Journal of the Mathematical Society of Japan Vol. 2, Nos. 3-4, March, 1951.

On Conformal Representation of Multiply Connected Polygonal Domain.

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(Received Oct. 1, 1949)

It is known, that a function $w(z)$ is schlicht and star-shaped with respect to $w(0)=0$ in $|z|<1$, when, and only when, it can be expressed in the form

$$w(z)=\text{const.}\cdot z\cdot \exp.\ 2\int_{|\zeta|=1}\log\frac{\zeta}{\zeta-z}d\mu(\zeta),$$

where μ denotes a positive distribution of total mass 1 on the unit circle. This formula can also be written in the form

$$w(z)=\text{const. }\exp.\int_{|\zeta|=1}\log\frac{z}{\left(1-\frac{z}{\zeta}\right)^2}d\mu(\zeta),$$

and here comes out *Koebe's extremal function.* The argument of this function is equal to a constant on $|z|=1$ except the point ζ, and jumps by $+2\pi$ when z passes ζ in positive direction on $|z|=1$. Then, the above formula shows: The star-shaped function $w(z)$, whose argument is non-decreasing for z moving on $|z|=1$ in positive direction, can be constructed *from such elements as a sort of geometrical mean.*

We shall prove in this paper an analogue of this fact for n-ply connected domain, and, as an application thereof, treat the conformal representation of n-ply connected polygonal domain.

In order to simplify the wording, we call a half straight-line $\text{Arg}\,\Omega=\text{const.}$, $|\Omega|\geqq \text{const.}>0$ an "*infinite radial slit*", and a segment $\text{Arg}\,\Omega=\text{const.}$, $\text{const.}\geqq|\Omega|\geqq\text{const.}>0$ a "*radial slit*", respectively.

§ 1.

Let D be a domain on z-plane bounded by n analytic closed curves $\Gamma_1,\cdots\cdots,\Gamma_n$, whose sum we denote by Γ, and let z_0 be a fixed point in D.

For any point ζ on Γ, we denote by $\Omega(z,\zeta)$ the function which satisfies the conditions $\Omega(z_0,\zeta)=0$, $\Omega'(z_0,\zeta)=1$ and maps D conformally on the whole Ω-plane cut along an infinite radial slit and $(n-1)$ radial slits, so that the boundary point ζ of D corresponds to the bodunary point

$\Omega=\infty$. The existence, uniqueness and the continuity in ζ of such functions are to be proved afterwards in Lemmas 1. and 3.

We will now formulate the theorem to be proved as follows:

Theorem 1. *Let $w(z)$ be a function which satisfies the following three conditions:*

1. *$w(z)$ is regular and does not vanish in D except at z_0, where it has an expansion of the form*

$$w(z)=(z-z_0)^{\alpha}\left\{1+c_1(z-z_0)+\cdots\cdots\right\} \qquad (\alpha\gtreqless 0).$$

2. *$|w(z)|$ is one-valued in D.*

3. *Any branch of* Arg $w(z)$ *is bounded in the neighbourhood of Γ, and the limiting value*

$$\lim_{z\to\zeta^*} \operatorname{Arg} w(z)=\theta(\zeta^*)$$

exists for each ζ^* on Γ except at most an enumerable infinity of points and is of bounded variation, as function of ζ^* on Γ, on the set where it exists.

A necessary and sufficient condition for this, is that $w(z)$ can be expressed in the form

$$(1) \qquad w(z)=\exp.\int_{\Gamma}\log\, \Omega(z,\,\zeta)\;d\sigma(\zeta),$$

where σ is a distribution of bounded variation of total mass α on Γ, determined by the function of bounded variation $\dfrac{1}{2\pi}\theta(\zeta^*)$.

We shall make some preparations and prove some lemmas.

Definition of the Riemann surface Φ. Let $\tilde{D}$ be another sheet of D. We put $\tilde{D}$ on D and identify the corresponding boundary points of D and $\tilde{D}$. This closed surface can be regarded as a closed Riemann surface Φ of genus $n-1$, since we can define a local parameter $t(p)$ for each point p on Φ: for a point of $\tilde{D}$ by taking conjugate complex, and for a point on Γ by reflection in Γ.

By interchanging the two sheets D and $\tilde{D}$, we obtain a transformation $p\to\tilde{p}$ which transforms Φ into itself conformally with inversion of angles.

Besides, we denote by $\omega_{q_1,q_2}(p)$ the elementary integral of third kind on Φ, which has the singularities $\log t(q_1)$ at q_1 and $-\log t(q_2)$ at q_2, and whose real part is one-valued on Φ. And by $\omega'_{q_1,q_2}(p)$ we denote the elementary integral of third kind, which has the singularities $-i\log t(q_1)$ at

q_1 and $i \log t(q_2)$ at q_2, and whose real part is one-valued on Φ cut along a curve connecting q_1 with q_2[1].

Lemma 1. *For each ζ on Γ, there exists one and only one function $\Omega(z, \zeta)$ with the mentioned properties.*

Proof. We put

$$\omega_{z_0,\zeta}(p) + \omega_{\tilde{z}_0,\zeta}(p) = u(p) + iv(p),$$

and

$$\Omega(z, \zeta) = \text{const. exp.} \left\{ u(z) + iv(z) \right\}. \tag{2}$$

Since the one-valued potential function $u(\tilde{p})$ has on Φ the same singularities as $u(p)$, and takes on Γ the same value as $u(p)$, we have

$$u(p) \equiv u(\tilde{p}),$$

i.e. $u(p)$ takes the same value at $\tilde{p}$ as at p. Hence we have at each point on Γ except ζ,

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{consequently} \quad \frac{\partial v}{\partial \tau} = 0,$$

where ν and τ denote the normal and tangent to Γ. Therefore, v takes a constant value on each Γ_k. It follows from this, that $\Omega(z, \zeta)$ is one-valued in D.

On the other hand, u is finite at each point of Γ except ζ, where u is positively infinite. Hence, the image of Γ by $\Omega(z, \zeta)$ consists of an infinite radial slit and $n-1$ radial slits.

Let Ω_0 be a point of Ω-plane, which does not belong to these n slits. Since

$$\text{Arg}\left\{ \frac{1}{\Omega(z, \zeta)} - \frac{1}{\Omega_0} \right\}$$

remains unchanged when z moves on Γ once around and returns to the original value, and since $1/\Omega(z, \zeta)$ has one and only one pole in D, $\Omega(z, \zeta)$ takes each value Ω_0 once and only once in D. Therefore, $\Omega(z, \zeta)$ provides the required mapping, when the constant factor in (2) is determined by the condition $\Omega'(z_0, \zeta) = 1$.

The uniqueness of the mapping function can be proved as follows. Let $\Omega_1(z, \zeta)$ be another mapping function with the mentioned properties. When we continue $\Omega_1(z, \zeta)$ analytically across Γ on Φ by the principle of reflection, we obtain a one-valued potential function $\log |\Omega_1(p, \zeta)|$ on Φ, since $|\Omega_1|$ remains unchanged by reflection in a radial slit. Moreover $\log |\Omega_1(p, \zeta)|$ has the same singularities as $u(p)$. Therefore, by the normalisation $\Omega_1'(z_0, \zeta)=1$, $\Omega_1(z, \zeta)$ must be identical with $\Omega(z, \zeta)$.

Remark. By the same idea as in the above proof, we can construct the function $\Omega(z, z^*)$ for $z^* \in D$, which maps D conformally on the whole Ω-plane cut along n radial slits and satisfies $\Omega(z_0, z^*)=0$, $\Omega'(z_0, z^*)=1$, $\Omega(z^*, z^*)=\infty$. For this purpose, we have only to put

$$(3) \qquad \Omega(z, z^*)=\text{const. exp.} \left\{\omega_{z_0 z^*}(z)+\omega_{\tilde{z}_0, \tilde{z}^*}(z)\right\}.$$

We cut the domain D by n curves, each of which connects z_0 respectively with an arbitrarily fixed point ζ_k on Γ_k, and which do not cross each others. We denote by D_0 the resulting simply connected domain, in which $\operatorname{Arg} \Omega(z, \zeta)$ is one-valued. We can assume that D_0 contains wholly in it a line element dx at z_0 with direction of positive real axis. We take the branch of $\operatorname{Arg} \Omega(z, \zeta)$ which vanishes at z_0+dx and put

$$\theta(z, \zeta)=\operatorname{Arg} \Omega(z, \zeta)$$

for $z \in D_0$.

As function of z with fixed ζ, $\theta(z, \zeta)$ has the following properties.

Lemma 2. *$\theta(z, \zeta)$ is bounded in D_0, aud the limiting value*

$$\lim_{z \to \zeta^*} \theta(z, \zeta)=\theta(\zeta^*, \zeta)$$

exists for each ζ^* on Γ except ζ. $\theta(\zeta^*, \zeta)$ is equal to a constant on each arc of Γ which contains neither ζ nor $\zeta_1, \cdots\cdots \zeta_n$, and jumps by $+2\pi$ at ζ when ζ^* moves on Γ in pocitive direction.

Proof. This is obvious from the shape of the image of D by $\Omega(z, \zeta)$.

As function of ζ with fixed z, $\log \Omega(z, \zeta)$ and $\theta(z, \zeta)$ have the following properties.

Lemma 3. *$\log \Omega(z, \zeta)$ is one-valued and continuous, and its imaginary part $\theta(z, \zeta)$ is uniformly bounded for the parameter z in D_0.*

Proof. The constant factor in (2), which is to be determined by the condition $\Omega'(z_0, \zeta)=1$, depends naturally on ζ. When we write (2) in the form

$$\log \Omega(z, \zeta)=\int_{z_0^*}^{z} d\omega_{z_0, \zeta}+\int_{z_0^*}^{z} d\omega_{\tilde{z}_0, \zeta}+c(\zeta),$$

where z_0^* denotes an arbitrarily fixed point in D, the condition $\Omega'(z_0, \zeta)=1$

is given by

$$c(\zeta)=\lim_{z_1\to z_0}\left\{-\int_{z_0^*}^{z_1} d\omega_{z_0,\zeta}-\int_{z_0^*}^{z_1} d\omega_{z_0,\zeta}+\log(z_1-z_0),\right\},$$

and we obtain the following definite form of log $\Omega(z, \zeta)$,

$$\log \Omega(z, \zeta)=\lim_{z_1\to z_0}\left\{\int_{z_1}^{z} d\omega_{z_0,\zeta}+\int_{z_1}^{z} d\omega_{z_0,\zeta}+\log(z_1-z_0)\right\}.$$

We assume that ζ lies on Γ_k, and consider the difference

$$\log \Omega(z, \zeta)-\log \Omega(z, \zeta_k)$$

$$=\lim_{z_1\to z_0}\left\{\int_{z^1}^{z} d\omega_{z_0,\zeta}-\int_{z_1}^{z} d\omega_{z_0,\zeta_k}+\int_{z_1}^{z} d\omega_{z_0,\zeta}-\int_{z_1}^{z} d\omega_{z_0,\zeta_k}\right\}$$

$$=\lim_{z_1\to z_0} 2\int_{z_1}^{z} d\omega_{\zeta_k,\zeta}=2\int_{z_0}^{z} d\omega_{\zeta_k,\zeta}.$$

By the *theorem of interchange of argument and parameter*,[2] we can write this in the form

$$\log \Omega(z, \zeta)-\log \Omega(z, \zeta_k)=2\left\{\Re\int_{\zeta_k}^{\zeta} d\omega_{z_0,z}+i\Re\int_{\zeta_k}^{\zeta} d\omega'_{z_0,z}\right\}.$$

This proves the mentioned property of log $\Omega(z, \zeta)$.

While taking the imaginary part of this formula, we have

$$\Theta(z, \zeta)-\Theta(z, \zeta_k)=2\Re\int_{\zeta_k}^{\zeta} d\omega'_{z_0,z}.$$

Since the right-hand side is certainly uniformly bounded for z in D_0, and since $\Theta(z, \zeta_k)$ is, by Lemma 2, bounded in D_0, $\Theta(z, \zeta)$ is uniformly bounded for z in D_0.

Lemma 4. *Let $f(z)$ be a function one-valued and regular in D, whose imaginary part is bounded. If the limiting value*

$$\lim_{z\to\zeta} \Im f(z)$$

exists for each ζ on Γ except at most an enumerable infinity of points, and if this limiting value is equal respectively to a constant on each Γ_k, then $f(z)$ is identically equal to a constant.

Proof. In the first place, since $\Im f(z)$ is bounded, there exist in fact no exceptional points. Then, we can continue $f(z)$ analytically on Φ across each Γ_k, by the principle of reflection. Since $\Re f(z)$ remains unchanged by reflection in a straight-line parallel to the real axis, we obtain, by this continuation, a one-valued potential function $\Re f(p)$ everywhere regular on Φ, which must be identically a constant.

Q. E. D.

Now we will prove Theorem 1.

Proof of Theorem 1.

Sufficiency. Since log $\Omega(z, \zeta)$ is continuous as function of ζ by Lemma 3,

$$w(z)=\exp.\int_{\Gamma}\Omega(z, \zeta)d\sigma(\zeta)$$

represents an analytic function of z, which obviously satisfies the conditions 1 and 2 The property 3 can be proved as follows.

For a branch of Arg $w(z)$ one-valued in D_0, we have

$$\text{Arg } w(z)=\int_{\Gamma}\theta(z, \zeta)d\sigma(\zeta)$$

By Lemma 3, this function of z is bounded in D_0. When z approaches to a point ζ^* on Γ, which is a point of continuity of the distribution σ, we have, by Lemmas 2, 3 and by Lebesgue's theorem,

$$\theta(\zeta^*)=\lim_{z\to\zeta^*}\text{Arg } w(z)=\lim_{z\to\zeta^*}\int_{\Gamma}\theta(z, \zeta)d\sigma(\zeta)=\int_{\Gamma}\theta(\zeta^*, \zeta)d\sigma(\zeta).$$

Therefore, the limiting value $\theta(\zeta^*)$ certainly exists for a point of continuity of σ.

Let C be an arc of positive direction on Γ_k, which does not contain the point ζ_k, and whose starting and ending points ζ_1^*, ζ_2^* are both points of continuity of σ.

Then, we have

$$\theta(\zeta_2^*)-\theta(\zeta_1^*)=\int_{\Gamma}\left\{\theta(\zeta_2^*, \zeta)-\theta(\zeta_1^*, \zeta)\right\}d\sigma(\zeta).$$

On the other hand, we have by Lemma 2,

$$\theta(\zeta_2^*, \zeta)-\theta(\zeta_1^*,\zeta)=\begin{cases} 2\pi & \zeta \in C, \\ 0 & \zeta \notin C. \end{cases}$$

Therefore we obtain

$$\theta(\zeta_2^*)-\theta(\zeta_1^*)=2\pi\sigma(C).$$

This proves the last part of 3 and the mentioned relation between θ and σ.

Necessity. Let $w(z)$ be a function which satisfies the conditions 1, 2 and 3. We define by $\frac{1}{2\pi}\theta(\zeta^*)$ a distribution σ of bounded variation on Γ. Obviously σ has the total mass α. We put

$$w_1(z)=\exp.\int_\Gamma \log \Omega(z, \zeta)\ d\sigma(\zeta),$$

$$\theta_1(\zeta^*)=\lim_{z\to\zeta^*} \text{Arg } w_1(z)$$

and
$$f(z)=\log\frac{w(z)}{w_1(z)}.$$

$f(z)$ is one-valued and regular in D.

Let C be such an arc of Γ_k, as mentioned in the first part of this proof. Then we have

$$\lim_{z\to\zeta_2^*} \Im f(z)-\lim_{z\to\zeta_1^*} \Im f(z)$$

$$=\left\{\theta(\zeta_2^*)-\theta_1(\zeta_2^*)\right\}-\left\{\theta(\zeta_1^*)-\theta_1(\zeta_1^*)\right\}$$

$$=\left\{\theta(\zeta_2^*)-\theta(\zeta_1^*)\right\}-\left\{\theta_1(\zeta_2^*)-\theta_1(\zeta_1^*)\right\}$$

$$=2\pi\sigma(C)-2\pi\sigma(C)=0.$$

Therefore, $\Im f(z)$ has a constant limiting value on each Γ_k respectively. Further, $\Im f(z)$ is bounded in D, since Arg $w(z)$ and Arg $w_1(z)$ are both bounded in D_0. Consequently by Lemma 4 we obtain

$$f(z)\equiv\text{const.},$$

and the normalisation in condition 1 gives

$$w(z)\equiv w_1(z).$$

Q. E. D.

Remark 2 Making use of conformal representation, Theorem 1 finds itself valid, in the form as it stands, for any n-ply connected Jordan domain.

§ 2.

If D is the interior of a circle or a circular ring-shaped domain, we can write down the explicit forms of $\Omega(z, \zeta)$ and $\Omega(z, z^*)$.[3)]

The case where D is the interior of the unit circle $|z|<1$ and z_0 is the origin $z=0$.

By reflection in $|z|=1$, the Riemann surface Φ represents itself conformally on the whole z-plane. And the elementary integral w_{z_0,z_1} is given by

$$w_{z_0,z_1}(z)=\log\frac{z}{z-z_1}+\text{const.}.$$

While giving suitable values to z_1 and combining them, we obtain by (2) and (3)

$$\Omega(z,\zeta)=\frac{z}{(1-\frac{z}{\zeta})^2}$$

and

$$\Omega(z,z^*)=\frac{z}{\left(1-\frac{z}{z^*}\right)\left(1-\bar{z}^*z\right)}$$

under consideration of the normalisation $\Omega'(0)=1$.

The case where D is the ring-shaped domain $q<|z|<1$ and z_0 is real and positive.

By repeated reflections in the boundary curves, and by the transformation

$$u=u(z)=-i\log\frac{z}{z_0}$$

the universal covering surface of Φ is mapped conformally on the whole finite u-plane. Then, putting $u_1=u(z_1)$, ω_{z_0,z_1} is given by

$$\omega_{z_0,z_1}=\log\frac{\sigma(u)}{\sigma(u-u_1)}-\left(\frac{\eta_1}{\omega_1}\Re u_1+i\frac{\eta_3}{\omega_3}\Im u_1\right)u+\text{const.},$$

where σ denotes the Weierstrass' σ-function with primitive periods

$$2\omega_1=2\pi,\quad 2\omega_3=-2i\log q$$

and η_1 and η_3 have the ordinary significations.

While giving suitable values to z_1 and combining them, we obtain by (3), after simple calculations,

$\Omega(z,z^*)=$

$$-iz_0\frac{\sigma\left(i\log\frac{z_0}{z^*}\right)\sigma(i\log\bar{z}^*z_0)}{\sigma(2i\log z_0)}\cdot\frac{\sigma\left(i\log\frac{z}{z_0}\right)\sigma(i\log z_0z)}{\sigma\left(i\log\frac{z}{z^*}\right)\sigma(i\log\bar{z}^*z)}\cdot\left(\frac{z}{z_0}\right)^{2i\frac{\eta_1}{\pi}\operatorname{Arg}z^*}$$

under consideration of the normalisation $\Omega'(z_0)=1$.

When we replace z^* by ζ in the above formula, we obtain the expression for $\Omega(z, \zeta)$. But it can be a little simplified by separating the two cases $|\zeta|=1$ and $|\zeta|=q$. In fact, we have

$\Omega(z, e^{i\varphi})=$

$$-\frac{iz_0}{\sigma(2i\log z_0)}\cdot\frac{\sigma(i\log z_0+\varphi)^2}{\sigma(i\log z+\varphi)^2}\cdot\sigma\Big(i\log\frac{z}{z_0}\Big)\sigma(i\log z_0 z)\cdot\Big(\frac{z}{z_0}\Big)^{2i\frac{\eta_1}{\pi}\varphi}$$

and

$\Omega(z, qe^{i\varphi})=$

$$-\frac{iz_0}{\sigma(2i\log z_0)}\cdot\frac{\sigma_3(i\log z_0+\varphi)^2}{\sigma_3(i\log z+\varphi)^2}\cdot\sigma\Big(i\log\frac{z}{z_0}\Big)\sigma(i\log z_0\, z)\cdot\Big(\frac{z}{z_0}\Big)^{2i\frac{\eta_1}{\pi}\varphi}$$

Remark. If D is the domain $|z|<1$, or if D is $q<|z|<1$ and $a=0$, while differentiating the logarithm of (1) and multiplying it by z, we obtain by the above expressions for $\Omega(z, \zeta)$ *the Poisson-Stieltjes' or the Villat-Stieltjes'*[4] *expression for* zw'/w. Further, it is easy to prove fiom Theorem 1 these two formulae in their perfect forms.

§ 3.

As an application of Theorem 1, we shall give an expression for the mapping function of n-ply connected polygonal domain, an analogue of Schwarz-Christoffel's formula. Here, by the word "*n-ply connected polygonal domain*", we mean an *n-ply connected Riemann surface P of planar character (schlichtartig), whose boundary consists of a finite number of segments or half straight-lines.* P may contain in it a finite number of points of ramification, and may cover the point at infinity a finite number of times.

We assume that n is greater than 1. Let D be a concentric circular ring $R_1<|z|<R_2$ with $n-2$ concentric circular slits, whose $2(n-2)$ end points we denote by s_k $(k=1,\cdots\cdots)$. And we fix a point z_0 in D arbitrarily.

Let $f(z)$ be the function which maps D conformally on P. We denote by ζ_k the boundary point of D, which corresponds by this function to a vertex of P with the interior angle $a_k\pi$. If the vertex lies on the point at infinity, we agree to give a_k negative sign.

In the first place, we assume that P contains in it neither points of ramification nor points lying at infinity.

Then, $zf'(z)$ is regular and does not vanish in D, and when z moves on the boundary of D in positive direction, the variations of its argument are as follows:

$$d\text{Arg } zf'(z)=d\text{ Arg }\frac{df(z)}{d\log z}=d\text{ Arg } df(z)-d\text{ Arg } d\log z,$$

$$d\text{Arg } df(z)=(1-\alpha_k)\pi \quad \text{at } \zeta_k \text{ and } \quad =0 \quad \text{elsewhere,}$$

$$d\text{Arg } d\log z=-\pi \quad \text{at } s_k \text{ and } \quad =0 \quad \text{elsewhere.}$$

Therefore, while defining the distribution σ by

$$\sigma(\zeta_k)=\frac{1-\alpha_k}{2},\quad \sigma(s_k)=\frac{1}{2} \text{ and } \sigma\equiv 0 \text{ elsewhere,}$$

we obtain by theorem 1.

$$zf'(z)=z_0f'(z_0)\prod_k \Omega(z,\zeta_k)^{\frac{1-\alpha_k}{2}}\prod_k \Omega(z,s_k)^{\frac{1}{2}}$$

Thus, we have the following expression for the mapping function.

$$f(z)=z_0f'(z_0)\int_{z_0}^{z}\prod_k \Omega(z,\zeta_k)^{\frac{1-\alpha_k}{2}}\prod_k \Omega(z,s_k)^{\frac{1}{2}}\frac{dz}{z}+f(z_0).$$

In the general case, we denote by z_k the point of D which corresponds to a point of ramification of m_k-th order ($m_k>0$) on P lying in the finite part of the plane, and by z_k' the point which corresponds to a point of ramification of m_k'-th order ($m_k'\geqq 0$) lying at infinity.

Then, $zf'(z)$ has a zero of m_k-th order at z_k and a pole of $(m_k'+2)$-th order at z_k'. We can apply Theorem 1 to the function

$$z\cdot f'(z)\cdot\frac{\prod_k \Omega(z,z_k)^{m_k}}{\prod_k \Omega(z,z_k')^{m'_k+2}}$$

and, since we have

$$d\text{ Arg } \Omega(z,z^*)\equiv 0$$

on the boundary of D, the distribution σ can be so determined as before. Therefore, we have

$$zf'(z)=C_1\prod_k \Omega(z,\zeta_k)^{\frac{1-\alpha_k}{2}}\cdot\frac{\prod_k \Omega(z,z_k')^{m_k'+2}}{\prod_k \Omega(z,z_k)^{m_k}}\cdot\prod_k \Omega(z,s_k)^{\frac{1}{2}},$$

C_1 being a suitable constant.

Thus, we obtain the following

Theorem 2. *The function which maps D conformally on P is given by*

$$f(z)=C_1\int^z \prod_k \Omega(z,\zeta_k)^{\frac{1-\alpha_k}{2}}\cdot\frac{\prod_k \Omega(z,z_k')^{m_k'+2}}{\prod_k \Omega(z,z_k)^{m_k}}\cdot\prod_k \Omega(z,s_k)^{\frac{1}{2}}\cdot\frac{dz}{z}+C_2,$$

where C_1 and C_2 are constants depending on position and magnitude of P and on the lower bound of the integration.

Remark 1. If one of the points z_k or z_k' coincides with z_0, we have to understand $\Omega(z, z_0)$ to be $\equiv 1$ in the above formula.

Remark 2. Though we have deduced Theorem 2 under the assumption $n\geqq 2$, it is also valid for $n=1$, as can be seen easily, if D is the domain $|z|<R$ and z_0 is the origin $z=0$.

By the expressions for $\Omega(z, \zeta)$ given in § 2., we can easily see that, in case P is schlicht, the formula of Theorem 2 coincides for $n=1$ with the ordinary Schwarz-Christoffel's formula, and for $n=2$ with the formula given by Mr. Y. Komatu[5].

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