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On the Multivalency of Analytic Functions

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The purpose of the present note is to extend NOSHIRO's theorem[1] (generalization of DEUDONNÉ's theorem[2]) concerning the univalency of analytic functions to the case of n-valence. First, assuming that $\varphi(z)=c_0+\ldots\ldots$, $c_0\neq 0$, is regular and $\varphi(z)\subset D$[3] in $|z|<1$, where D is a given connected domain, we obtain a general theorem on the multivalency and the star-shapedness of the function $f(z)\equiv z^n\varphi(z)$, according to K. NOSHIRO's method with the aid of KAKEYA's principle[4]. Then, we shall give some consequences of this theorem, taking some special domains as D, one of which gives a result obtained by Lynn H. LOOMIS[5].

Lemma. *Suppose that $\varphi(z)=c_0+\ldots\ldots$, $c_0\neq 0$, is regular in $|z|<1$. Then $f(z)\equiv z^n\varphi(z)$ is n-valent and starshaped with respect to the origin for $|z|<1$, provided that*

$$\Re\left(z\frac{f'(z)}{f(z)}\right)>0. \qquad (|z|<1). \tag{1}$$

Proof. If $\Re\left(z\frac{f'(z)}{f(z)}\right)>0$ $(|z|<1)$, then $f(z)$ does not vanish in the unit circle except at the origin and so there exists a function $h(z)$ which is regular in $|z|<1$ and satisfies

$$f(z)\equiv z^n\varphi(z)=[h(z)]^n.$$

Consequently

$$f'(z)=n[h(z)]^{n-1}h'(z), \qquad z\frac{f'(z)}{f(z)}=nz\frac{h'(z)}{h(z)}.$$

By (1) we have

$$\Re\left(z\frac{h'(z)}{h(z)}\right)>0 \qquad (|z|<1). \tag{2}$$

As is well known, (2) is a sufficient condition in order that a function

$h(z)$, regular in $|z|<1$ with $h(0)=0$, $h'(0)\neq 0$, be univalent and starshaped with respect to the origin for $|z|<1$[6]. This proves our lemma.

Now we shall consider the n-valency of the function $f(z)\equiv z^n\varphi(z)$, using NOSHIRO's method.

Let $\varphi(z)=c_0+\ldots\ldots$, $c_0\neq 0$, be regular and $\varphi(z)\subset D$ in $|z|<1$, where D is a given connected domain. Let us denote by $g(z)$ an arbitrary branch of a function mapping D conformally on $|z|<1$, and put

$$\frac{1-|g(a)|^2}{|g'(a)|}\equiv\Omega(a,\ D) \qquad (a \in D)\ ;$$

here the positive quantity $\Omega(a,\ D)$ depends only on a and D, and neither on the selection of a mapping function nor on that of a branch $g(z)$ of a mapping function[7]. Then, applying a theorem of E. LANDAU,[8] we obtain

$$\left|z\frac{\varphi'(z)}{\varphi(z)}\right|\leqq\frac{|z|}{1-|z|^2}\cdot\frac{\Omega(\varphi(z),D)}{|\varphi(z)|} \qquad (|z|<1),^{9)} \tag{3}$$

provided that $\varphi(z)$ does not vanish at a point $z(|z|<1)$.

Thus we have the following extension of NOSHIRO's theorem to the case of the n-valency:

Theorem 1. *Under these assumptions, $f(z)\equiv z^n\varphi(z)$ is n-valent and starshaped with respect to the origin for $|z|<\rho_n\leqq 1$, provided that*

$$\frac{|z|}{1-|z|^2}\cdot\frac{\Omega(\varphi(z).\ D)}{|\varphi(z)|}<n \qquad (|z|<\rho_n). \tag{4}$$

Proof. Putting

$$f(z)\equiv z^n\varphi(z),$$

we have

$$f'(z)=nz^{n-1}\varphi(z)+z^n\varphi'(z),\quad z\frac{f'(z)}{f(z)}=n+z\frac{\varphi'(z)}{\varphi(z)}.$$

Hence, $\Re\left(z\frac{f'(z)}{f(z)}\right)>0 \quad (|z|<\rho_n)$ whenever

$$\left|z\frac{\varphi'(z)}{\varphi(z)}\right|<n \qquad (|z|<\rho_n). \tag{5}$$

While, by (3) and (4), the inequality (5) is obtained, our theorem is proved.

Theorem 2. *Let* $\varphi(z)=c_0+\dots\dots$, $c_0\neq 0$, *be regular and bounded in* $|z|<1$: $|\varphi(z)|<M$. *Then* $f(z)\equiv z^n\varphi(z)$ *is n-valent and starshaped with respect to the origin for* $|z|<\rho_n\leqq 1$, *if*

$$\left|\frac{\varphi(z)}{M}\right|>\frac{-n(1-r^2)+\sqrt{n^2(1-r^2)^2+4r^2}}{2r}\quad(|z|=r<\rho_n).\tag{6}$$

Proof. Take a circular domain $|z|<M$ as D in the preceding theorem. Since $g(z)=\frac{z}{M}$ is a mapping function, we see easily that (4) takes the form

$$\frac{r}{1-r}\cdot\frac{1-\left|\frac{\varphi(z)}{M}\right|^2}{\left|\frac{\varphi(z)}{M}\right|}<n\qquad(|z|=r),$$

provided that $\varphi(z)$ does not vanish at a point $z(|z|<1)$. Consequently we get (6) and the theorem is proved.

The particular case $c_0=1$ of Theorem 2 gives LOOMIS' extension of DIEUDONNÉ's theorem to the n-valency:

Theorem 3. (LOOMIS' theorem). *Suppose that* $f(z)=z^n+\dots\dots\dots$ *is regular and bounded in* $|z|<1$: $|f(z)|<M$. *Then*

(*a*) *The radius* ρ_n *of n-valence and starshapedness of* $f(z)$[10] *is given by*

$$\rho_n=M_n-\sqrt{M_n^2-1},\quad M_n\equiv\frac{1}{2}\left[\left(1+\frac{1}{n}\right)M+\left(1-\frac{1}{n}\right)\frac{1}{M}\right].$$

(*b*) *The modulus* m_n *of n-valence of* $f(z)$[11] *is given by*

$$m_n=\rho_n^n\cdot\frac{M(1-M\rho_n)}{M-\rho_n}.$$

(*c*) *These limits* (*a*) *and* (*b*) *are attained by the function*

$$f_0(z)=\frac{Mz^n(1-Mz)}{M-z}.$$

Proof. Put $c_0=1$ or $\varphi(0)=1$ in Theorem 2 and consider the function

$$F(z)=\frac{1-M\frac{\varphi(z)}{M}}{M-\frac{\varphi(z)}{M}}.$$

Since $F(z)$ is regular and $|F(z)|<1$ in $|z|<1$, $F(0)=0$, SCHWARZ's lemma gives $|F(z)|\leqq|z|$, whence

$$\frac{1-Mr}{M-r}\leqq\left|\frac{\varphi(z)}{M}\right|\leqq\frac{1+Mr}{M+r}\quad(|z|=r). \tag{7}$$

And so by (6) and (7) the radius ρ_n of n-valence may be calculated from the inequality:

$$\frac{1-Mr}{M-r}>\frac{-n(1-r^2)+\sqrt{n^2(1-r^2)^2+4r^2}}{2r}. \tag{8}$$

Multiplying the both sides by $-n(1-r^2)-\sqrt{n^2(1-r^2)^2+4r^2}$, we have

$$\frac{1-Mr}{M-r}[-n(1r^2)-\sqrt{n^2(1-r^2)^2+4r^2}]<\frac{-4r^2}{2r}.$$

Hence

$$-\frac{M-r}{1-Mr}>\frac{-n(1-r^2)-\sqrt{n^2(1-r^2)^2+4r^2}}{2r}. \tag{9}$$

Adding (8) and (9), we have

$$\frac{1-Mr}{M-r}-\frac{M-r}{1-Mr}>\frac{-n(1-r^2)}{r}.$$

Namely

$$n>\frac{r(M^2-1)}{(M-r)(1-Mr)}, \tag{10}$$

which is the same as an inequality obtained by LOOMIS. And (10) reduces, for $r<\frac{1}{M}$, to

$$r^2-\left[\left(1+\frac{1}{n}\right)M+\left(1-\frac{1}{n}\right)\frac{1}{M}\right]r+1>0,$$

which is satisfied (as we see by factoring the left member) if

$$r < M_n - \sqrt{M_n^2 - 1} = \rho_n, \quad M_n \equiv \frac{1}{2}\left[\left(1+\frac{1}{n}\right)M + \left(1-\frac{1}{n}\right)\frac{1}{M}\right].$$

Notice that $\rho_n < \frac{1}{M}$. Using the radius ρ_n, we have from (7)

$$|f(z)| \geqq \frac{M\rho_n^n(1-M\rho_n)}{M-\rho_n} = m_n \qquad (|z| = \rho_n).$$

Thus, LOOMIS' theorem is proved, if we consider the function

$$f_0(z) = \frac{Mz^n(1-Mz)}{M-z},$$

such that

$$f_0(\rho_n) = m_n, f_0'(\rho_n) = \frac{M^2 n \rho_n^{n-1}}{(M-\rho_n)^2}\left[\rho_n^2 - \left\{\left(1+\frac{1}{n}\right)M + \left(1-\frac{1}{n}\right)\frac{1}{M}\right\}\rho_n + 1\right] = 0.$$

Theorem 4. *Let $\varphi(z)$ be regular and $\Re(\varphi(z)) > 0$ for $|z| < 1$. Then, the radius ρ_n of n-valence and starshapedness of $f(z) \equiv z^n \varphi(z)$ is given by $\rho_n = \frac{\sqrt{n^2+1}-1}{n}$. This limit can be attained by the function*

$$f_0(z) = cz^n \frac{1-z}{1+z}, \qquad (c > 0).$$

Proof. Considering a half-plane $D: \Re(z) > 0$ and taking a mapping function $g(z) = \frac{1-z}{1+z}$, we have

$$\Omega(a, D) = 2\Re(a) \text{ and } \left| z\frac{\varphi'(z)}{\varphi(z)} \right| \leqq \frac{2r}{1-r^2} \cdot \frac{\Re(\varphi(z))}{|\varphi(z)|} \leqq \frac{2r}{1-r^2} \quad (|z| = r).$$

Hence, (4) holds whenever $\frac{2r}{1-r^2} < n$. Finally, this reduces to

$$nr^2 + 2r - n < 0,$$

which is satisfied if

$$r < \frac{\sqrt{n^2+1}-1}{n} = \rho_n.$$

Our theorem is proved, if we consider the function

$$f_0(z)=cz^n\frac{1-z}{1+z}, \qquad (c>0),$$

with

$$f'_0(\rho_n)=\frac{c\rho_n^{n-1}}{(1+\rho_n)^2}\cdot(n-2\rho_n-n\rho_n^2)=0.$$

Theorem 5. *Let $\varphi(z)=c_0+\ldots\ldots$ be regular and $0<|\varphi(z)|<M$ for $|z|<1$. Then $f(z)\equiv z^n\varphi(z)$ is n-valent and starshaped with respect to the origin for $|z|<\rho_n<1$, if*

$$\left|\frac{\varphi(z)}{M}\right|>e^{-\frac{n(1-r^2)}{2r}} \qquad (|z|=r). \tag{11}$$

Proof. Considering a pricked (*punktiert*) circular domain D: $0<|z|<M$, and taking a mapping function

$$g(z)=\frac{1+\log\frac{z}{M}}{1-\log\frac{z}{M}},$$

we see that

$$\frac{2r}{1-r^2}\log\left|\frac{M}{\varphi(z)}\right|<n \qquad (|z|=r).$$

The theorem is thus proved.

Theorem 6. *Let $f(z)=z^n+\ldots\ldots$ be regular for $|z|<1$ and $0<|f(z)|<M$ for $0<|z|<1$. Then*

(*a*) *The radius ρ_n of n-valence and starshapedness of $f(z)$ is given by*

$$\rho_n=\frac{\log e^nM-\sqrt{(\log e^nM)^2-n^2}}{n}$$

(*b*) *The modulus m_n of n-valence of $f(z)$ is given by*

$$m_n=\left[\frac{\log e^nM-\sqrt{(\log e^nM)^2-n^2}}{n}\right]^n\cdot M^{-\frac{2n}{\log M+\sqrt{(2n+\log M)\log M}}}$$

(*c*) *These limits* (*a*) *and* (*b*) *are attained by the function*

$$f_0(z)=z^n\cdot M^{\frac{2e^{i\theta}z}{1+e^{i\theta}z}}$$

Proof. Put $c_0=1$ in Theorem 5 and consider the function

$$F(z)=\frac{\log\dfrac{\varphi(z)}{M}-\log\dfrac{1}{M}}{\log\dfrac{\varphi(z)}{M}+\log\dfrac{1}{M}},$$

taking a branch of $\log z$ such that $\log 1=0$. Then the following inequalities result from SCHWARZ's lemma:

$$M^{-\frac{1+r}{1-r}}\leqq\left|\frac{\varphi(z)}{M}\right|\leqq M^{-\frac{1-r}{1+r}}\qquad(|z|=r<1).\tag{12}$$

So by (11) and (12) the radius ρ_n of n-valence and starshapedness is calculated from the inequality:

$$e^{-\frac{n(1-r^2)}{2r}}<M^{-\frac{1+r}{1-r}},$$

or

$$n>\frac{2r}{(1-r)^2}\log M.$$

Hence

$$nr^2-2(\log e^nM)\cdot r+n>0,$$

$$r<\frac{\log e^nM-\sqrt{(\log e^nM)^2-n^2}}{n}=\rho_n.$$

And from (12)

$$|f(z)|\geqq\rho_n^n\cdot M^{-\frac{2\rho_n}{1-\rho_n}}=m_n\qquad(|z|=\rho_n).$$

Our theorem is proved, if we consider the function

$$f_0(z)=z^n\cdot M^{\frac{2e^{i\theta}z}{1+e^{i\theta}z}}$$

with

$$|f_0(-\rho_n e^{-i\theta})|=\rho_n^n\cdot M^{-\frac{2\rho_n}{1-\rho_n}}=m_n,\quad |f'_0(-\rho_n e^{-i\theta})|=\rho_n^{n-1}\cdot M^{-\frac{2\rho_n}{1-\rho_n}}.$$

$$\left|n-\frac{2\rho_n}{(1-\rho_n)^2}\log M\right|=0.$$

Lastly, we consider as a further application of Theorem 1 a ring-domain $D: m<|z|<M$, which can be mapped on $|z|<1$ by a function

$$g(z)=\left(e^{i\frac{\pi}{2}\cdot\frac{\log\frac{z}{\sqrt{mM}}}{\log\sqrt{\frac{M}{m}}}}-1\right):\left(e^{i\frac{\pi}{2}\cdot\frac{\log\frac{z}{\sqrt{mM}}}{\log\sqrt{\frac{M}{m}}}}+1\right).$$

In this case a discussion similar as above gives

$$\frac{2}{\pi}\cdot\frac{r}{1-r^2}\left(\log\frac{M}{m}\right)\cos\frac{\pi\log\frac{\varphi(z)}{\sqrt{mM}}}{\log\frac{M}{m}}<n \qquad (|z|=r).$$

And we obtain

Theorem 7. *Let $\varphi(z)$ be regular and $m<|\varphi(z)|<M$ for $|z|<1$. Then $f(z)\equiv z^n\varphi(z)$ is n-valent and starshaped for $|z|<\rho_n<1$ if*

$$\cos\frac{\pi}{2}\frac{\log\frac{|\varphi(z)|^2}{mM}}{\log\frac{M}{m}}<\frac{n\pi}{2}\cdot\frac{\frac{1-r}{r}}{\log\frac{M}{m}} \qquad (|z|=r).$$

Remark. Putting $n=1$ in Theorem 3, we get DIEUDONNÉ's theorem since $M_n=M$ and further, putting $n=1$ in all results of this paper, NOSHIRO's results. LOOMIS' theorem (see Theorem 3) can be enunciated, with a slight modification, as follows: *Let $f(z)=c_0z^n+\ldots\ldots$, $|c_0|$ given and $\neq 0$, be regular and $|f(z)|<M(M>|c_0|R^n)$ for $|z|<R$. Then the radius ρ_n of n-valence and starshapedness of $f(z)$ and the modulus m_n are given by*

$$\rho_n = R\sigma \quad and \quad m_n = \frac{M\sigma^n(|c_0|R^n - M\sigma)}{M - |c_0|R^n\sigma}$$

respectively, where

$$\sigma \equiv M_n - \sqrt{M_n^2 - 1}, \quad M_n \equiv \frac{1}{2}\left[\left(1+\frac{1}{n}\right)\frac{M}{|c_0|R^n} + \left(1-\frac{1}{n}\right)\frac{|c_0|R^n}{M}\right]^{12)}.$$

It seems to me very difficult to generalize this theorem to the case where p initial coefficients c_0 $c_1, \ldots\ldots, c_{p-1}(c^0 \neq 0)$ are given. We can treat the case where only two initial coefficients c_0, $c_1(c_0 \neq 0)$ are given, by means of the same method as in § IV of NOSHIRO's paper, but we shall refrain from entering into this case.

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NOTES

1) K. Noshiro, *On the univalency of certain analytic functions,* Journ. Fac. Sci. Hokkaido Imp. Univ. (1) **2**, Nos. 1–2 (1934), pp. 89–101.

2) J. Dieudonné, *Recherches sur quelques problemes relatifs aux polynômes et aux fonctions bornées d'une variable complexe,* Thèse de Paris; Ann. Sci. École Norm. Sup. vol. 48 (1931), pp. 248–358.

3) We mean by $\varphi(z) \subset D$ that the set of values taken by $\varphi(z)$ in $|z|<1$ belongs to the domain D.

4) S. Kakeya, *On the zero-points of a limited function,* Sci. Rep. of Tokyo Bunrika Daigaku, Section A, I, No. 14 (1931), pp. 159–165.

5) L. H. Loomis, *The radius and modulus of n-valence for analvitc functions whose first n–1 derivatives vanish at a point,* Bull. Amer. Math. Soc. vol. 46, No. 6 (1940), pp. 496–561.

6) A. Kobori, *Über die notwendige und hinreichende Bedingung dafür, dass eine Potenzreihe den Kreisbereich auf den schlichten sternigen bzw. konvexen Bereich abbildet,* Mem. Coll. Sci. Kyoto Imp. Univ. (A) **15** (1932) pp. 279–291.

7) See 1), foot-notes at p. 90.

8) See 4), p. 161.

9) See 1), p. 91.

10) The radius of n–valence of the function $f(z)$ means the radius of the largest circle within which $f(z)$ assumes no value more than n times, and assumes at least one value n times.

11) The modulus of n–valence of $f(z)$ means the radius of the largest circle whose interior is covered exactly n times by the map under $f(z)$ of $|z|<\rho$ where ρ is the above radius of n–valence.

12) Cf. Corollary 1 of 5).