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On the mapping functions of Riemann surfaces.

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Let W be a simply connected infinitely many sheeted open Riemann surface, whose singularities are all logarithmic and lie only on a finite number of base-points $x_1, x_2, \ldots, x_n$ ($n \geqq 3$), and W^∞ be its universal covering surface.

Let $x = m(z)$ be the function which maps W^∞ one-to one and conformally on the unit-circle $|z| < 1$. The properties of the function $m(z)$ are well known. Let $x = \varphi(w)$ be the function which maps W one-to-one and conformally on the finite plane $w \fallingdotseq \infty$ or the unit-circle $|w| < 1$ according as W is parabolic or hyperbolic and $\varphi^{-1}(x)$ be its inverse function. We shall obtain some properties of the function $\varphi^{-1}(m(z))$, which is regular in $|z| < 1$.

Let

$$w = f(z) = \varphi^{-1}(m(z)) \tag{1}$$

and R be the Riemann surface on which the unit-circle $|z| < 1$ is mapped one-to-one and conformally by $w = f(z)$. If W is of parabolic type, then R is a Riemann surface spread over the w-plane. If W is of hyperbolic type, then R is a Riemann surface spread over the unit-circle $|w| < 1$. Let B be the boundary of the domain of $\varphi(w)$. The set B consits of only the point at infinity or the all points on the circumference $|w| = 1$ according as the Riemann surface W is parabolic or hyperbolic.

Lemma 1. *The set M of points on the w-plane, which are the projections of the branch points of R is enumerable and the set M' of the limiting points of M is contained in the set B.*

Proof. Since the branch points of the universal covering surface W^∞ lie only on the base-points $x_1, x_2, \ldots, x_n$ and $f^{-1}(w) = m^{-1}(\varphi(w))$ by the relation (1), we have a regular functional element of $f^{-1}(w)$ at the point w, if $\varphi(w) \fallingdotseq x_i (i = 1, 2, \ldots, n)$. Hence the projections of the branch points of R on the w-plane are the zero-points of $\varphi(w) - x_i$ ($i = 1, 2, \ldots, n$). As the zero-points of an analytic function is enumerable, the set M is enumerable.

Since the limiting point of the zero-points of an analytic function lies on the boundary of the domain of definition, the set M' is contained in the set B.

Lemma 2. *If there exists the limit*

$$f(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta}), \tag{2}$$

then the point $w = f(e^{i\theta})$ belongs to the set $M+B$.

Proof. As the set M' is contained in the set B by the lemma 1, $M+B$ is a closed set. Let w_1 be a point in the domain of $\varphi(w)$ not belonging to the set M. Then there exists a small circle $|w-w_1| < \rho$ such that the Riemann surface R has no branch points on this circle and the domains in the unit-circle $|z| < 1$ which correspond to the circle $|w-w_1| < \rho$ are bounded by a simple closed curve in $|z| < 1$. Hence w_1 is not an asymptotic value of $f(z)$ at any point on the circumference $|z|=1$. Therefore w_1 is not equal to any of the limit (2). Hence the point $w=f(e^{i\theta})$ belongs to the set $M+B$.

Theorem 1. *If the Riemann surface W is of parabolic type, then we have*

$$T(r) = 0(\log\frac{1}{1-r}), \fallingdotseq 0(1), \tag{3}$$

where $T(r)$ is the characteristic function of $f(z)$.

Proof. Since W is of parabolic type, R is a Riemann surface spread over the finite plane $w \fallingdotseq \infty$ and the set $M+B$ is enumerable and closed by the lemma 1.

If we assume that the function $f(z)$ is 'beschränktartig', then there exists the limit (2) for almost every θ by a theorem of Fatou[1]. These limits belong to the set $M+B$ by the lemma 2. Being enumerable and closed, the set $M+B$ is of capacity zero. Hence the set of the limits (2) is of capacity zero. Therefore $f(z)$ is equal to a constant by a theorem of Tsuji[2]. Hence the function $f(z)$ is not 'beschränktartig'.

Let $w_{i\nu}(\nu=1,2,\ldots)$ be the zero-points of $\varphi(w)-x_i(i=1,2,\ldots,n)$. Since the Riemann surface W is of parabolic type and $n \geqq 3$, there exists infinitely many such a point $w_{i\nu}$ by a theorem of Picard-Borel. As the function $f(z)$ does not take the infinitely many values $w_{i\nu}$, we have (3) by the second fundamental theorem of Nevanlinna for meromorphic functions.

Theorem 2. *If the Riemann surface W is of hyperbolic type, then the function $f(z)$ is equal to a Blaschke's product, that is*

$$f(z)=e^{ia}\pi(z), \quad \pi(z)=\prod_{k=1}^{\infty}\frac{|z_k|}{z_k}\frac{z_k-z}{1-\bar{z}_k z}, \tag{4}$$

where a is a real constant and z_k are the zero-points of $f(z)$.

Proof. Since W is of hyperbolic type, R is a Riemann surface spread over the unit-circle $|w|<1$ and the set B is the circumference $|w|=1$.

As $|f(z)|<1$ in $|z|<1$, we have by a theorem of Nevanlinna[(3)]

$$f(z)=\psi(z)\pi(z),$$

where the function $\psi(z)$ is regular, bounded and has no zeros in $|z|<1$, and there exists the limit (2) for almost every θ by the theorem of Fatou.

The limit (2) belongs to the set $M+B$ by the lemma 2. The set M is enumerable by the lemma 1. The set of θ for which the limit (2) is equal to a constant is of measure zero by a theorem of Riesz[(4)]. The sum of an enumerable number of sets of measure zero is a set of measure zero. Therefore the limit (2) belongs to the set B for almost every θ, that is, the absolute value of the limit (2) is equal to *1* for almost every θ.

We know by Nevanlinna that a Blaschke's product has the radial limits of absolute value *1* for almost every points on the circumference $|z|=1$[5)]. Hence the function $\psi(z)$ has the radial limits of absolute value *1* for almost every points on $|z|=1$. As $\psi(z)$ is bounded in $|z|<1$, $\psi(z)$ is equal to a constant, which is of absolute value *1*. Hence we have (4).

Theorem 3. *The function $f(z)$ is automorphic with respect to a group G of linear transformations which make the unit-circle invariant and G can be produced by parabolic transformations only.*

Proof. Without loss of generality we may assume that the number of base-points is three. Let x_1, x_2, x_3 be the base-points and 1_1, 1_2, 1_3 be the arcs $\widehat{x_1x_2}$, $\widehat{x_2x_3}$, $\widehat{x_3x_1}$ of the circle which pass through x_1, x_2, x_3. If W has a inner point on $x_i (i=1,2,3)$, then we cut W from this point along 1_i. After cutting in this way we obtain a simply connected surface W_0. We take infinitely many same samples $W_i (i=1,2,\ldots)$ as W_0 and connect them along the opposite shores of the cuts in the well known way and obtain a universal covering Riemann surface W^∞.

Since the function $\varphi^{-1}(x)$ is one-valued on W. The function $f(z)$ is automorphic with respect to a group of linear transformations U: $z'=U(z)$ which make $|z|<1$ invariant, where z, z' correspond to the same point of W.

Let x, x' be two points on W^∞ which correspond to z, z' respectively. If x and x' lie on W_i and W_j respectively, and W_i and W_j are connected to each other by the arc which lies on I_k, then W^∞ has only one invariant point on its boundary lying on x_k by the transformation of W^∞ in itself. which corresponds to U. Hence U has only one invariant point on $|z|=1$, U is parabolic. It is well known that G can be produced by such a transformation.

The inner points of W which lie on the points x_i correspond to the zero-points of $\varphi(w)-x_i$ and the function $f(z)$ does not take the values of the set M which consists of all the zero-points of $\varphi(w)-x_i(i=1, 2, 3)$. Hence *$R$ is the universal covering Riemann surface of the domain which is bounded by the set $M+B$.*

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Reference.

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[3] R. Nevanlinna: loc. cit. 174–178.

[4] R. Nevanlinna: loc. cit. 197–198.

[5] R. Nevanlinna: loc. cit. 196–197.