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On Maximal Proper Sublattices.

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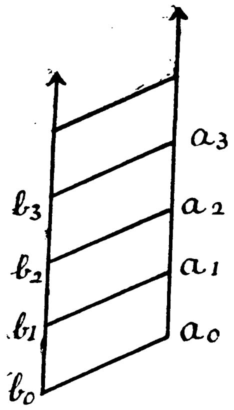

Fig. 1.

G. Birkhoff has proposed the following problem in his revised edition of "Lattice Theory".

Problem 18: Prove or disprove that every proper sublattice S of a lattice L can be extended to a maximal proper sublattice. He suggests: The answer may be yes for distributive lattices.

In this paper we shall prove that the answer is yes for any Boolean algebra (with I and O). But this will be disproved for the distributive lattice $\{a_n, b_n\ (n=0,1,2,\cdots\cdots)\}$ with the Hasse diagram as Fig. 1. Consider, in fact, the sublattice $S=\{a_n (n=0,1,2,\cdots\cdots)\}$. Since the sublattice generated by S and b_n contains all $b_m (m \geqq n)$, S cannot be extended to a maximal proper sublattice.

Let L be a lattice, S a proper sublattice of L and x an element of $L-S$. $M_x(S)$ denotes a maximal subset among all the subsets of L containing S, such that the sublattices generated by them do not contain x. We shall write M_x for any $M_x(\phi)$, where ϕ is the empty set. The existence of $M_x(S)$ is assured by Zorn's lemma and it is evidently one of M_x.

Lemma: A maximal proper sublattice of L is characterized as a maximal subset of L among all the subsets M satisfying the following condition.

(*) There exists an element of L which is not contained in the sublattice generated by M.

Proof: A maximal M is a proper sublattice. Since every proper sublattice satisfies the condition (*), a maximal M is a maximal proper sublattice.

A maximal proper sublattice N satisfies the condition (*). Since every sublattice generated by a subset satisfying the condition (*) is a proper sublattice, N is a maximal M. Q. E. D.

Corollary: A maximal proper sublattice is characterized as a maximal element of the set of all M_x, $x \in L$.

Theorem: Let L be a Boolean algebra with I and O. Then every proper sublattice S of L can be extended to a maximal proper sublattice.

Proof: We divide the proof into four steps.

(I) Withoul loss of generality, L is supposed to contain an element which is neither I nor O. Then $L-S$ contains an element which is neither I nor O, for I (or O) $\in S$ and $x \in S$ together would imply $x' \in S$, where x' denotes the complement of x.

(II) Let $x \notin S$, $x \subsetneqq I, O$. Then, $M_x(S)$ exists and it is one of M_x. If M_x is not a maximal proper sublattice, then, by the corollary of the previous lemma, there exists M_y which satisfies $M_x \subsetneqq M_y$. Since $y \notin M_x$, x is an element of the sublattice generated by M_x and y. Hence, by the distributivity of L, x must be expressed in one of the following forms.

(i) $x = a \cup y$

(ii) $x = a \cap y$ where $a, b, c \in M_x$.

(iii) $x = (b \cup y) \cap c$

We shall show that each case will lead to a contradiction.

(III) The case (i) (or, dually, (ii)).
First, we prove that $a' \cup y \in M_x$. If $a' \cup y \notin M_x$, then, as before, x must be expressed in one of the following forms.

(1) $x = d \cup a' \cup y$

(2) $x = d \cap (a' \cup y)$ where $d, e, f \in M_x$.

(3) $x = (e \cup a' \cup y) \cap f$

The case (1): $I = x \cup (d' \cap a \cap y') = x$, since $x > a$.

The case (2): $I = x \cup d' \cup (a \cap y') = x \cup d'$, since $x > a$.

$O = x \cap \{d' \cup (a \cap y')\} = x \cap d'$.

Hence, $x = d \in M_x$.

The case (3): $I = x \cup (e' \cap a \cap y') \cup f' = x \cup f'$, since $x > a$.

$O = x \cap \{(e' \cap a \cap y') \cup f'\} = x \cap f'$.

Hence, $x = f \in M_x$.

In any case we have a contradiction, and therefore $a' \cup y \in M_x$. Since $x \in M_y$ and $a' \cup y \in M_y$, we have $x \cap (a' \cup y) = y \in M_y$, in contradiction to the definition of M_y.

(IV) The case (iii).
Since $b \cup y \notin M_x$ (for, $x \in M_y$ would follow from $b \cup y \in M_x$ and $c \in M_x$), $M_{b \cup y}$ (M_x) surely exists and it is one of $M_{b \cup y}$.

The case $M_{b \cup y} = M_x$: Since $b \cup y \neq I, O$ (for, $b \cup y = I$ would imply $x = c$ and $b \cup y = O$ would imply $x = O$), if we write $z = b \cup y$ then $M_z \subsetneqq M_y$, and the case is reduced to the case (i).

The case $M_{b\cup y}\supset M_x$: If we write $z=b\cup y$ then $M_z\supsetneqq M_x$, and the case is reduced to the case (ii). Q. E. D.

After my investigation had been completed (in July 1950), Mr. J. Hashimoto communicated me that he had also obtained a similar result. See his forthcoming paper: Ideal Theory in Lattices.

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