively n-simple when $\sigma_n(Z, Y, *) \cong \pi_n(Z, Y, *) \oplus \pi_1(Y, *)$. Evident'y the pair (Z, Y) is relatively simple in any dimension n for $n \geqq 3$, if Y is simply connected.

4. *Case* $n \geqq 2$.

In case of $n=1$ the definition of the relative homotopy group $\pi_1(Z, Y, *)$ is inapplicable unless $Y=*$, and when $Y=*$ and $n=1$, the discussions are reduced to M. Abe's ones. When $n=2$, the same results as in case $n=3$ will hold true if the definition of $\sigma_2(Z, Y, *)$ is slightly changed as follows. Both homotopy and multiplication are defined as usual among the set of all the mappings, each of which satisfies the conditions: $f(x^2) = *$ when $x_0(x_0-1) = 0$ and $f(x^2) \in Y$ when $\prod_{i=0}^{1} x_i(x_i-1) = 0$. Thus the homotopy classes, together with the multiplication, constitute a group $\sigma_2(Z, Y, *)$, in which all the theorems mentioned above are proved in an analogus way as in case $n \geqq 3$.

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Bibliography

1) Abe, M., Über die stetigen Abbildingen der n-Sphäre in einen metrischen Raum. Jap. J. Math. **16** (1940) 169-176.